ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 11, Number 2, Spring 1981

A NOTE ON ALGEBRAS BETWEEN L^∞ AND H^∞

CARL SUNDBERG.

Let L^∞ denote the usual Lebesgue space of essentially bounded functions on the unit circle, and let H^∞ denote the closed subalgebra of L^∞ consisting of functions on the circle that are radial limits a.e. of bounded analytic functions of the unit disc. In this note we prove the following theorem.

THEOREM. *Let B be a closed proper subalgebra of L^∞ containing H^∞. Then L^∞ is not countably generated as a closed algebra over B. In particular, L^∞/B is not a separable Banach space.*

This theorem, which generalizes the well-known fact that L^∞/H^∞ is not separable (see [2] for seven different proofs!), also implies that H^∞ is not contained in any maximal subalgebra of L^∞, a fact first proven by Hoffman and Singer in [5]. To see this, simply note that if B were such a maximal subalgebra and $f \in L^\infty \backslash B$, then L^∞ would be generated as a closed algebra over B by the single function f.

The proof of the theorem follows after a few preliminary remarks and a lemma.

It is well known (see [6]) that the linear span of H^∞ and the continuous functions, $H^\infty + C$, is a closed subalgebra that is contained in every subalgebra of L^∞ properly containing H^∞. Hence we may assume without loss of generality that the algebra B in the statement of the theorem contains $H^\infty + C$. From now on, B will denote a proper closed subalgebra of L^∞ containing $H^\infty + C$. For functions $f_1, f_2, \ldots \in L^\infty$ we denote by $B[f_1, f_2 \ldots]$ the closed subalgebra of L^∞ generated over B by $f_1, f_2, \ldots$, i.e., the smallest closed subalgebra of L^∞ containing B and the functions $f_1, f_2, \ldots$.

LEMMA. *If $f_1, f_2, \ldots \in L^\infty$, then there is a Blaschke product b_0 such that $B[f_1, f_2, \ldots] \subseteq B[\bar{b}_0]$.*

PROOF. By [3, Theorem 2], we can approximate each f_n by a function of the form $g\bar{b}$, where $g \in H^\infty$ and b is an inner function. Since every inner function is a uniform limit of Blaschke products (see [4, pp. 175–177]), we may assume that b is in fact a Blaschke product. Hence we can find a countable family of Blaschke products $b_1, b_2, \ldots$ such that $B[f_1, f_2, \ldots] \subseteq B[\bar{b}_1, \bar{b}_2, \ldots]$. Following Axler [1] we write each b_n in the form $b_n = b_n' b_n''$

Received by the editors on March 29, 1979, and in revised form on August 1, 1979.
Copyright © 1981 Rocky Mountain Mathematics Consortium

where b'_n is a finite Blaschke product and the infinite product $b_0 = \prod_{n=1}^{\infty} b''_n$ converges. Since $b_0\bar{b}''_n \in H^\infty$, we have that $b_0\bar{b}_n = (b_0\bar{b}''_n)\bar{b}'_n \in H^\infty + C$. Since B contains $H^\infty + C$, it thus follows that $\bar{b}_n = (b_0\bar{b}_n)\bar{b}_0 \in B[\bar{b}_0]$ for every n. Hence $B[f_1, f_2, \ldots] \subseteqq B[\bar{b}_1, \bar{b}_2, \ldots] \subseteqq B[\bar{b}_0]$.

Proof of the theorem. By the lemma all we need show is that $B[\bar{b}_0] \neq L^\infty$ for any Blaschke product b_0. Assume the contrary, and write

$$b_0(z) = \prod_{n=1}^{\infty} \frac{|\alpha_n|}{\alpha_n} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z}.$$

Let p_n be a sequence of positive integers such that $p_n \to \infty$ but $\sum_{n=1}^{\infty} p_n(1 - |\alpha_n|) < \infty$. Then

$$b_1(z) = \prod_{n=1}^{\infty} \left(\frac{|\alpha_n|}{\alpha_n} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z} \right)^{p_n}$$

converges and it is easy to see that $b_1 \bar{b}_0^n \in H^\infty + C \subseteqq B$ for every positive integer n. Now if $\bar{b}_1$ were in $B[\bar{b}_0]$, we could write, for any $\varepsilon > 0$,

$$\| f_0 + f_1\bar{b}_0 + \cdots + f_n\bar{b}_0^n - \bar{b}_1\|_\infty < \varepsilon$$

for some $f_0, \ldots f_n \in B$. But

$$\begin{aligned} &\| f_0 + f_1\bar{b}_0 + \cdots + f_n\bar{b}_0^n - \bar{b}_1\|_\infty \\ &\qquad = \|b_1\bar{b}_0(f_0 + f_1\bar{b}_0 + \cdots + f_n\bar{b}_0^n - \bar{b}_1)\|_\infty \\ &\qquad = \| f_0 b_1\bar{b}_0 + f_1 b_1\bar{b}_0^2 + \cdots + f_n b_1\bar{b}_0^{n+1} - \bar{b}_0\|_\infty . \end{aligned}$$

Since $f_0 b_1\bar{b}_0 + f_1 b_1\bar{b}_0^2 + \cdots + f_n b_1\bar{b}_0^{n+1} \in B$, this last expression shows that we could approximate $\bar{b}_0$ by elements of B. Hence $\bar{b}_0$ would be in B, so we would have $L^\infty = B[\bar{b}_0] = B$, contradicting the assumption that B is proper. This completes the proof.

As a further example of how far below L^∞ any proper subalgebra containing H^∞ must lie, we mention the following easily proven fact.

Proposition. *Let $B_1, B_2, \ldots$ be closed algebras such that $H^\infty \subseteqq B_1 \subsetneqq B_2 \subsetneqq \cdots \subseteqq L^\infty$. Then*

$$\overline{\bigcup_{n=1}^{\infty} B_n} \neq L^\infty.$$

Proof. Since $B_n \neq L^\infty$, we can use [**3**, Theorem 2] once more to conclude that there exists a Blaschke product b_n such that $\bar{b}_n \notin B_n$. As before we can assume $H^\infty + C \subseteqq B_n$, so we can arrange things so that $b = \prod_{n=1}^{\infty} b_n$ converges. Then $\bar{b} \notin B_n$ for any n. By elementary Banach algebra theory, this forces $\|\bar{b} - f\|_\infty \geqq 1$ for $f \in B_n$ for any n. Therefore $\bar{b} \notin \overline{\bigcup B_n}$.

REFERENCES

1. S. Axler, *Factorization of L^∞ functions*, Annals of Mathematics **106** (1977), 567–572.

2. E. Berkson and L. A. Rubel, *Seven different proofs that L^∞/H^∞ is not separable*, Rocky Mountain Journal **5** (1975), 237–245.

3. R. G. Douglas and W. Rudin, *Approximation by inner functions*, Pacific Journal of Math. **31** (1969), 313–320.

4. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, Englewood Cliffs, N.J., 1962.

5. ——— and I. M. Singer, *Maximal algebras of continuous functions*, Acta Math. **103** (1960), 217–241.

6. D. Sarason, *Algebras of functions on the unit circle*, Bull. Amer. Math. Soc. **79** (1973), 286-299.

MATHEMATICS DEPARTMENT, UNIVERSITY OF TENNESSEE, KNOXVILLE, TN 37916.