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ON THEOREMS OF B. H. NEUMANN CONCERNING *FC*-GROUPS

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1. **Introduction**. The theorems we are concerned with here are the characterizations of central-by-finite groups and finite-by-abelian groups given by B. H. Neumann [6]. He proved that a group G is central-by-finite if and only if each subgroup has only finitely many conjugates or, equivalently, U/U_G is finite for each subgroup U of G. Here U_G denotes the core of U, that is, the largest normal subgroup of G contained in U; we use U^G to denote the normal closure of U in G. The "dual" characterization given by Neumann was that G is finite-by-abelian if and only if $|U^G:U|$ is finite for each subgroup U of G.

It was indicated by Eremin [3] that it is only necessary to consider the abelian subgroups of G in the first of these theorems. Although one of the apparent simplifications in Eremin's proof is incorrect, his strategy of concentrating on direct products of cyclic groups does give a slightly simpler proof of Neumann's results and we use this in the generalizations that we give here.

Our main concern is to consider FC-groups in which $|G'| < \mathfrak{m}$ or $|G/Z| < \mathfrak{m}$; here as throughout the paper $\mathfrak{m}$ denotes an infinite cardianl. We prove the following theorem.

THEOREM A. *Let G be an FC-group. Then $|G'| < \mathfrak{m}$ if and only if $|U^G: U| < \mathfrak{m}$ for each $U \leqq G$.*

The results for G/Z cannot be proved for all FC-groups but hold in large subclasses. We define $\mathfrak{Z}_\mathfrak{m}$ to be the class of FC-groups in which $|G: C_G(U)| < \mathfrak{m}$ whenever U is generated by fewer than $\mathfrak{m}$ elements. [If G is periodic or $\mathfrak{m}$ is uncountable, U being generated by fewer than $\mathfrak{m}$ elements simply means $|U| < \mathfrak{m}$]. In [9], we defined $\mathfrak{Z}$ to be the class of locally finite groups G satisfying the condition: if $\mathfrak{m}$ is an infinite cardinal and $H \leqq G$ such that $|H| < \mathfrak{m}$, then $|G: C_G(H)| < \mathfrak{m}$. It is clear that $\mathfrak{Z} \subseteqq \mathfrak{Z}_\mathfrak{m}$ for each $\mathfrak{m}$ and all the evidence we have suggests that $\mathfrak{Z}$ is a very large subclass of the class of periodic FC-groups. It should also be noted that if $\mathfrak{m} = \aleph_0$, then $\mathfrak{Z}_\mathfrak{m}$ is the class of all FC-groups and so Neumann's result is a special case of the following theorem.

THEOREM B. *Let $G \in \mathfrak{Z}_\mathfrak{m}$. Then the following are equivalent*:

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(i) $|G/Z| < \mathfrak{m}$.
(ii) $|U/U_G| < \mathfrak{m}$, *for each* $U \leqq G$.
(iii) $|\mathrm{Cl}(U)| < \mathfrak{m}$, *for each* $U \leqq G$.
(iv) $|A/A_G| < \mathfrak{m}$, *for each abelian subgroup* A *of* G.
(v) $|\mathrm{Cl}(A)| < \mathfrak{m}$, *for each abelian subgroup* A *of* G.

If we work within the class $\mathfrak{Z}_{\mathfrak{m}}$, then we are also able to restrict attention to abelian subgroups when considering the derived subgroup.

THEOREM C. *Let* $G \in \mathfrak{Z}_{\mathfrak{m}}$. *Then* $|G'| < \mathfrak{m}$ *if and only if* $|A^G: A| < \mathfrak{m}$, *for each abelian subgroup* A *of* G.

Suppose that $\mathfrak{m}$ is a limit cardinal and U is generated by $\mathfrak{n}$ elements where $\mathfrak{n} < \mathfrak{m}$. If $\mathfrak{n}$ is finite, then $|G: C_G(U)|$ is finite and if $\mathfrak{n}$ is infinite, then $|G: C_G(U)| \leqq 2^{\mathfrak{n}}$. With the assumption of GCH, $2^{\mathfrak{n}} = \mathfrak{n}^+ < \mathfrak{m}$ and so for any limit cardinal $\mathfrak{m}$, $\mathfrak{Z}_{\mathfrak{m}}$ is the class of all FC-groups. Thus Theorem C shows that if $\mathfrak{m}$ is a limit cardinal, then assuming GCH we can strengthen Theorem A and consider only abelian subgroups. Later (5.2) we shall give an example (also assuming GCH) which shows that this cannot be done if $\mathfrak{m}$ is a non-limit.

After the elementary relationships between the different cardinalities $|U^G: U|$, $|U/U_G|$ and $|\mathrm{Cl}(U)|$ have been discussed in §2, the bulk of the proof consists in obtaining inside G a group of a rather special type and discussing this in detail. These groups were considered by Neumann **[6]** and we therefore use the term N-group.

An N-group of cardinality $\mathfrak{m}$ is a group generated by elements a_α, b_α, $\alpha < \rho$, where ρ is the least ordinal of cardinality $\mathfrak{m}$, subject to relations

$$(1) \qquad \begin{gathered}[a_\alpha, a_\beta] = [b_\alpha, b_\beta] = [a_\alpha, b_\beta] = 1, \text{ if } \alpha \neq \beta, \\ [a_\alpha, b_\alpha] = c_\alpha \neq 1.\end{gathered}$$

It is a consequence of these relations that

$$(1') \qquad a_\alpha a_\beta^{-1} \text{ is non-central, if } \alpha \neq \beta.$$

Further conditions will be imposed on these groups when appropriate.

In proving Theorem A, the construction inside G of a Neumann group of cardinality $\mathfrak{m}$ depends largely on a combinatorial result proved in **[8]**. The special case of the result that we use here did not require the assumption of GCH for its proof.

THEOREM 1.1. **[8]** *Let* X_i, $i \in I$, *be finite sets with* $|I| = \mathfrak{m}$, *where* $\mathfrak{m}$ *is an uncountable cardinal. Then there is a subset* J *of* I *such that* $|J| = \mathfrak{m}$ *and* $|\bigcup_{j \neq k \in J}(X_j \cap X_k)| < \mathfrak{m}$.

[Note: It has been pointed out to me that this result can be readily deduced from the Marczewski-Erdös-Rado Theorem on Δ-systems **[2]**.]

One of the applications of this result concerns sets of commutators in an *FC*-group. If S and T are subsets of a group G, then $[S, T] = \langle [s, t];\ s \in S,\ t \in T\rangle$. If $\{[s, t];\ s \in S,\ t \in T\}$ is infinite, then $|[S, T]| = |\{[s, t];\ s \in S,\ t \in T\}$; but we need to distinguish between these if the set of commutators is finite.

COROLLARY 1.2. *Let G be an FC-group. If S is a subset of G such that $|[S, G]| = \mathfrak{m}$ is uncountable, then there is a subset T of G such that $|[T, G]| = \mathfrak{m}$ but $|[T, T]| < \mathfrak{m}$.*

PROOF. For each $s \in S$, $X_s = \{[s, g];\ g \in G\}$ is finite and so we have $\mathfrak{m}$ finite subsets X_s of $[S, G]$. There is a subset S_1 of S such that $|S_1| = \mathfrak{m}$ and there are distinct elements $x_s \in X_s$, $s \in S_1$. The commutator $[s, t]$ is in the intersection $X_s \cap X_t$. By the theorem, there is a subset T of S_1, such that $|T| = \mathfrak{m}$ and hence $[T, G] = \mathfrak{m}$ but $|\bigcup_{s \neq t \in T}(X_s \cap X_t)| < \mathfrak{m}$. Hence $[T, T] = \langle \bigcup_{s \neq t \in T}[s, t]\rangle$ has cardinality less than $\mathfrak{m}$.

The notation we use is basically that given in [7]. Scott's book also contains some of the elementary results on *FC*-groups which we use frequently. For ease of reference we summarize some of these results.

Let G be an FC-group. Then
(1.3) *G' is periodic*, [7, 15.1.7]
(1.4) *$G/Z(G)$ is periodic*, [7, 15.1.16]
(1.5) *If S is a set of elements in G such that $|S| < \mathfrak{m}$, where $\mathfrak{m}$ is an infinite cardinal, then $\langle S^G\rangle$ is generated by fewer than $\mathfrak{m}$ elements. In particular if $H \lhd G$ such that $|G/H| < \mathfrak{m}$, then there is a normal subgroup N generated by fewer than $\mathfrak{m}$ elements such that $NH = G$.*

Although we have restricted our attention entirely to *FC*-groups in this paper, the statements of the theorems are meaningful without that restriction. It is possible to obtain some general results although the methods used here are not always applicable and some restriction on the cardinal $\mathfrak{m}$ which appears is often necessary. A discussion of these more general questions will appear in a forthcoming paper with V. Faber.

2. **Preliminary results.**

LEMMA 2.1 (cf. [9], Lemma 2.1). *$\mathfrak{Z}_{\mathfrak{m}}$ is a* QS-*closed class of groups.*

PROOF. Let $G \in \mathfrak{Z}_{\mathfrak{m}}$ and $H \lhd G$. If U/H is generated by fewer than $\mathfrak{m}$ elements, then by (1.5) there is a subgroup K of G such that $KH = U$ and K is generated by fewer than $\mathfrak{m}$ elements. Since $G \in \mathfrak{Z}_{\mathfrak{m}}$, $|G\colon C_G(K)| < \mathfrak{m}$. But $C_G(K) \leqq C_G(U/H)$ and hence $|G\colon C_G(U/H)| < \mathfrak{m}$. Thus $\mathfrak{Z}^{\mathfrak{m}}$ is Q-closed. The S-closure of $\mathfrak{Z}_{\mathfrak{m}}$ is straightforward.

LEMMA 2.2 ([7], 15.1.13 and 15.1.24). *If G is any group such that $|G/Z| < \mathfrak{m}$, then $|G'| < \mathfrak{m}$.*

PROOF. Let $|G/Z| = \mathfrak{n} < \mathfrak{m}$; then each element of G has at most $\mathfrak{n}$ conjugates. Thus there is a normal subgroup N of G such that $NZ = G$ and N is generated by $\mathfrak{n}^2$ elements. Thus $|N| < \mathfrak{m}$ if $\mathfrak{m}$ is uncountable and N is a finitely generated FC-group if $\mathfrak{m} = \aleph_0$. In both cases $|N'| < \mathfrak{m}$ and since $G' = N'$, this completes the proof.

LEMMA 2.3. *Let U be a subgroup of the FC-group G.*
(i) *If* $|U/U_G| < \mathfrak{m}$, *then* $|U^G: U| < \mathfrak{m}$.
(ii) *If* $|\mathrm{Cl}(U)| < \mathfrak{m}$, *then* $|U^G: U| < \mathfrak{m}$.
(iii) *If* $G \in \mathfrak{Z}_{\mathfrak{m}}$, *then* $|U/U_G| < \mathfrak{m}$ *if and only if* $|\mathrm{Cl}(U)| < \mathfrak{m}$.

PROOF. (i) If $|U/U|_G < \infty$, then U has only finitely many conjugates and so $|U^G/U_G| < \infty$. If $|U/U_G|$ is infinite, then $U/U_G = \langle F_i/U_G; i \in I\rangle$, where each F_i/U_G is finitely generated and $|I| = |U/U_G|$. For each $i \in I$, F_i^G/U_G is finitely generated and $U^G = \langle F_i^G; i \in I\rangle$. Hence $|U^G/U_G| = |I| < \mathfrak{m}$.

(ii) If $|\mathrm{Cl}(U)| < \infty$, then $U/U_G < \infty$ and the result follows from (i). If $|\mathrm{Cl}(U)|$ is infinite, let T be a transversal to $N_G(U)$ in G and $t \in T$. There is a finitely generated normal subgroup F_t of G such that $U^t \leqq UF_t$ and so $U^G \leqq U\langle F_t;\ t \in T\rangle$. Hence $|U^G:U| \leqq |T| < \mathfrak{m}$.

(iii) Let $|U/U_G| < \mathfrak{m}$; then $|G: C_G(U/U_G)| < \mathfrak{m}$ and hence $|G: N_G(U)| < \mathfrak{m}$.

Conversely, suppose that $|\mathrm{Cl}(U)| < \mathfrak{m}$; then G has a normal subgroup N such that $G = NN_G(U)$ and N is generated by fewer than $\mathfrak{m}$ elements (1.5). Since $G \in \mathfrak{Z}_{\mathfrak{m}}$, $|G: C_G(N)| < \mathfrak{m}$ and so $|U: C_U(N)| < \mathfrak{m}$. But $C_U(N) \lhd NN_G(U) = G$ and so $|U/U_G| < \mathfrak{m}$.

It should be noted that both directions of (iii) are false for an arbitrary FC-group. For, let G be the extraspecial p-group generated by the two elementary abelian subgroups $X = \mathrm{Dr}_{i=1}^{\infty}\langle x_i\rangle$ and $Y = \prod_{i=1}^{\infty}\langle y_i\rangle$ such that $[x_i, y_j] = 1$ if $i \neq j$ and $[x_i, y_i] = z$, where z is a generator of the centre of order p. Then $|X/X_G| = \aleph_0$ but $|\mathrm{Cl}(X)| = 2^{\aleph_0}$. Also $|\mathrm{Cl}(Y)| = \aleph_0$ but $|Y/Y_G| = 2^{\aleph_0}$.

The converses of (i) and (ii) are false even for $\mathfrak{Z}$-groups. For, let G be a countable extraspecial p-group and let X be an elementary abelian subgroup maximal with respect to $X \cap Z = 1$. Then $|X^G: X| = p$ but $|X/X_G| = |\mathrm{Cl}(X)| = \aleph_0$.

LEMMA 2.4. *Let U be a subgroup of the FC-group G such that* $|G: U| < \mathfrak{m}$.
(i) *If* $|U'| < \mathfrak{m}$, *then* $|G'| < \mathfrak{m}$.
(ii) *If* $G \in \mathfrak{Z}_{\mathfrak{m}}$ *and* $|U/Z(U)| < \mathfrak{m}$, *then* $|G/Z(G)| < \mathfrak{m}$.

PROOF. (i) There is a normal subgroup N of G such that $NU = G$ and N is generated by fewer than $\mathfrak{m}$ elements (1.5). Clearly $G' \leqq U'P$, where P is the periodic subgroup of N(1.3) and hence $|G'| \leqq |U'|\,|P| < \mathfrak{m}$.

(ii) There is a normal subgroup N of G such that $NZ(U) = G$ and N is generated by fewer than $\mathfrak{m}$ elements (1.5). Since $G \in \mathfrak{Z}_{\mathfrak{m}}$, $|Z(U)$: $Z(U) \cap$

$C_G(N)| < \mathfrak{m}$ and so $|G: Z(U) \cap C_G(N)| < \mathfrak{m}$. But $Z(U) \cap C_G(N) \leqq Z(G)$ and the result follows.

Part (ii) of Lemma 2.4 is false in general as the example following Lemma 2.3 shows. For. Y is abelian and $|G: Y| = \aleph_0$, but $|G/Z| = 2^{\aleph_0}$.

3. **Proof of Theorems *A* and *C*.** It is clear that if $|G'| < \mathfrak{m}$, then $|U^G: U| < \mathfrak{m}$ for each $U \leqq G$, since $U^G \leqq UG'$. So in this section we consider an FC-group G such that $|G'| \geqq \mathfrak{m}$ and have to construct a subgroup U such that $|U^G: U| \geqq \mathfrak{m}$.

The N-group $\langle a_\alpha, b_\alpha; \alpha < \rho\rangle$ of cardinality $\mathfrak{m}$ is called an N_1-group if it satisfies the additional condition

$$(2) \qquad c_\alpha \neq c_\beta, \text{ whenever } \alpha \neq \beta.$$

THEOREM 3.1. (i) *If G is a $\mathfrak{Z}_\mathfrak{m}$-group with $|G'| \geqq \mathfrak{m}$, then G contains a subgroup which is an N_1-group of cardinality* $\mathfrak{m}$.

(ii) *If G is any FC-group with $|G'| \geqq \mathfrak{m}$, then G has a normal subgroup F with $|F| < \mathfrak{m}$, such that G/F contains a subgroup which is an N_1-group of cardinality* $\mathfrak{m}$.

PROOF. (i) Suppose that we have defined the elements $a_\beta, b_\beta \in G$, for all $\beta < \alpha$ (some $\alpha < \rho$), such that

$$[a_\beta, a_\gamma] = [b_\beta, b_\gamma] = [a_\beta, b_\gamma] = 1, \text{ if } \beta \neq \gamma,$$
$$[a_\beta, b_\beta] = c_\beta \notin \langle a_\gamma, b_\gamma; \gamma < \beta\rangle.$$

Let $S_\alpha = \langle a_\beta, b_\beta; \beta < \alpha\rangle$ and let $C_\alpha = C_G(S_\alpha)$. Since $G \in \mathfrak{Z}_\mathfrak{m}$ and S_α is generated by fewer than $\mathfrak{m}$ elements, $|G: C_\alpha| < \mathfrak{m}$, By Lemma 2.4, $|C_\alpha'| \geqq \mathfrak{m}$. Also, since $G' \cap S_\alpha$ is periodic (1.3), $|G' \cap S_\alpha| < \mathfrak{m}$. Therefore C_α contains elements a_α, b_α such that $[a_\alpha, b_\alpha] = c_\alpha \notin S_\alpha$. Thus we can construct the N_1-group $\langle a_\alpha, b_\alpha; \alpha < \rho\rangle$.

(ii) If $\mathfrak{m} = \aleph_0$, then the result follows from (i) and so we may assume that $\mathfrak{m}$ is uncountable. By Corollary 1.2, there is a subset T of G such that $|[T, G]| = \mathfrak{m}$ and $|[T, T]| < \mathfrak{m}$.

Factoring out $[T, T]^G = F_1$, we have an abelian subgroup $A(=\langle T\rangle F_1/F_1)$ such that $|[A, G]| = \mathfrak{m}$ (writing G in place of G/F_1).

We can choose elements a_i, $i \in I$, of A and $b_i \in G$ such that $|I| = \mathfrak{m}$ and $[a_i, b_i] = c_i$, with $c_i \neq c_j$ whenever $i \neq j$. By Corollary 1.2, there is a subset $B = \{b_i; i \in I_1\}$ of $\{b_i; i \in I\}$ such that $|B| = \mathfrak{m}$ and $|[B, B]| < \mathfrak{m}$. Factoring out $[B, B]^G = F_2$, we have two abelian subgroups $A_1 = \langle a_i; i \in I_1\rangle$ and $B_1 = \langle b_i; i \in I_1\rangle$ with the commutators $c_i = [a_i, b_i]$, $i \in I_1$, being all distinct.

Now let $X_i = [a_i, B_i] \cup [b_i, A_1]$, then since G is an FC-group X_i is a finite set containing c_i. There is a subset J of I_1 such that $|J| = \mathfrak{m}$ and $|\bigcup_{j\neq k\in J}(X_j \cap X_k)| < \mathfrak{m}$. Factoring out $(\bigcup_{j\neq k\in I}(X_j \cap X_k))^G = F_3$, we have elements $a_j, b_j, j \in J$ such that

$$[a_j,\ a_k] = [b_j b_k] = [a_j,\ b_k] = 1, \text{ if } j \neq k$$
$$[a_j,\ b_j] = c_j \text{ and } c_j \neq c_k, \text{ whenever } j \neq k.$$

Well-ordering the set J, we can relabel the elements a_j, b_j, $j \in J$, as a_α, b_α, $\alpha < \rho$.

Theorems A and C will now both follow from a result about N_1-groups.

THEOREM 3.2 *An N_1-group G of cardinality $\mathfrak{m}$ which is also an FC-group contains an abelian subgroup X such that $|X^G \colon X| = \mathfrak{m}$.*

PROOF. If $A = \langle a_\alpha; \alpha < \rho\rangle$; then $A^G \geqq \langle a_\alpha,\ c_\alpha;\ \alpha < \rho\rangle$. If $c_\alpha c_\beta^{-1} \notin A$ whenever $\alpha \neq \beta$, then we have $|A^G \colon A| = \mathfrak{m}$. We need to show that G has a subgroup H which is an N_1-group and satisfies the extra condition

$$c_\alpha c_\beta^{-1} \notin A, \text{ whenever } \alpha \neq \beta. \tag{3}$$

Let $I_1 = \{\alpha < \rho;\ [a_\alpha,\ c_\alpha] \neq 1\}$ and let $\alpha,\ \beta \in I_1$. Then $[a_\alpha,\ c_\alpha c_\beta^{-1}] = [a_\alpha,\ c_\alpha] \neq 1$ and so $c_\alpha c_\beta^{-1} \notin A$. If $|I_1| = \mathfrak{m}$, then it follows that $|A^G \colon A| = \mathfrak{m}$. Similarly, if $I_2 = \{\alpha < \rho;\ [b_\alpha,\ c_\alpha] \neq 1\}$ has cardinality $\mathfrak{m}$, then $|B^G \colon B| = \mathfrak{m}$. Therefore we may assume that $|I_1| < \mathfrak{m}$ and $|I_2| < \mathfrak{m}$. If $I = \{\alpha < \rho;\ [a_\alpha,\ c_\alpha] = [b_\alpha,\ c_\alpha] = 1\}$, then $|I| = \mathfrak{m}$ and $S = \langle a_\alpha,\ b_\alpha;\ \alpha \in I\rangle$ is nilpotent of class two.

S has a maximal torsion-free subgroup F contained in its centre. It is sufficient to show that the group S/F contains an abelian subgroup $\bar{X} = X/F$ such that $|\bar{X}^{\bar{S}} \colon \bar{X}| = \mathfrak{m}$. For $X' \leqq F$ and since X' is periodic (1.3), we have $X' = 1$ so that X is an abelian subgroup of S such that $|X^S \colon X| = \mathfrak{m}$. We may therefore assume that $S = \langle a_\alpha,\ b_\alpha;\ \alpha \in I\rangle$ is periodic.

The group $C = \langle c_\alpha;\ \alpha \in I\rangle$ is abelian and $|C| = \mathfrak{m}$. It is clear that $|\mathrm{Soc}(C)| = |C|$ unless $\mathrm{Soc}(C)$ is finite and $\mathfrak{m} = \aleph_0$. In this case C contains a subgroup Q of type C_p^∞. If $Q \leqq A = \langle a_\alpha,\ \alpha \in I\rangle$, then $A = Q \times A_1$ and $A_1^S \geqq Q$ so that $|A_1^S \colon A|$ is infinite. If $Q \nleqq A$, then $A^S \geqq Q$ and $A \cap Q$ is finite so that $|A^S \colon A| \geqq |AQ \colon A| = |Q \colon A \cap Q|$ is infinite. Thus we may assume that $|\mathrm{Soc}(C)| = \mathfrak{m}$.

We can now choose elements $d_\alpha \in A$, $e_\alpha \in B$, for all $\alpha < \rho$, such that $[d_\alpha,\ e_\beta] = 1$, if $\alpha \neq \beta$, $[d_\alpha,\ e_\alpha] = f_\alpha$ has prime order $p(\alpha)$, $f_\alpha \neq f_\beta$, if $\alpha \neq \beta$, and d_α and e_α are $p(\alpha)$-elements. For, suppose that we have defined d_β, e_β for all $\beta < \alpha$. There is a subset $I_\alpha \subseteqq I$ such that $|I_\alpha| < \mathfrak{m}$ and $\langle d_\beta,\ e_\beta;\ \beta < \alpha\rangle \subseteqq \langle a_\gamma,\ b_\gamma;\ \gamma \in I_\alpha\rangle$. Let $C^* = \langle c_\gamma;\ \gamma \notin I_\alpha\rangle$; then $|C^*| = \mathfrak{m}$ and, as above, we may assume that $|\mathrm{Soc}(C^*)| = \mathfrak{m}$. Thus there is an element $c \in \mathrm{Soc}(C^*) - \langle a_\gamma,\ b_\gamma;\ \gamma \in I_\alpha\rangle$ of prime order $p(\alpha)$. We let $f_\alpha = c$. Since S is nilpotent we can find $p(\alpha)$-elements $d_\alpha \in \langle a_\gamma;\ \gamma \notin I_\alpha\rangle$ and $e_\alpha \in \langle b_\gamma;\ \gamma \notin I_\alpha\rangle$ such that $[d_\alpha,\ e_\alpha] = f_\alpha$.

Now let $I_p = \{\alpha < \rho;\ p(\alpha) = p\}$ and let $|I_p| = \mathfrak{m}_p$. Then $\sum_p \mathfrak{m}_p = \mathfrak{m}$.

Case (a): $\mathfrak{m}_p < \infty$ *for all* p. In this case $\mathfrak{m} = \aleph_0$ and we need to find an abelian subgroup X such that $|X^G: X|$ is infinite. Let $N_\alpha = \langle d_\alpha, e_\alpha \rangle$; then if $p(\alpha) \neq 2$, N_α contains elements x_α, y_α such that $[x_\alpha, y_\alpha] = f_\alpha$ but $f_\alpha \notin \langle x_\alpha \rangle$. For, otherwise f_α would be contained in the subgroup generated by each non-central element. If $Z_\alpha = Z(N_\alpha)$ then $Z_\alpha = C \times D$, where C is a cyclic group containing f_α In the group N_α/D, $\langle f_\alpha \rangle D/D$ is the unique subgroup of order $p(\alpha)$ and hence $p(\alpha) = 2$ (and N_α/D is generalized quaternion [7, 9.7.3]).

For each prime $p \neq 2$, choose $\alpha(p) \in I_p$ and write x_p for $x_{\alpha(p)}$ etc. Let $X = \langle x_p;\ p \text{ an odd prime} \rangle$; then X is abelian, $c_p \in X^S$, for all p, and $c_p c_q^{-1} \notin X$. For if $c_p c_q^{-1} \in X$, then taking the q*th* power we see that $c_p \in X$ and the Sylow p-subgroup of X is $\langle x_p \rangle$ which does not contain c_p.

Case (b): $\mathfrak{m}_p$ *infinite for some* p. Let $P = \{p;\ \mathfrak{m}_p \text{ is infinite}\}$; then $\sum_{p \in P} \mathfrak{m}_p = \mathfrak{m}$. For each $p \in P$, let $T_p = \langle d_\alpha, e_\alpha;\ \alpha \in I_p \rangle$ so that T_p is a p-group and $T = \langle d_\alpha, e_\alpha;\ \alpha < \rho \rangle = \mathrm{Dr}\, T_p$. If we can show that T_p contains an abelian subgroup A_p such that $|A_p^T: A_p| = \mathfrak{m}_p$, then if $A = \mathrm{Dr}_p A_p$, we have $|A^T: A| = \sum \mathfrak{m}_p = \mathfrak{m}$.

It is therefore sufficient to consider the case in which $T = \langle d_i, e_i;\ i \in I \rangle$ is a p-group with $|I| = \mathfrak{m}$, $f_i = [d_i, e_i]$ has order p and $f_i \neq f_j$, if $i \neq j$. We show that T has an abelian subgroup X such that $|X^T: X| = \mathfrak{m}$.

We write $D = \langle d_i;\ i \in I \rangle$. The conditions on T imply that $D^p \leqq Z(T)$ and that the elements d_i belong to different cosets of $Z(T)$. Thus $D/D^p = \mathrm{Dr}_{i \in I} \langle \bar{d}_i \rangle$, where $\bar{d}_i = d_i D^p$. Let X be a basic subgroup of the abelian group D. Then X is a direct product of cyclic groups $X = \mathrm{Dr}_{j \in J} \langle x_j \rangle$ and $XD^p = D$([5], pp. 139, 144) so that $|J| = |D/D^p| = \mathfrak{m}$. Also $X \cap D^p = X^p$ and so the elements x_j, $j \in J$, belong to different cosets of D^p in D. Using bars to denote the images of elements modulo D^p, each of the distinct elements $\bar{x}_j$ is uniquely expressible in the form

$$\bar{x}_j = \bar{d}_{i_1}^{\alpha_1} \cdots \bar{d}_{i_n}^{\alpha_n} \quad (\alpha_i = 1, \cdots, p - 1).$$

We now show that we can find elements $x_k \in X$ and $y_k \in E = \langle e_i;\ i \in I \rangle$, $k \in K$, such that $|K| = \mathfrak{m}$,

$$\begin{aligned} \langle x_k;\ k \in K \rangle &= \mathrm{Dr}_{K \in k} \langle x_k \rangle, \\ [x_k, y_\ell] &= 1, \text{ if } k \neq \ell, \\ [x_k, y_k] &= z_k \neq 1, \\ z_k &\neq z_\ell, \text{ if } k \neq \ell. \end{aligned}$$

Case (i): $\mathfrak{m} = \aleph_0$. Suppose that we have defined the elements $x_1, \ldots, x_{n-1}$ and $y_1, \ldots, y_{n-1}$. The elements $x_1, \ldots, x_{n-1}$ are contained in a finite direct factor X_1 of X, so that $X = X_1 \times X_2$ and X_2 is infinite. Let $I_n = \{i \in I;\ \bar{d}_i \text{ occurs as a component of } \bar{x}_k \text{ for some } k < n\}$; then $|I_n| < \infty$.

Now $|X_2: C_{x_2}(y_1, \ldots, y_{n-1})| < \infty$ and so there is an element $x \in C_{x_2}(y_1, \ldots, y_{n-1})$ such that $\bar{x}$ has a non-trivial component $\bar{d}_i$ for some $i \notin I_n$. We put $x_n = x$ and $y_n = e_i$.

Case (ii): $\mathfrak{m} > \aleph_0$. If

$$\bar{x}_j = \bar{d}_{i_1}^{\alpha_1} \cdots \bar{d}_{i_n}^{\alpha_n},$$

let $X_j = \{i_1, \ldots, i_n\}$. We have $\mathfrak{m}$ finite subsets of I indexed by J. By Theorem 1.1, there is a subset K of J such that $|K| = \mathfrak{m}$ and $|\bigcup_{k \neq \ell \in K}(X_k \cap X_\ell)| < \mathfrak{m}$. Thus $|\bigcup_{k \in K} X_k - \bigcup_{k \neq \ell \in K}(X_k \cap X_\ell)| = \mathfrak{m}$ and K can be chosen so that for each $k \in K$, $X_k - \bigcup_{\ell \neq k} X_\ell \neq \emptyset$. Corresponding to each element $k \in K$, there is an element x_k and we can choose an element $y_k = e_i$ where $i \in X_k - \bigcup_{\ell \neq k} X_\ell$. Then, if $\ell \neq k$, we have $[x_\ell, y_k] = 1$ and $[x_k, y_k] = z_k$, where z_k is a non-trivial power of f_i. Since the x_k's are a subset of the x_j's, the group generated by the x_k's is the direct porduct of the groups $\langle x_k \rangle$, $k \in K$.

This completes the construction of the elements x_k, y_k, $k \in K$ for the two cases.

Finally we obtain elements u_α, v_α, $\alpha < \rho$ such that

$$[u_\alpha, u_\beta] = [v_\alpha, v_\beta] = [u_\alpha, v_\beta] = 1, \text{ if } \alpha \neq \beta,$$
$$[u_\alpha, v_\alpha] = w_\alpha \neq 1,$$
$$w_\alpha w_\beta^{-1} \notin \langle u_\alpha;\ \alpha < \rho \rangle.$$

and

Observe first that XZ is a direct product of cyclic groups,

$$XZ = \mathrm{Dr}_{k \in K}\langle x_k \rangle \times \mathrm{Dr}_{\ell \in L}\langle t_\ell \rangle,$$

where t_ℓ are elements of $Z = \langle z_k;\ k \in K \rangle$. [$\mathrm{Dr}_{\ell \in L}\langle t_\ell \rangle$ is just a complement to $X \cap Z$ in Z.]

Suppose that we have obtained the elements u_β, v_β for all $\beta < \alpha$. The group $\langle u_\beta, v_\beta;\ \beta < \alpha \rangle$ has cardinality less than $\mathfrak{m}$ and so if $K_\alpha \cup L_\alpha$ is the set of elements of $K \cup L$ corresponding to non-trivial components of elements of $XZ \cap \langle u_\beta, v_\beta;\ \beta < \alpha \rangle$, then $|K_\alpha \cup L_\alpha| < \mathfrak{m}$.

There are therefore $\mathfrak{m}$ values of $m \in K - K_\alpha$ such that z_m has a non-trivial component outside $K_\alpha \cup L_\alpha$. Choose such an m with $|\langle x_m \rangle|$ minimal. Now let $K_\alpha^* \cup L_\alpha^*$ be the set of elements of $K \cup L$ corresponding to non-trivial components of lements of $XZ \cap \langle x_m, z_m, u_\beta, v_\beta;\ \beta < \alpha \rangle$. There is an $n \in K - K_\alpha^*$ such that z_n has a non-trivial component outside $K_\alpha^* \cup L_\alpha^*$. By the choice of m, $|\langle x_m \rangle| \leqq |\langle x_n \rangle|$ and so any non-trivial power of $x_m x_n$ will have a non-trivial component in $\langle x_n \rangle$.

Let $u_\alpha = x_m x_n$ and $v_\alpha = y_m$ so that $w_\alpha = z_m$. It is clear that if $\beta < \alpha$, then $[u_\alpha, v_\beta] = [u_\beta, v_\alpha] = 1$. Since w_α involves a component outside $K_\alpha \cup L_\alpha$ but does not have a component in $\langle x_n \rangle$, we have

$$w_\alpha w_\beta^{-1} \notin \mathrm{Dr}_{k \in K_\alpha} \langle x_k \rangle \times \langle x_m x_n \rangle$$

and hence $w_\alpha w_\beta^{-1} \notin \langle u_\beta;\ \beta \leqq \alpha \rangle$.

Also if $\beta < \gamma < \alpha$, then $w_\gamma w_\beta^{-1}$ has all its components in $K_\alpha \cup L_\alpha$. If

$$w_\gamma w_\beta^{-1} \in \langle u_\delta;\ \delta \leqq \alpha \rangle = \langle u_\delta;\ \delta < \alpha \rangle \times \langle x_m x_n \rangle,$$

then it would follow that $w_\gamma w_\beta^{-1} \in \langle u_\delta;\ \delta < \alpha \rangle$ contrary to the inductive hypothesis in our construction.

Thus we can obtain the elements u_α, v_α for all $\alpha < \rho$ and the conditions $w_\alpha w_\beta^{-1} \notin U = \langle u_\alpha;\ \alpha < \rho \rangle$ ensures that $|U^T\colon U| = \mathfrak{m}$.

4. **Proof of Theorem B**. If $|G/Z| < \mathfrak{m}$, then it is clear that $|U/U_G| < \mathfrak{m}$ for each $U \leqq G$. So we take a $\mathfrak{Z}_\mathfrak{m}$-group G with $|G/Z| > \mathfrak{m}$ and show that there is an abelian subgroup A of G such that $|A/A_G| = \mathfrak{m}$. Certainly if $|G'| \geqq \mathfrak{m}$, then by Theorem *C* there is an abelian subgroup A such that $|A^G\colon A| = \mathfrak{m}$ and so by Lemma 2.3, $|A/A_G| \geqq \mathfrak{m}$. It is therefore sufficient to prove the following theorem.

THEOREM 4.1. *Let G be a $\mathfrak{Z}_\mathfrak{m}$-group such that $|G'| < \mathfrak{m}$ and $|G/Z| \geqq \mathfrak{m}$. Then there is an abelian subgroup A of G such that $|A/A_G| = \mathfrak{m}$.*

Again the proof of this result will depend on obtaining a subgroup of G which is an N-group of cardinality $\mathfrak{m}$. We require the following lemma.

LEMMA 4.2. *Let $S = \langle a_\alpha, b_\alpha;\ \alpha < \rho \rangle$ be an N-group of cardinality $\mathfrak{m}$. If the set $I = \{\alpha < \rho;\ c_\alpha \notin A\}$ has cardinality $\mathfrak{m}$, then $|\mathrm{Cl}(A)| = \mathfrak{m}$.*

PROOF. Suppose $\alpha, \beta \in I$: then $b_\alpha^{-1} A b_\alpha \neq b_\beta^{-1} A b_\beta$. For if these subgroups were equal, then $b_\alpha b_\beta^{-1} \in N_S(A)$ and so $b_\beta b_\alpha^{-1} a_\alpha b_\alpha b_\beta^{-1} = a_\alpha c_\alpha \in A$, contrary to $c_\alpha \notin A$.

PROOF OF THEOREM 4.1. We begin by showing that G has a subgroup which is an N-group of cardinality $\mathfrak{m}$. Suppose we have defined a_β, b_β for all $\beta < \alpha$ and let $S_\alpha = \langle a_\beta, b_\beta;\ \beta < \alpha \rangle$. If $C_\alpha = C_G(S_\alpha)$, then $|G\colon C_\alpha| < \mathfrak{m}$ and so, by Lemma 2.4 (ii), $|C_\alpha/Z(C_\alpha)| \geqq \mathfrak{m}$. In particular, there are elements $a_\alpha, b_\alpha \in C_\alpha$ such that $[a_\alpha, b_\alpha] = c_\alpha \neq 1$.

Let $I_1 = \{\alpha < \rho;\ c_\alpha \notin A\}$ and $I_2 = \{\alpha < \rho;\ c_\alpha \notin B\}$. If $|I_1| = \mathfrak{m}$, then by Lemma 4.2, $|\mathrm{Cl}(A)| = \mathfrak{m}$ and if $|I_2| = \mathfrak{m}$, then $|\mathrm{Cl}(B)| = \mathfrak{m}$. Therefore we may assume that $|I_1| < \mathfrak{m}$ and $|I_2| < \mathfrak{m}$. Let $I = \{\alpha < \rho;\ c_\alpha \in A \cap B\}$; then $|I| = \mathfrak{m}$. Since A and B are abelian, $c_\alpha \in Z(S)$ for all $\alpha \in I$. Let $T = \langle a_\alpha, b_\alpha;\ \alpha \in I \rangle$; then T is nilpotent of calss two.

Let F be a maximal torsion-free subgroup of $Z(T)$. Then T/F is periodic

(1.4) and if T/F contains an abelian subgroup X/F such that $|X/X_T| = \mathfrak{m}$, then X will be the required abelian subgroup of T. So by considering T/F, we may assume that T is a periodic FC-gorup.

For each $\alpha \in I$, choose a prime P_α dividing the order of c_α. Then by replacing a_α and b_α by appropriate powers, we may assume that c_α has order P_α and $\langle a_\alpha, b_\alpha \rangle$ is a finite p_α-group.

Let $I_p = \{\alpha \in I;\ c_\alpha$ has order $p\}$ and let $|I_p| = \mathfrak{m}_p$ so that $\sum_p \mathfrak{m}_p = \mathfrak{m}$. If each $\mathfrak{m}_p$ is finite, then $\mathfrak{m} = \aleph_0$; but in this case we have infinitely many distinct elements c_α, contrary to G' being finite. Thus $\mathfrak{m}_p$ is infinite for some primes p. Let the set of distinct c_α's of order p have cardinality $\mathfrak{n}_p$. Then $\sum_p \mathfrak{n}_p < \mathfrak{m}$ and so the set $P = \{p;\ \mathfrak{m}_p$ is finite and $\mathfrak{n}_p < \mathfrak{m}_p\}$ is nonempty. If $Q = \{P;\ \mathfrak{m}_p$ is infinite and $\mathfrak{n}_p = \mathfrak{m}_p\}$, then $\sum_{p \in Q} \mathfrak{m}_p = \sum_{p \in Q} \mathfrak{n}_p < \mathfrak{m}$. Therefore $\sum_{p \in P} \mathfrak{m}_p = \mathfrak{m}$.

For each $p \in P$, let $T_p = \langle a_\alpha, b_\alpha;\ \alpha \in I_p \rangle$; we show that T_p has an abelian subgroup X_p such that $|X_p/(X_p)_T| = \mathfrak{m}_p$ and hence if $X = \mathrm{Dr}_{p \in P} X_p$, we have $|X/X_T| = \sum_{p \in P} \mathfrak{m}_p = \mathfrak{m}$. We may therefore assume that $T = \langle a_\alpha, b_\alpha;\ \alpha \in I \rangle$ is a p-group, $[a_\alpha, b_\alpha] = c_\alpha$ has order p, $|I| = \mathfrak{m}$ and the set of distinct c_α's has cardinality less then $\mathfrak{m}$.

Let D be a basic subgroup of the abelian p-group $A = \langle a_\alpha;\ \alpha \in I \rangle$. Then D is a direct product of cyclic groups $D = \mathrm{Dr}_{j \in J} \langle d_j \rangle$ and $DA^p = A$ ([5], pp. 139, 144) so that $|J| = |A/A^p| = \mathfrak{m}$. Since there are fewer than $\mathfrak{m}$ distinct c_α's, those c_α's which are contained in D involve fewer than $\mathfrak{m}$ components. Thus there is a subset J^* of J with $|J - J^*| < \mathfrak{m}$ and if $D^* = \mathrm{Dr}_{j \in J^*} \langle d_j \rangle$, then no element of $C = \langle c_\alpha;\ \alpha \in I \rangle$ is contained in D^*. Also $|D: D^*| < \mathfrak{m}$ and so if $B = \langle b_\alpha;\ \alpha \in I \rangle$,

$$|D^*: D^* \cap C(B)| = |D: D \cap C(B)| = |A/A^p| = \mathfrak{m}.$$

We now define elements $x_\alpha \in D^*$, $y_\alpha \in B$, for $\alpha < \rho$, such that

$$[x_\alpha, x_\beta] = [y_\alpha, y_\beta] = [x_\alpha, y_\beta] = 1, \text{ if } \alpha \neq \beta,$$
$$[x_\alpha, y_\alpha] = z_\alpha \in C - \{1\}.$$

Suppose that we have obtained the elements x_β, y_β for all $\beta < \alpha$ and let $S_\alpha = \langle x_\beta, y_\beta;\ \beta < \alpha \rangle$; then $|S_\alpha| < \mathfrak{m}$ and if $C_\alpha = C_T(S_\alpha)$, then $|T: C_\alpha| < \mathfrak{m}$.

Since $|B: B \cap C_\alpha| < \mathfrak{m}$, there is a normal subgroup E of T such that $B \leqq E(B \cap C_\alpha)$ and $|E| < \mathfrak{m}$. Now $|D^*: D^* \cap C(E)| < \mathfrak{m}$ and $|D^*: D^* \cap C(B)| = \mathfrak{m}$. Therefore $|D^*: D^* \cap C(B \cap C_\alpha)| = \mathfrak{m}$. Also $|D^*: D^* \cap C_\alpha| < \mathfrak{m}$ and so there is an element $x_\alpha \in (D^* \cap C_\alpha) - C(B \cap C_\alpha)$. That is, there is also an element $y_\alpha \in B \cap C_\alpha$ such that $[x_\alpha, y_\alpha] \neq 1$.

Thus the elements x_α, y_α, $\alpha < \rho$, can be constructed. By the condition on D^*, no element z_α is in $X = \langle x_\alpha;\ \alpha < \rho \rangle$. By Lemma 4.2, $|X/X_T| = \mathfrak{m}$, as required.

5. **Examples.** We give two examples to show that, in a sense, our results are best possible for the class of *FC*-groups.

EXAMPLE 5.1. *If* $\mathfrak{m}$ *is a non-limit, then there is an FC-group G with* $|U/U_G| < \mathfrak{m}$ *for all subgroups U of G but* $|G/Z| = \mathfrak{m}$.

Ehrenfeucht and Faber **[1]** constructed an extraspecial p-gorup of order $\mathfrak{m}$ in which each abelian subgroup has order less than $\mathfrak{m}$. In this group any subgroup of order $\mathfrak{m}$ contains G' and so is normal; thus $|U/U_G| < \mathfrak{m}$ for all $U \leqq G$.

In this group the abelian subgroups A maximal subject to $A \cap G' = 1$ each have $\mathfrak{m}$ conjugates and so we do not have a counterexample to the possible conjecture: $|G/Z| < \mathfrak{m}$ if and only if $|\mathrm{Cl}(A)| < \mathfrak{m}$ for all (abelian) subgroups A of G. The main problem here is the seemingly difficult one of relating the two conditions $|G\colon N_G(U)| < \mathfrak{m}$ for all $U \leqq G$ and $|G\colon C_G(U)| < \mathfrak{m}$ for all $U \leqq G$. Some of the difficulties involved in this question are discussed in Theorem $D(i)$ and §5 of **[9]**.

EXAMPLE 5.2. *If* $\mathfrak{m}$ *is a non-limit, then there is an FC-group G such that* $|A^G\colon A| < \mathfrak{m}$ *for each abelian subgroup A of G, but* $|G'| = \mathfrak{m}$.

Let $X = \langle x_\alpha;\ \alpha < \rho\rangle$ and $Y = \langle y_\alpha;\ \alpha < \rho\rangle$ be two Ehrenfeucht-Faber p-groups of cardinality $\mathfrak{m}$ with central elements x_0, y_0, respectively. Define G to be the join of X and Y in which

$$[x_\alpha, y_\beta] = \begin{cases} 1, & \alpha \neq \beta, \\ z_\alpha, & \alpha = \beta, \end{cases}$$

where z_α is a central element of order p.

Suppose U is a subgroup such that $|UG'/G'| = \mathfrak{m}$. Then the projection of U onto either XG'/G' or YG'/G' has cardinality $\mathfrak{m}$ and so $U' \neq 1$. Thus every abelian subgroup A satisfies the condition $|AG'/G'| < \mathfrak{m}$ and hence $|A/(A \cap G')| < \mathfrak{m}$. But $A \cap G' \lhd G$ and so $|A/A_G| < \mathfrak{m}$. Hence, by Lemma 2.3, $|A^G\colon A| < \mathfrak{m}$ for each abelian subgroup A. But $G' \geqq \langle z_\alpha; \alpha < \rho\rangle$ has cardinality $\mathfrak{m}$.

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