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THE HAM SANDWICH THEOREM AND SOME RELATED RESULTS

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ABSTRACT. Using integral transforms, a new and elementary proof of the ham sandwich theorem is presented. The proof requires a corollary of the Borsuk-Ulam theorem. Conversely, it is shown that the ham sandwich theorem implies this corollary. In the course of establishing the converse implication, a weak L^1 inversion theorem for the Radon transform is obtained.

It is our purpose in this paper to establish an interrelation between two known results in topology and geometry. The technique of proof of our theorems involves the theory of integral transforms which is of considerable independent interest.

Given a set E in n-dimensional Euclidean space $\mathbf{R}^n$, we shall call any positive measure μ with support contained in E a *mass density* of E. A measure, μ, not assumed to be necessarily positive, will be called a *signed mass density*. We shall assume throughout that the mass densities have zero absolute mass on any hyperplane of $\mathbf{R}^n$. Given any measure μ whose total mass on $\mathbf{R}^n$ satisfies

$$\int_{\mathbf{R}^N} d\mu(x) \neq 0, \tag{1}$$

it is evident that a hyperplane defined by $\{x \in \mathbf{R}^n | \langle \xi, x \rangle = p\}$ may be chosen which divides the mass of E into two equal parts. Indeed, given any $\xi \in \mathbf{R}^n$, $\xi \neq 0$, the mass of the set

$$E \cap \{x \in \mathbf{R}^n | \langle \xi, x \rangle \leqq p\} \tag{2}$$

is a continuous monotonic function of p. For each fixed ξ, the mass of the set described by (2) approaches either zero or the value of the integral in (1) as $|p| \to \infty$. Thus, the division of the mass of E, as defined by (1), into two equal parts is a simple consequence of the intermediate value theorem. Since no restriction has been made on the value of ξ, it is reasonable to conjecture that n sets $E_1, \ldots, E_n$ with mass densities satisfying (1) can be simultaneously divided into two equal parts by a fixed hyperplane. Actually, a somewhat stronger assertion can be shown which we now state for later reference.

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PROPOSITION 1 *Consider n sets E_j $(j = 1, \ldots, n)$ with signed mass densities $\mu_1, \ldots, \mu_n$ each having zero absolute mass on any hyperplane in $\mathbf{R}^n$. If, at least one of the $\mu_1, \ldots, \mu_n$ satisfies* (1), *then there exists a hyperplane $\{x \in \mathbf{R}^n | \langle \xi, x \rangle = p\}$ which simultaneously divides the mass of each set into two equal parts.*

Before obtaining a proof of Proposition 1 we note the special case where each set E_j $(j = 1, \ldots, n)$ is of finite measure and $d\mu_j$ is the characteristic function of E_j. The fact that these sets can be simultaneously divided in two equal parts by a hyperplane is the statement of the ham sandwich theorem. The name is derived from the interpretation, in R^3, of E_1 and E_2 as slices of bread and E_3 as a slice of ham; the sandwich can be cut into two equal halves.

Our proof of Proposition 1 will depend on a well-known result concerning the variety, i.e., the set of common zeroes, of a collection of continuous functions defined on the unit sphere S^n of $\mathbf{R}^{n+1}$.

PROPOSITION 2. (cf. [3, p. 93]) *Given n continuous functions $\phi_1(\theta), \ldots \phi_n(\theta)$, defined on S^n, which satisfy the symmetry condition*

$$\phi_j(\theta) = -\phi_j(-\theta) \quad \text{for } j = 1, 2, \ldots n \text{ and } \theta \in S^n, \tag{3}$$

there is a zero common to all the ϕ_j's.

We shall prove the following theorem.

THEOREM 1. *Proposition* 2 *implies Proposition* 1.

PROOF. We denote by $H(q)$ the Heaviside function defined by

$$H(q) = \begin{cases} 1 \text{ if } q > 0 \\ 0 \text{ if } q \leqq 0 \end{cases} \tag{4}$$

where $q \in \mathbf{R}^1$. Given a signed mass density μ on $\mathbf{R}^n$ its Heaviside transform is defined by

$$\hat{\mu}(\xi, p) = \int_{\mathbf{R}^n} H(p - \langle \xi, x \rangle) d\mu(x) \tag{5}$$

where $\xi \in \mathbf{R}^n \backslash \{0\}$, $p \in \mathbf{R}^1$. The domain of definition of $\hat{\mu}$ is then extended to $p = \pm\infty$ by taking the limit of $\hat{\mu}(\xi, p)$ as p approaches $\pm\infty$ respectively. We shall consider the Heaviside transform of each signed mass density $\mu_j(x)$. It follows immediately from (5) that each $\hat{\mu}_j(\xi, p)$ is homogenous of degree zero on $\mathbf{R}^{n+1}$, i.e.,

$$\hat{\mu}(\alpha\xi, \alpha p) = \hat{\mu}(\xi, p), \alpha > 0. \tag{6}$$

Consequently, each $\hat{\mu}_j(\xi, p)$ can be identified with a function on S^n. Next, we note that for all $j = 1, \ldots, n$

$$\hat{\mu}_j(\xi, p) + \hat{\mu}_j(-\xi, -p) = \int_{\mathbf{R}^n} d\mu_j(x). \tag{7}$$

Finally, we note that each $\hat{\mu}_j(\xi, p)$ is continuous in ξ and p. This verification is left to the reader. We define $\phi_j(\xi, p)$ to be the odd component of $\hat{\mu}_j(\xi, p)$, i.e.,

$$\phi_j(\xi, p) = (1/2)[\hat{\mu}_j(\xi, p) - \hat{\mu}_j(-\xi, -p)].$$

It is evident from (7) that the even component of $\hat{\mu}$ is equal to one half of the total mass of μ_j on $\mathbf{R}^n$. Thus

$$\phi_j(\xi, p) = \hat{\mu}_j(\xi, p) - (1/2) \int_{\mathbf{R}^n} d\mu_j(x). \tag{8}$$

The functions $\phi_1, \ldots, \phi_n$ satisfy all the hypotheses of Proposition 2. If (ξ_0, p_0) denotes the zero common to all the ϕ_j's, then it follows from (8) that for $j = 1, \ldots, n$

$$\hat{\mu}_j(\xi_0, p_0) = (1/2) \int_{\mathbf{R}^n} d\mu_j(x).$$

We cannot have $|p_0| = \infty$ since this would imply that the total mass of μ_j over $\mathbf{R}^n$ is zero for all j. This contradicts our hypothesis that at least one μ_j satisfies (1). Consequently, the odd components ϕ_j of $\hat{\mu}_j$ vanish simultaneously at a finite value of p_0. This concludes the proof.

We now seek a converse to Theorem 1. Rather than showing that Proposition 1 implies Proposition 2 as stated, we shall establish a statement equivalent to Proposition 2.

PROPOSITION 2'. *Let $\phi_1(\theta), \ldots, \phi_n(\theta)$ be infinitely differentiable functions defined on S^n which are constant on some neighborhood of the north pole of S^n. If, in addition, each $\phi_j(\theta)$ satisfies the symmetry condition.*

$$\phi_j(\theta) = -\phi_j(-\theta), \theta \in S^n,$$

then there is a zero common to all the ϕ_j's.

It is evident that Proposition 2 implies Proposition 2'. Reversing the implication consists of showing that each continuous ϕ_j in Proposition 2 is the uniform limit of a sequence of infinitely differentiable functions $\{\phi_{jm}\}$ as in Proposition 2'. Indeed, once such an approximation is obtained, then Propositions 2' guarantees the existence of a common zero of the ϕ_{jm}'s for each m, and consequently, their limits the ϕ_j's. The method by which we obtain the approximation is, essentially, a routine convolution argument (cf. [**2**, p. 143]). We now proceed to establish Proposition 2' on the basis of Proposition 1. As such, all ϕ_j's referred to in the sequel will be assumed to satisfy the hypotheses of Proposition 2'. This is phrased in terms of the ξ, p coordinates as follows. Let $\eta = (\eta_1, \ldots, \eta_n)$ and q be coordinates on S^n where

$$(9) \qquad \eta_\ell = \xi_\ell\Big(\sum_{j=1}^{n} \xi_j^2 + p^2\Big)^{-1/2}, \qquad q = p\Big(\sum_{j=1}^{n} \xi_j^2 + p^2\Big)^{-1/2}.$$

The hypotheses of Proposition 2′ require that ϕ be infinitely differentiable in η and q as well as being constant for $|q|$ sufficiently close to 1. In terms of the ξ, p coordinates this implies that ϕ is infinitely differentiable and constant for p sufficiently large and $\xi \neq 0$ restricted to a compact subset of $\mathbf{R}^n$.

Suppose that each ϕ_j is the odd component of the Heaviside transform of some μ_j. Then ϕ_j differs from $\hat{\mu}_j$ by a constant which is numerically equal to half the total mass of μ_j. Surprisingly enough, we can obtain an inverse Heaviside transform to determine $\mu_j(x)$ from only a knowledge of $\phi_j(\xi, p)$. To accomplish this, we differentiate the Heaviside transform, with respect to p, under the integral. This yields

$$(10) \qquad \frac{\partial\hat{\mu}(\xi, p)}{\partial p} = \int_{\mathbf{R}^n} \frac{\partial H(p - \langle\xi, x\rangle)}{\partial p} d\mu(x) = \int_{\mathbf{R}^n} \delta(p - \langle\xi, x\rangle) d\mu(x) = \check{\mu}(\xi, p),$$

where $\delta(p - \langle\xi, x\rangle)$ denotes the Dirac mass concentrated on the hyperplane $\langle\xi, x\rangle = p$. The function $\check{\mu}(\xi, p)$ so obtained is the Radon transform of the signed mass density μ (cf. [**1**, **5**, **6**]). If the Radon-Nikodym derivative of μ is identifiable with an infinitely differentiable, rapidly decreasing function f, i.e., $d\mu(x) = f(x)dx$, then we may recover f from its Heaviside transform. For spaces of odd dimension $\mathbf{R}^{2k+1} (k \geqq 1)$, the inversion formula is

$$(11) \qquad f(x_0) = \frac{(-1)^k}{2(2\pi)^{2k}} \int_\Gamma \hat{f}^{(2k+1)}(\xi, \langle\xi, x_0\rangle)\omega(\xi),$$

where $\hat{f}^{(2k+1)}(\xi, \langle\xi, x_0\rangle)$ denotes the $(2k + 1)$-st partial derivative of $\hat{f}$, with respect to p, evaluated at $p = \langle\xi, x_0\rangle$. The surface Γ is taken to be the unit sphere of $\mathbf{R}^{2k+1}$ and $\omega(\xi)$ is the induced differential form of volume on this surface. This is given by

$$(12) \qquad \omega(\xi) = \sum_{j=1}^{2k+1} (-1)^{j-1} \xi_j \prod_{m \neq j} d\xi_m.$$

More generally, Γ may be chosen to be any manifold (or collection of manifolds) which is the boundary of a neighborhood of the origin (cf. [**1**]).

The validity of (11) is an immediate consequence of the analogous result for the Radon transform and (10) (cf. [**1**, p. 11]). In obtaining a converse to Theorem 1 we shall use (11) as follows. Suppose that $\hat{f}(\xi, p)$ is a Heaviside transform with odd component $\phi(\xi, p)$. Then since

$$\frac{\partial^{2k+1}\hat{f}(\xi, p)}{\partial p^{2k+1}} = \frac{\partial^{2k+1}\phi(\xi, p)}{\partial p^{2k+1}}, \tag{13}$$

we may recover $f(x)$ from only a knowledge of the odd component of its Heaviside transform.

A limitation of the above inversion formula is the assumption of odd dimensionality of $\mathbf{R}^n$. We remark that a somewhat more complicated inversion formula may be obtained for $\mathbf{R}^n$ with n even. In the present discussion, however, it is more appropriate to extend the functions $\phi_1, \ldots, \phi_{2k}$ on S^{2k} to a system of $2k + 1$ functions on S^{2k+1}. This is accomplished by defining

$$\psi_j(\xi_1, \ldots, \xi_{2k+1}, p) = \phi_j(\xi_1, \ldots, \xi_{2k}, p)$$

for $j = 1, \ldots, 2k$ and

$$\psi_{2k+1}(\xi_1, \ldots, \xi_{2k+1}, p) = \frac{p}{\|\xi\|} \exp\left[-(1 - p^2/\|\xi\|^2)^{-1}\right] \tag{14}$$

where

$$\|\xi\|^2 = \sum_{k=1}^{2k+1} \xi_j^2.$$

The exponential function defined above is taken to be zero for $|p| \geqq \|\xi\|$. It is evident that the functions $\psi_1, \ldots, \psi_{2k+1}$ are infinitely differentiable, odd, and homogeneous of degree zero. Thus, they are identifiable with odd, infinitely differentiable functions on S^{2k+1}. Further, a common zero of the ψ_j's, if it exists, is also a common zero of the ϕ_j's. Finally, it is not difficult to see that each ψ_j is constant for p sufficiently large and $\xi \neq 0$ restricted to a compact subset of $\mathbf{R}^{2k+1}$.

Our next result deals with the question of when an infinitely differentiable, odd function on S^{2k+1} is the odd component of a Heaviside transform. Initially, our attention will be restricted to the space of infinitely differentiable functions with compact support, denoted $C_0^\infty(\mathbf{R}^{2k+1})$. We now characterize the space of functions on S^{2k+1} which are odd components of Heaviside transforms of functions in $C_0^\infty(\mathbf{R}^{2k+1})$. It is easily verified that given $f(x) \in C_0^\infty(\mathbf{R}^{2k+1})$ its Heaviside transform $\hat{f}(\xi, p)$ is infinitely differentiable in p and $\xi \neq 0$. The compact support of $f(x)$ implies that $\hat{f}(\xi, p)$ is constant for $|p|$ sufficiently large, and ξ restricted to a compact set. The odd component of $\hat{f}(\xi, p)$ is also infinitely differentiable and constant for $|p|$ sufficiently large. The latter statement follows immediately by recalling that the even component of $\hat{f}(\xi, p)$ is a constant. The following set of homogeneity conditions complete the characterization. For every non-negative integer m,

$$(15)\qquad \int_{-\infty}^{\infty} \frac{\partial \hat{f}(\xi, p)}{\partial p}\, p^m dp = \int_{-\infty}^{\infty} \check{f}(\xi, p) p^m dp = Q_m(\xi)$$

yields a polynomial in ξ which is homogeneous of degree m. We remark that this also includes the possibility that $Q_m(\xi) \equiv 0$. It is easily verified that $\hat{f}(\xi, p)$ satisfies (15). Indeed, for the Radon transform $\check{f}$ we have

$$(16)\qquad \int_{-\infty}^{\infty} \check{f}(\xi, p) p^m dp = \int_{\mathbf{R}^{2k+1}} f(x) \langle \xi, x\rangle^m dx.$$

The right hand side of (16) yields a polynomial $Q_m(\xi)$ as described above. Since the even component of $\hat{f}$ is constant, we may replace $\hat{f}$ by its odd component in (15).

To show that these conditions do, in fact, characterize the Heaviside transform, suppose that $\phi(\xi, p)$ is an odd function, defined for $\xi \in \mathbf{R}^{2k+1}\backslash \{0\}$, $p \in \mathbf{R}^1$ which is homogeneous of degree zero. If, in addition, $\phi(\xi, p)$ is infinitely differentiable in p and $\xi \neq 0$, constant for $|p|$ sufficiently large with ξ restricted to a compact set, and (15) is valid for every integer $m \geqq 0$, then $\partial\phi/\partial p$ is the Radon transform of some $f(x) \in C_0^\infty(\mathbf{R}^{2k+1})$. The proof of this last statement is an immediate consequence of the analogous characterization for the Radon transform as given in [5].

The hypotheses of Proposition 2′ imply that each $\phi(\xi, p)$ satisfies all the above mentioned conditions with the exception of (15) for every integer $m \geqq 0$. Indeed, $\partial\phi/\partial p$ is infinitely differentiable in p and $\xi \neq 0$ and has compact support (in p) for each ξ restricted to a compact set. Applying the inversion theorem for square summable functions as obtained in [5], there exists a function $f(x) \in L^2(\mathbf{R}^{2k+1})$ such that $\partial\phi/\partial p$ is the Radon transform of $f(x)$. Since the inversion formula given in (11) is valid for square summable functions, we may compute successive partial derivatives of $f(x)$ by differentiating the Radon transform under the integrals. Since $\check{f}(\xi, p)$ has compact support with ξ restricted so that $\|\xi\| = 1$, this is permissible, and we conclude that $f(x)$ is infinitely differentiable. We cannot assert, however, that $f(x)$ has compact support unless (15) is valid for every $m \geqq 0$.

While ϕ need not satisfy (15) for every $m \geqq 0$, it is valid for $m = 0$. To prove this, we recall the conditions which ϕ satisfies when viewed as a function defined on S^{2k+1}. These are:

(i) $\phi(\eta, q)$ is an infinitely differentiable odd function on S^{2k+1}.

(ii) $\phi(\eta, q)$ is constant for $|q|$ sufficiently close to 1.

It is implicit in the statement of (i) that the limit as $q \to \pm 1$ of $\phi(\eta, q)$ is well defined. Equivalently, $\phi(\xi, \infty)$ and $\phi(\xi, -\infty)$ are well defined, i.e., independent of ξ. Consequently, we may write

$$\phi(\xi, p) - \phi(\xi, -\infty) = \int_{-\infty}^{p} \frac{\partial\phi(\xi, t)}{\partial t}\, dt$$

and, in the limit as $p \to \infty$,

$$\phi(\xi, \infty) - \phi(\xi, -\infty) = \int_{-\infty}^{\infty} \frac{\partial\phi(\xi, t)}{\partial t}\, dt. \tag{17}$$

The right hand side of (17) is $Q_0(\xi)$ as previously defined. Since $\phi(\xi, \infty)$ and $\phi(\xi, -\infty)$ are numerical values independent of ξ, $Q_0(\xi)$ is a polynomial homogeneous of degree zero, i.e., a constant. We summarize the remarks of the last two paragraphs as follows.

LEMMA 1. *Suppose that* $\phi(\eta, q)$ *is a function defined on* S^{2k+1} *satisfying* (i) *and* (ii). *If, in addition,* ϕ *is an infinitely differentiable function of* p *and* $\xi \neq 0$, *then* $\partial\phi/\partial p = \check{f}$ *where* $\check{f}$ *is the Radon transform of an infinitely differentiable, square summable function* $f(x)$. *Further,*

$$\int_{-\infty}^{\infty} \check{f}(\xi, p)dp = a \text{ constant.} \tag{18}$$

It is known that the above conditions on $\check{f}(\xi, p)$ also imply that $f(x)$ is a pseudofunction (cf. [**6**, p. 1258]). This means that the Fourier transform of $f(x)$ is continuous and vanishes at infinity. We improve upon this by establishing that $f(x)$ is integrable. i.e., $f(x) \in L^1(\mathbf{R}^{2k+1})$. By the Riemann-Lebesgue Lemma, our result immediately implies that the Fourier transform of $f(x)$ is continuous and vanishes at infinity.

LEMMA 2. *Suppose that* $\check{f}(\xi, p)$ *is infinitely differentiable in* p, *and* $\xi \neq 0$, *and has compact support in* p *for* ξ *restricted to a compact set. If, in addition,* $\check{f}(\xi, p)$ *satisfies* (18), *then* $f(x) \in L^1(\mathbf{R}^{2k+1})$.

PROOF. Following [**1**, p. 17], we choose the surface Γ in the inversion formula to be the hyperplanes $\xi_{2k+1} = \pm 1$. This choice of Γ is permissible since the integrand of (11) is homogeneous of degree zero. Consequently, any surface which encloses the origin may be used in place of the unit sphere of $\mathbf{R}^{2k+1}$. A surface Γ is said to enclose the origin if for every $\xi \neq 0$, $\alpha\xi \in \Gamma$ for some $\alpha > 0$. Strictly speaking, the set of hyperplanes $\xi_{2k+1} = \pm 1$ do not enclose the origin in that any ξ of the form $\xi = (\xi_1, \ldots, \xi_{2k}, 0)$ satisfies $\alpha\xi \notin \Gamma$ for all $\alpha > 0$. However, since this set of directions which is not enclosed by Γ is of measure zero in the space of homogeneous coordinates of the form $(\xi_1, \ldots, \xi_{2k+1})$, we may still evaluate the integral over the remaining set of directions.

If Γ is so chosen, then a simple computation yields

$$f(rx_0) = r^{-1} \frac{(-1)^k}{(2\pi)^{2k}} \int_\Gamma \check{f}^{(2k)}(pr^{-1}, \xi_2, \ldots, \xi_{2k}, 1; p)dpd\xi_2 \ldots d\xi_{2k}$$

where $x_0 = (1, 0, 0, \ldots 0)$ and r is a real number. Since the integrand is an infinitely differentiable function of r^{-1}, we can expand $\check{f}^{(2k)}$ in an asymptotic series in powers of r^{-1}. Thus,

$$f(rx_0) \sim (-1)^k (2\pi)^{-2k} \sum_{j=0}^{\infty} \frac{r^{-j-1}}{j!} \times$$
$$\left[\frac{\partial^j}{\partial \xi_1^j} \int_{\Gamma} \check{f}^{(2k)}(\xi_1, \xi_2, \ldots \xi_{2k}; p) p^j dp \, d\xi_2 \ldots d\xi_{2k} \right]_{\xi_1 = 0}.$$

Of course, the series approximates $f(rx_0)$ to any desired accuracy outside of any compact neighborhood K of the origin. Since $f(rx_0)$ is integrable on K, the proof hinges on showing that the first $2k + 1$ terms of the series vanish. Indeed, suppose the asymptotic series of $f(rx_0)$ begins with a constant multiple of r^{-2k-2}. Then, since r^{-2k-2} is integrable on $\mathbf{R}^{2k+1} \backslash K$, so is $f(rx_0)$. We now show that

$$\frac{\partial^j}{\partial \xi_1^j} \int_{-\infty}^{\infty} \check{f}^{(2k)}(\xi, p) p^j dp = 0 \tag{19}$$

for $0 \leqq j \leqq 2k$. For $j \leqq 2k - 1$ this follows immediately. Indeed, for $j \leqq 2k - 1$ we have, by virtue of the compact support of $\check{f}(\xi, p)$ for fixed ξ,

$$\int_{-\infty}^{\infty} \check{f}^{(2k-j)}(\xi, p) \, dp = 0.$$

By successive integration by parts, this implies that the integral in (19) must also vanish. It is only for $j = 2k$ that we require the validity of (18). It follows from (18) that

$$\frac{\partial^{2k}}{\partial \xi_1^{2k}} \int_{-\infty}^{\infty} \check{f}(\xi, p) dp = 0.$$

Again, by successive integration by parts, we obtain (19) for $j = 2k$. To complete the proof we extend the above argument to all rotations of x_0. Letting U denote a unitary matrix, this requires computing the asymptotic series for $f(Ux_0)$ as we did for $f(x_0)$. The computation is facilitated by applying a result obtained in [**1**, p. 6] which states that the Radon transform of $f(Ux)$ is given by $\check{f}(U^{-1}\xi, p)$. The validity of (18) and (19) are invariant under U^{-1}, the rotation inverse to U. Since U is arbitrary, the lemma is established.

We have assembled all the details necessary to establish the converse of Theorem 1.

THEOREM 2. *Proposition* 1 *implies Proposition* 2 *or, equivalently, Proposition* 2′.

PROOF. Suppose that $\phi_1, \phi_2, \ldots, \phi_n$ are given to be infinitely differentiable odd functions on S^n each of which are constant for $|q|$ sufficiently close to 1. Since these functions on S^n may be extended to a system of $n + 1$ functions on S^{n+1} via (14), it is sufficient to determine a common zero of the ϕ_j's with n odd.

Each $\phi_j(\eta, q)$ satisfies (i) and (ii). Rewriting ϕ_j in terms of the ξ, p coordinates we have, by virtue of Lemma 1, $\partial\phi_j/\partial p$ is the Radon transform of an infinitely differentiable square summable function $f_j(x)$. By Lemma 2, $f_j(x)$ is, in fact, integrable. As previously noted, each $f_j(x)$ need not necessarily have compact support. We now invoke the hypothesis that (1) is satisfied for at least one $f_j(x)$. In terms of their Heaviside transforms this requires that at least one $\hat{f}_j(\xi, p)$ does not vanish at $p = \pm\infty$. Suppose to the contrary that $\hat{f}_j(\xi, \infty) = 0$ for each j. Since this is equivalent to saying that each $\hat{f}_j(\xi, p)$ is odd, we have $\phi_j(\xi, p) = \hat{f}_j(\xi, p)$ for each j. Thus, all of the ϕ_j's must vanish at $p = \infty$. In terms of the η, q coordinates on S^n, this asserts the existence of a common zero at $q = \pm 1$. In this case Proposition 2′ is trivially valid. Ruling out this case, we have established the validity of (1) for at least one $f_j(x)$. Applying the Ham Sandwich Theorem to the f_j's we obtain

$$\begin{aligned} 0 &= \hat{f}(\xi_0, p_0) - (1/2)\int_{\mathbf{R}^n} f_j(x)dx \\ &= (1/2)[\hat{f}_j(\xi_0, p_0) - \hat{f}_j(-\xi_0, -p_0)] = \phi_j(\xi_0, p_0) \end{aligned}$$

for some ξ_0 and p_0 and all j. This concludes the proof and establishes the equivalence of Proposition 1 and 2.

Several remarks are in order concerning the above results. First, it is worth noting that our methods give constructive solutions to what are usually presented as merely existence theorems. This is a decided advantage in the use of integral transforms. Secondly, it is not difficult to show that Proposition 2 is equivalent to the statement that an odd continuous $\phi\colon S^n \to S^n$ wraps around the origin. This means that any ray in $\mathbf{R}^{n+1}$ meets the range of ϕ. The Borsuk-Ulam Theorem states that ϕ, as defined above, wraps around the origin an odd number of times, i.e., any ray in $\mathbf{R}^{n+1}$ meets the range of ϕ an odd number of times (cf. [**3**, p. 93]). This number is called the winding number of ϕ at the origin. Consequently, the Borsuk-Ulam Theorem is stronger than both Propositions 1 and 2 by an amount which is described in terms of winding numbers. This provides some perspective on the usual proofs of the Ham Sandwich Theorem which require the Borsuk-Ulam Theorem.

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