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EXAMPLES OF FIXED POINT FREE MAPS FROM CELLS ONTO LARGER CELLS AND SPHERES

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ABSTRACT. Let $n \geqq 2$. Let A and B be closed n dimensional balls in Euclidean space such that $A \subset B$ and $A \neq B$. Two types of fixed point free maps, f and g, from A onto B are obtained—f is space filling on ∂A, i.e., $f(\partial A) = B$, and g preserves the boundaries of A and B i.e., $g^{-1}(\partial B) = \partial A$. A fixed point free map from a Hilbert cube onto a larger Hilbert cube is obtained which preserves their pseudo-boundaries. Two fixed point free maps with special properties from the bottom half of S^n onto the n sphere S^n, $n \geqq 2$, are obtained. Under the first one the preimage of the North Pole is the Equator. Under the second one the preimage of the South Pole is the Equator and, in addition, the second one is monotone. In relation to these last two examples, the following theorem is proved. If K is a proper nonseparating subcontinuum of S^2 and if f is a monotone mapping from K onto S^2 such that $f[\mathrm{Bd}(K)] \not\subset K$, then f has a fixed point. This theorem is compared with the Knaster-Kuratowski-Mazurkiewicz fixed point theorem.

1. **Introduction.** Any continuous function from an arc onto a larger arc has a fixed point. This simple observation leads to the following question. Does every continuous function from a ball in a Euclidean space onto a larger ball have a fixed point? In various papers, conditions have been imposed on mappings f from certain types of subcontinua of balls onto balls which imply that f has a fixed point (for example, see [**1**], [**2**], [**4**], [**6**], [**9**], [**10**]). However, there seem to be no examples in the literature to show that the question above has a negative answer. In this paper we give such examples. In addition to being continuous and fixed point free, our mappings have certain special properties which lead to an example involving Hilbert cubes and some examples involving spheres. We then prove a fixed point theorem concerning the 2-sphere. This theorem is discussed in the light of our examples and a theorem in [**6**].

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2. **Notation and terminology.** We let $\mathbf{R}^n$ denote Euclidean n-space with its usual norm, $\|x\|_n[\sum_{i=1}^n x_i^2]^{1/2}$ for each $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ We let $B^n = \{x \in \mathbf{R}^n \colon \|x\|_n \leqq 1\}$, $B^n(1/2) = \{x \in \mathbf{R}^n \colon \|x\|_n \leqq 1/2\}$, $S^{n-1} = \{x \in \mathbf{R}^n \colon \|x\|_n = 1\}$, and $S^{n-1}(1/2) = \{x \in \mathbf{R}^n \colon \|x\|_n = 1/2\}$. We let $S^n_- = \{(x_i)_{i=1}^{n+1} \in S^n \colon x_{n+1} \leqq 0\}$, $S^n_+ = \{(x_i)_{i=1}^{n+1} \in S^n \colon x_{n+1} \geqq 0\}$, and $E_n = \{(x_i)_{i=1}^{n+1} \in S^n \colon x_{n+1} = 0\}$. The symbol P_n denotes the North Pole of S^n, i.e., $P_n = (0, \ldots, 0, +1)$, and then, of course, $-P_n$ is the South Pole of S^n. By an *n-cell* we mean a space which is homeomorphic to B^n. If Δ^n is an n-cell, then $\partial\Delta^n$ denotes the manifold boundary of Δ^n (i.e., $\partial\Delta^n$ is the part of Δ^n which corresponds to S^{n-1} under a homeomorphism from Δ^n onto B^n). The symbol Q^∞ denotes the cartesian product of the closed interval $[-1, +1]$ with itself countably many times (Q^∞ is assumed to have the product topology), $Q^\infty(1/2) = \{(x_i)_{i=1}^\infty \in Q^\infty \colon -1/2 \leqq x_i \leqq +1/2$ for each $i = 1, 2, \ldots\}$, $Q^\infty_n = \{(x_i)_{i=1}^\infty \in Q^\infty \colon x_i = 0$ for each $i \geqq n + 1\}$, and $Q^\infty_n(1/2) = \{(x_i)_{i=1}^\infty \in Q^\infty(1/2) \colon x_i = 0$ for each $i \geqq n + 1\}$. Any space which is homeomorphic to Q^∞ is called a *Hilbert cube*. The *pseudo-boundary* of Q^∞ [resp., of $Q^\infty(1/2)$] is denoted by bQ^∞ [resp., by $bQ^\infty(1/2)$] and is defined to be all those points of Q^∞ [resp., of $Q^\infty(1/2)$] which have some coordinate $= \pm 1$ [resp., $= \pm 1/2$]. The symbol cl denotes closure. The symbol Bd denotes topological boundary.

By a *mapping* we mean a continuous function. An onto mapping $f\colon X \to Y$ is *monotone* provided that $f^{-1}(y)$ is connected for each $y \in Y$. If $f\colon X \to Y$ is a mapping and if $Z \subset X$, then $f|Z$ denotes the restriction of f to Z.

3. **Examples and a fixed point theorem.** In (3.1) through (3.6) we give examples of fixed point free mappings. The first three examples involve n-cells. Then we give an example for Hilbert cubes and two examples concerning n-spheres. Our fixed point theorem is discussed after (3.6) and is stated and proved in (3.7).

We begin with the following example of a general nature.

EXAMPLE 3.1. Fix $n \geqq 2$ and let K be any compact uncountable proper subset of B^n. We show that there is an onto fixed point free mapping $f\colon K \to B^n$ such that $f[\mathrm{Bd}(K)] = B^n$, where $\mathrm{Bd}(K)$ denotes the boundary of K in B^n [i.e., $\mathrm{Bd}(K) = K \cap \mathrm{cl}(B^n - K)$]. To show this, first note that $\mathrm{Bd}(K)$ is uncountable (use [5, Cor. 2, p. 48] if K has nonempty interior). Hence, since $\mathrm{Bd}(K)$ is compact, $\mathrm{Bd}(K)$ contains a Cantor set C [7, Cor. 1, p. 445]. Let $c_1, c_2 \in C$ such that $c_1 \neq c_2$ and let $p \in (B^n - K)$ such that p, c_1, and c_2 are not collinear (this choice of p is possible since $B^n - K$ is a nonempty open subset of B^n and $n \geqq 2$). Then, there is an $(n - 1)$-dimensional hyperplane H in $\mathbf{R}^n$ such that $p \in H$ and such that, for each $i = 1$ and 2, $c_i \in U_i$ where U_1 and U_2 are the two components of $\mathbf{R}^n - H$. Clearly there are Cantor subsets C_1 and C_2 of C such that $C_1 \subset U_1$ and $C_2 \subset U_2$.

Let $A_1 = (U_2 \cup H) \cap B^n$ and let $A_2 = (U_1 \cup H) \cap B^n$. Since every compact metric space is a continuous image of a Cantor set [**8**, Cor. 3a, p. 23], there is a mapping ψ_i from C_i onto A_i for each $i = 1$ and 2. For each $i = 1$ and 2, extend ψ_i to $\psi_i^*\colon C_i \cup (H \cap K) \to A_i$ by letting $\psi_i^*(x) = p$ for each $x \in H \cap K$ (note that, since $H \cap C_i = \emptyset$, ψ_i^* is an onto function and is continuous). Note that

$$C_1 \cup (H \cap K) \subset A_2 \cap K, \quad C_2 \cup (H \cap K) \subset A_1 \cap K$$

and that A_1 and A_2 are each homeomorphic to B^n. Hence, by [**7**, Cor. 6c, p. 151], we can extend ψ_1^* and ψ_2^* to mappings $f_1\colon A_2 \cap K \to A_1$ and $f_2\colon A_1 \cap K \to A_2$ respectively. Define $f\colon K \to B^n$ by

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in A_2 \cap K \\ f_2(x), & \text{if } x \in A_1 \cap K. \end{cases}$$

Clearly, f is continuous. Since ψ_i maps $C_i \subset \mathrm{Bd}(K)$ onto A_i for each $i = 1$ and 2, we have that $f[\mathrm{Bd}(K)] = B^n$. Finally, we see that f is fixed point free since $f[H \cap K] \subset \{p\}$, $p \notin K$, and, for each $i = 1$ and 2, $f[U_i \cap K] = f_i[U_i \cap K] = A_i$ and $U_i \cap A_i = \emptyset$.

Let us note the following special case of Example 3.1.

EXAMPLE 3.2. For each $n \geqq 2$, there is an onto fixed point free mapping $f_n\colon B^n(1/2) \to B^n$ such that $f_n[S^{n-1}(1/2)] = B^n$. Hence, not only are there fixed point free mappings of balls onto larger balls, but also such mappings can be chosen to be space filling on the boundary of the smaller ball.

In contrast to the space filling behavior of f_n on $S^{n-1}(1/2)$ in Example 3.2, we have the following example which shows that mappings can be fixed point free and yet preserve boundaries.

EXAMPLE 3.3. We show that for each $n \geqq 2$, there is a fixed point free mapping g_n from $B^n(1/2)$ onto B^n such that $g_n^{-1}(S^{n-1}) = S^{n-1}(1/2)$. We first show that g_2 exists. Let $p \in S^1(1/2)$ and $q \in S^1$ be the points given by $p = (-1/2, 0)$ and $q = (-1, 0)$. For each $x \in S^1(1/2)$, let L_x denote the convex arc (or point if $x = p$) in $B^2(1/2)$ from p to x. For each $y \in S^1$, let M_y denote the convex arc (or point if $y = q$) in B^2 from q to y. Define an onto mapping $f\colon S^1(1/2) \to S^1$ by the following formula. If $x = (1/2)e^{i\theta} \in S^1(1/2)$ where $0 \leqq \theta \leqq 2\pi$, then

$$(f(x) = \begin{cases} e^{i(2\theta+\pi)}, & \text{if } 0 \leqq \theta \leqq \pi/2 \\ e^{i(-2\theta+3\pi)}, & \text{if } \pi/2 \leqq \theta \leqq 3\pi/2 \\ e^{i(2\theta-3\pi)}, & \text{if } 3\pi/2 \leqq \theta \leqq 2\pi. \end{cases}$$

Let $F\colon B^2(1/2) \to B^2$ be the mapping which, for each $x \in S^1(1/2)$, sends L_x linearly onto $M_{f(x)}$ with p going to q, i.e., for each $x \in S^1(1/2)$ and $0 \leqq \lambda \leqq$ $0 \leqq \lambda \leqq 1$,

$$f(\lambda \cdot x + [1 - \lambda] \cdot p) = \lambda \cdot f(x) + [1 - \lambda] \cdot q.$$

It is easy to verify that F is a fixed point free mapping from $B^2(1/2)$ onto B^2; however, $F^{-1}(S^1) \neq S^1(1/2)$ since $F(t, 0) = q$ for each $(t, 0) \in B^2(1/2)$. To correct this deficiency and obtain our desired mapping g_2, let

$$\eta = \inf\{\|F(z) - z\| : z \in B^2(1/2)\}.$$

Note that $\eta > 0$ since F is fixed point free and $B^2(1/2)$ is compact. Let u: $B^2 \to [1 - \eta/2, 1]$ be a mapping such that $u^{-1}(1) = S^1(1/2)$. Now, let $g_2(z) = u(z) \cdot F(z)$ for each $z \in B^2(1/2)$. Clearly, g_2 is a continuous function from $B^2(1/2)$ into B^2. By using the geometry of F and its linearity on each L_x, it follows that g_2 maps $B^2(1/2)$ onto B^2. Since $u^{-1}(1) = S^1(1/2)$, it follows that $g_2^{-1}(S^1) = S^1(1/2)$. To see that g_2 is fixed point free, suppose that $g_2(z) = z$ for some $z \in B^2(1/2)$. Then

$$\begin{aligned}\|F(z) - z\| &= \|F(z) - g_2(z)\| = \|F(z) - u(z) \cdot F(z)\| \\ &= [1 - u(z)]\, \|F(z)\| \leqq 1 - u(z) \leqq \eta/2.\end{aligned}$$

Hence, since $\eta > 0$, $\|F(z) - z\| < \eta$ which contradicts the definition of η. Thus, g_2 is fixed point free. This completes the proof that g_2 has the desired properties. We now show that g_n exists for each $n > 2$. Let h be a homeomorphism from Q_2^∞ onto B^2 such that $h[Q_2^\infty(1/2)] = B^2(1/2)$. Let

$$G_2 = h^{-1} \circ g_2 \circ (h | Q_2^\infty(1/2))$$

and note that G_2 is a fixed point free mapping from $Q_2^\infty(1/2)$ onto Q_2^∞ such that $G_2^{-1}(\partial Q_2^\infty) = \partial Q_2^\infty(1/2)$. For each $x = (x_i)_{i=1}^\infty \in Q^\infty(1/2)$, let

$$G_\infty(x) = G_2(x_1, x_2, 0, 0, \ldots) + (0, 0, 2x_3, 2x_4, \ldots).$$

The formula above defines an onto continuous function G_∞: $Q^\infty(1/2) \to Q^\infty$. Since G_2 is fixed point free, it follows using the formula for G_∞ that G^∞ is fixed point free. Choose and fix $n > 2$. Let $G_n = G_\infty | Q_n^\infty(1/2)$. Recall that G_2 maps $Q_2^\infty(1/2)$ onto Q_2^∞ and that $G_2^{-1}(\partial Q_2^\infty) = \partial Q_2^\infty(1/2)$. Hence, using the formula for G_∞, it follows easily that G_n maps $Q_n^\infty(1/2)$ onto Q_n^∞ and that $G_n^{-1}(\partial Q_n^\infty) = \partial Q_n^\infty(1/2)$. Let k be a homeomorphism from B^n onto Q_n^∞ such that $k[B^n(1/2)] = Q_n^\infty(1/2)$. Let

$$g_n = k^{-1} \circ G_n \circ (k | B^n(1/2)).$$

Then, it follows from the properties of G_n that g_n is a fixed point free mapping from $B^n(1/2)$ onto B^n such that $g_n^{-1}(S^{n-1}) = S^{n-1}(1/2)$. This completes Example 3.3.

In the example above we showed that there are fixed point free mappings from n-balls, $n \geqq 2$, onto larger n-balls which preserve their boundaries. Let us note the corresponding fact about Hilbert cubes and pseudo-boundaries.

EXAMPLE 3.4. There is a fixed point free mapping G_∞ from $Q^\infty(1/2)$ onto Q^∞ such that $G_\infty^{-1}(bQ^\infty) = bQ^\infty(1/2)$. As is easy to see, the mapping G^∞ defined in the proof in Example 3.3 has these properties.

Now we give our two examples involving spheres. In the first one we obtain a fixed point free mapping from the bottom half of S^n, $n \geqq 2$, onto S^n such that the preimage of the North Pole is the Equator (comp., Example 3.6 and Theorem 3.7).

EXAMPLE 3.5. For each $n \geqq 2$, there is a fixed point free mapping k_n from S^n_- onto S^n such that $k_n^{-1}(P_n) = E_n$. This easy to see by using the mappings g_n in Example 3.3 (and by using the fact that S^n is the quotient space obtained from B^n by shrinking the boundary of B^n to a point).

We mention the following generalization of Example 3.5. Many spaces of interest, including some manifolds, are quotient spaces of $B^n(n \geqq 2)$ obtained by identifying certain points of S^{n-1}. Let Z be such a space and let $\nu\colon B^n \to Z$ denote the quotient map. Then, as in Example 3.5, we can find fixed point free mappings f from certain n-cells $\Delta^n \subset Z$ onto Z such that $f^{-1}(\nu[S^{n-1}]) = \partial\Delta^n$. The reader may wish to examine this in the special case when Z is real projective 2-space (the space obtained from B^2 by identifying x with $-x$ for each $x \in S^1$). In connection with this, recall that real projective 2-space has the fixed point property [**3**, p. 31].

By replacing the North Pole in Example 3.5 with the South Pole, we can find fixed point free mappings which are monotone (comp., Theorem 3.7):

EXAMPLE 3.6. For each $n \geqq 2$, there is a fixed point free monotone mapping ψ_n from S^n_- onto S^n such that $\psi_n^{-1}(-P_n) = E_n$. To show this, fix $n \geqq 2$. Define $\alpha_n\colon S^n_- \to S^n_+$ by letting $\alpha_n(x) = -x$ for each $x \in S^n_-$. Let $\lambda_n\colon S^n_+ \to S^n$ denote the mapping which stretches S^n_+ onto S^n by doubling the angle each vector $v \in S^n_+$ makes with the vector P_n. Let $\psi_n = \lambda_n \circ \alpha_n$. It follows easily that ψ_n has the desired properties. We remark that for the case when $n = 1$, the procedure above gives a fixed point free mapping ψ_1 from S^1_- onto S^1 such that $\psi_1^{-1}(-P_1) = E_1$. Note that ψ_1 is not monotone. In fact, there is no monotone mapping from S^1_- onto S^1 since the monotone image of an arc is an arc [**11**, 1.1, p. 165].

In Theorem 3.7 we give a fixed point theorem for monotone mappings of nonseparating proper subcontinua of S^2 onto S^2. First, a brief discussion is appropriate.

The following theorem was proved in [**6**, p. 136]: If $f\colon B^n \to \mathbf{R}^n$ is a mapping such that f maps the boundary S^{n-1} of B^n back into B^n, then f has a fixed point. We see from Example 3.6 that this theorem's analogue for mappings of n-cells onto S^n is not valid even when the mappings are monotone. In fact, the "opposite" hypothesis concerning the boundary is needed; namely, Theorem 3.7 will show that fixed points exist for monotone mappings of 2-cells onto S^2 when the mapping sends a boundary point to a point outside the 2-cell. With respect to the condition of monotoneness, the reader should compare Theorem 3.7 with Example 3.5.

We will prove Theorem 3.7 by using the following special case of a result in [**9**]. If M is a compact subset of B^2 such that M does not separate $\mathbf{R}^2$ and if g is a monotone mapping from M onto B^2, then g has a fixed point. More general results than those in [**9**] are in [**10**, 3.2 and 3.4].

THEOREM 3.7. *Let K be a compact connected proper subset of S^2 such that K does not separate S^2. If f is a monotone mapping from K onto S^2 such that $f(x_0) \notin K$ for some point $x_0 \in \mathrm{Bd}(K)$, then f has a fixed point.*

PROOF. Let $y_0 = f(x_0)$. Since $y_0 \notin K$, it follows from the assumptions about K that there is a 2-cell Δ^2 in S^2 such that $K \subset \Delta^2$ and $y_0 \notin \Delta^2$. Let $M = f^{-1}(\Delta^2)$ and let $g = f|M$. Since f is a monotone mapping from K onto S^2, it follows easily that g is a monotone mapping from M onto Δ^2. Hence, once we show that M does not separate S^2, we can apply the result in [**9**] stated above to see that g (thus f) has a fixed point. Suppose that M separates S^2. Then, there exists a component U of $S^2 - M$ such that $y_0 \notin U$. Since $S^2 - K$ is a connected subset of $S^2 - M$ and since $y_0 \in (S^2 - K)$ and $y_0 \notin U$, it follows easily that $U \subset K$. In particular, $U \cap K \neq \emptyset$. Thus, there exists a point $z_0 \in S^2$ such that $f^{-1}(z_0) \cap U \neq \emptyset$. Let $W = S^2 - \Delta^2$. Since

$$U \subset S^2 - M = S^2 - f^{-1}(\Delta^2)$$

and since $f^{-1}(z_0) \cap U \neq \emptyset$, $z_0 \in W$ and we have that (i) $f^{-1}(W) \cap U \neq \emptyset$. Since $x_0 \in \mathrm{Bd}(K)$ and since U is an open subset of S^2 such that $U \subset K$, it is clear that $x_0 \notin U$. Also, since $y_0 \in W$ and $x_0 \in f^{-1}(y_0)$, $x_0 \in f^{-1}(W)$. Thus, (ii) $f^{-1}(W) \cap (S^2 - U) \neq \emptyset$. Since

$$f^{-1}(W) = f^{-1}(S^2) - f^{-1}(\Delta^2) = K - M,$$

we have that (iii) $f^{-1}(W) \subset (S^2 - M)$. Note that W is connected (use [**8**, Thm. 9, p. 475]). Hence, by the monotoneness of f and [**11**, 2.2. p.138], (iv) $f^{-1}(W)$ is connected. By (i) through (iv) we have a contradiction to the fact that U is a component of $S^2 - M$. Therefore, we have proved that M does not separate S^2. This completes the proof of Theorem 3.7.

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