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AMENABLE GROUPS FOR WHICH EVERY TOPOLOGICALLY LEFT INVARIANT MEAN IS RIGHT INVARIANT

PAUL MILNES[1]

Let G be an amenable locally compact group. A. L. T. Paterson proved that

(∗) every topologically left invariant mean on $L^\infty(G)$ is (topologically) right invariant if and only if G has relatively compact conjugacy classes,

provided G is discrete or compactly generated. He also conjectured that (∗) holds for all amenable locally compact groups. We are unable to deal with this conjecture completely, but can verify it for σ-compact groups and, with the word in parentheses removed, for groups with equivalent left and right uniform structures. Thus we generalize both cases dealt with by Paterson. Our method of proof is quite different from Paterson's and relies heavily on results of W. R. Emerson and C. Chou.

1. **Preliminaries.** Let G be a locally compact group with the usual (complex-valued) function spaces $C(G)$ of continuous functions, $LUC(G) = LUC$ of functions $f \in C(G)$ for which

$$s \to L_s f \colon G \to C(G)$$

is (norm-) continuous and $L^\infty(G) = L^\infty$ of essentially bounded measurable functions. (Here, and later, $L_s f(t) = f(st) = R_t f(s)$.) These spaces are C^*-algebras. A continuous linear functional m on one of them is called a *mean* if

$$\|m\| = m(1) = 1$$

(where 1 sometimes stands for the function in LUC everywhere equal to 1). A mean m is called a *left (right) invariant mean*, shortened to LIM (RIM), if $m(L_s f) = m(f)(m(R_s f) = m(f))$ for all $s \in G$ and all f in the space under consideration; m is called a *topologically left (right) invariant mean*, shortened to TLIM (TRIM) if $m(p * f) = m(f)\,(m(f * p) = m(f))$ for all f in the space under consideration and all $p \in L^1(G)$ satisfying $\|p\|_1 = \int p(s)ds = 1$. (Here $\check{p}(s) = p(s^{-1})$ and ds denotes a (fixed) left

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Haar measure on G.) [5] is a good source of information about invariant means. We note in particular that

(a) if any of the function spaces on G under consideration (i.e., $C(G)$, LUC, and L^∞) has an invariant mean of any of the kinds under consideration, ten all the spaces have invariant means of all kinds and, in fact, L^∞ has a mean simultaneously invariant in all the senses defined; G is then called *amenable*;

(b) TLIM's are always LIM's [**5**; Proposition 2.1.3];

(c) on $L^\infty(G)$ or $C(G)$ there can exist LIM's that are not TLIM's [**7**, **8**, **9**];

(d) on LUC all LIM's are TLIM's (**5**; proof of Lemma 2.2.2]

When G is amenable G admits a symmetric *Følner net* $\{U_\alpha\}$ of compact subsets of G: $\{U_\alpha\}$ satisfies ($|A|$ denoting the Haar measure of $A \subset G$)

(i) $\bigcup_\alpha U_\alpha = G$, $U_{\alpha_1} \supset U_{\alpha_2}$ if $\alpha_1 \geqq \alpha_2$,

(ii) $U_\alpha = U_\alpha^{-1} = \{s \in G \mid s^{-1} \in U_\alpha\}$,

(iii) $|sU_\alpha \cap U_\alpha|/|U_\alpha| \to 1$ uniformly on compact subsets of G.

The indices α can be of the form $(\nu, n) \in J \times N$, where J orders the σ-compact subgroups of G by inclusion, each subgroup $G_\nu = \bigcup_n K_{\nu,n}$ for a sequence of compact subsets of G_ν and

$$|sU_{(\nu,n)} \cap U_{(\nu,n)}| / |U_{(\nu,n)}| \geqq 1 - 1/n$$

for all $s \in K_{\nu,n}$. ($(\nu_1, n_1) \geqq (\nu_2, n_2)$ means $G_{\nu_1} \supset G_{\nu_2}$, $n_1 \geqq n_2$, $U_{(\nu_1,n_1)} \supset U_{(\nu_2,n_2)}$ and $K_{\nu_1,n_1} \supset K_{\nu_2,n_2}$. See [**1**; Theorem 4.4], and [**5**; Theorem 2] coupled with [**4**; Theorem 3]; one also needs to know that a closed subgroup of an amenable group is amenable [**5**; Theorem 2.3.2]. Implicit in the above is the fact that the net $\{U_\alpha\}$ can be a sequence (a *Følner sequence*) if G is σ-compact.

A locally compact group G with *relatively compact conjugacy classes* (i.e., $\{trt^{-1} \mid t \in G\}$ is relatively compact in G for each $r \in G$) is called an $[FC]^-$-group, written $G \in [FC]^-$. Such a group has finite conjugacy classes if its topology is discrete, and is called an $[FC]$-group. As noted by Paterson [**6**; near the end of §1], all $[FC]^-$-groups are amenable.

2. In this section we will prove, among other things, the following theorem.

THEOREM. *Let G be an amenable locally compact group.*

(i) *If G is σ-compact, then $G \in [FC]^-$ if and only if every* TLIM *on L^∞ is a* TRIM.

(ii) *If G has equivalent left and right uniform structures, then $G \in [FC]^-$ if and only if every* TLIM *on L^∞ is a* RIM.

This theorem generalizes the results of Paterson [**6**; Theorems 3.2 and 4.4], but does not completely resolve Paterson's conjecture: if G is an amenable locally compact group, then every TLIM on $L^\infty(G)$ is a TRIM if and only if $G \in [FC]^-$. The proof of the theorem will proceed via a number of lemmas.

We note first that we can assume we are dealing with unimodular groups. For

(a) all $[FC]^-$-groups are unimodular: this follows from [**3**; Theorem 13] (and is probably elsewhere in the literature);

(b) the proof of [**6**; Lemma 4.2] requires little modification to show that every amenable σ-compact group for which each TLIM is a TRIM is in the class $[IN]$ (i.e., has a compact neighbourhood D of the identity of G for which $sD = Ds$ for all $s \in G$), hence is unimodular;

(c) groups with equivalent left and right uniform structures are unimodular.

Following Chou [**1**] and, to some extent, also Emerson [**3**], we take a Følner net $\{U_\alpha\}$ for G and consider means x on L^∞ which are weak $*$ cluster points of nets $\{x_\alpha\}$ of means of the form

$$(1) \qquad \begin{aligned} x_\alpha(f) &= |U_\alpha|^{-1}\int_{U_\alpha} f(st_\alpha)ds = |U_\alpha|^{-1}\int_{U_\alpha t_\alpha} f(s)ds \\ &= |U_\alpha|^{-1}\chi_{U_\alpha} * f(t_\alpha),\ f \in L^\infty. \end{aligned}$$

(Here the left Haar measure ds can be assumed also right invariant and inversion invariant, since G is unimodular, $|U_\alpha|$ denotes the Haar measure of U_α, and χ_{U_α} is the characteristic function of U_α, $\chi_{U_\alpha}(s) = 1$ if $s \in U_\alpha$, $\chi_{U_\alpha}(s) = 0$ if $s \notin U$; and $\{t_\alpha\}$ is a net in G.) Let E' be the set of such means x and let E be the wead $*$ closed convex hull of E'.

LEMMA 1. *Let G be a unimodular amenable locally compact group, and write $\mathscr{L} = \{m \mid m$ is a TLIM on $L^\infty\}$. Then $\mathscr{L} = E$.*

PROOF. $\mathscr{L}$ is convex and weak $*$ compact. Also, a direct calculation shows $E' \subset \mathscr{L}$; hence $E \subset \mathscr{L}$. Suppose there is a TLIM $m \notin E$. By the Hahn-Banach theorem [**2**; Theorem 5.2.6, p. 416], there is a real-valued $g \in L^\infty$ with

$$(*) \qquad m(g) = a > b = \max\{m'(g) \mid m' \in E\}.$$

For each α, we have $m(|U_\alpha|^{-1}\chi_{U_\alpha} * g) = a$; hence, since m is a mean, there is a $t_\alpha \in G$ with

$$|U_\alpha|^{-1}\chi_{U_\alpha} * g(t_\alpha) \geqq (a + b)/2.$$

Any weak $*$ cluster point m' of the net $\{x_\alpha\}$,

$$x_\alpha(f) = |U_\alpha|^{-1}\chi_{U_\alpha} * f(t_\alpha),\ f \in L^\infty,$$

is in E' and satisfies $m'(g) \geqq (a + b)/2 > b$, contradicting $*$ above. Hence $\mathscr{L} \subset E$ and the lemma is proved.

REMARK. The hypothesis of unimodularity can be essentially avoided in Lemma 1.

LEMMA 2. *Let G be an amenable locally compact group. Then every member of E' is a* RIM *if $G \in [FC]^-$.*

PROOF. (following Emerson [3; Theorem 11]). If $\{U_\alpha t_\alpha\}$ is as in (1) above, we have for each $r \in G$

$$|U_\alpha t_\alpha r \cap U_\alpha t_\alpha| / |U_\alpha t_\alpha| = |U_\alpha t_\alpha r t_\alpha^{-1} \cap U_\alpha| / |U_\alpha| \to 1,$$

a consequence of (ii) and (iii) at the end of §1 and of the unimodularity of G. This implies each member of E' is a RIM.

LEMMA 3. *Let G be an amenable locally compact group that is discrete or σ-compact. Then every member of E' is a* TRIM *if $G \in [FC]^-$.*

PROOF. Even RIM is a TRIM if G is discrete. If G is σ-compact, G admits a Følner sequence $\{U_n\}$ and any sequence $\{U_n t_n\}$ satisfies

$$|U_n t_n r \cap U_n t_n| / |U_n t_n| \to 1$$

pointwise on G (Lemma 2), hence uniformly on compact subsets of G [3; Theorem 3]. The desired conclusion follows.

REMARK. If we could make the step from pointwise convergence on G to uniform convergence on compact subsets of G (as in the previous proof) for nets $\{U_\alpha t_\alpha\}$, we could conclude that every TLIM is a TRIM for any (amenable) locally compact $[FC]^-$-group G. This step could be made if $\{sKs^{-1} | s \in G\}$ were relatively compact for each compact $K \subset G$; we do not know if this is the case.

LEMMA 4. *Let G be an amenable locally compact σ-compact group. If $G \notin [FC]^-$, there is a member of E' that is not a* RIM.

PROOF. Let $\{U_n\}$ be a Følner sequence for G and let $r \in G$ have a conjugacy class that is not relatively compact. We can define inductively a sequence $\{t_n\} \subset G$ such that

$$t_n r t_n^{-1} \notin U_n^2,$$

$$t_n \notin U_n\left(\bigcup_{m=1}^{n-1} U_m t_m\right) r^{-1},$$

and

$$t_n \notin U_n\left(\bigcup_{m=1}^{n-1} U_m t_m\right) r,$$

from which it follows that

$$\left(\bigcup_{n=1}^{\infty} U_n t_n\right) r \cap \left(\bigcup_{n=1}^{\infty} U_n t_n\right) = \varnothing$$

and hence that no TLIM in E' derived from $\{U_n t_n\}$ (as near (1) above) can be a RIM.

LEMMA 5. Let G be an amenable locally compact group, $G \notin [FC]^-$.

(i) *There is a* LIM *on* $L^\infty(G)$ (or $C(G)$) *that is not a* RIM.

(ii) *If G has equivalent left and right uniform structures there is a* TLIM *on* L^∞ *that is not a* RIM.

PROOF. (following Chou [**1**; p. 453]). Suppose $r \in G$ and $\{trt^{-1} | t \in G\}$ is not relatively compact. Then G has a σ-compact open subgroup G_ν which contains r and for which $\{trt^{-1} | t \in G_\nu\}$ is not relatively compact. For, if H is any σ-compact open subgroup of G and $G = \bigcup Hs_\lambda$ is a disjoint union of cosets, then $\{trt^{-1} | t \in G\}$ either meets one Hs_λ in a set that is not relatively compact or meets infinitely many Hs_λ's. In either case we get the required subgroup G_ν. (For example, if $\{trt^{-1} | t \in G\} \cap Hs_\lambda$ is not relatively compact, we can get a sequence $\{t_n r t_n^{-1} \subset Hs_\lambda$ that is not relatively compact, and $\{r\} \cup \{t_n\}$ is contained in a σ-compact open subgroup of G.) Now G_ν is amenable [**5**; Theorem 2.3.2], σ-compact and not in $[FC]^-$. By Lemma 4, there is a TLIM m on $L^\infty(G_\nu)$ (or $C(G_\nu)$ or $LUC(G_\nu)$) that is not a RIM. On p. 453 of [**1**] is shown a way to lift m to a LIM m_0 on $L^\infty(G)$ (or $C(G)$), and it follows that m_0 is not a RIM. This proves (i).

If G also has equivalent left and right uniform structures one easily shows using Remark 2, p. 453, and also Lemma 2.2(3) of [**1**], that the extension m_0 of m as above is a TLIM on $LUC(G)$ (or $C(G)$ or $L^\infty(G)$) and is not a RIM.

The theorem is now proved: (i) follows from Lemmas 3 and 4, and (ii) follows from Lemmas 2 and 5 (ii).

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THE UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO CANADA, N6A5B9