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BANACH SPACES WHICH ARE NEARLY UNIFORMLY CONVEX

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ABSTRACT. A property which generalizes uniform convexity is defined in terms of sequences. Its relationships to uniform convexity and to weak and norm convergence on spheres are investigated.

1. **Introduction.** Let X be a (real) banach space with norm $\|\cdot\|$, let $B_\delta(x)$ (respectively, $\bar{B}_\delta(x)$) denote the open (closed) ball with center x and radius δ, and let $\mathrm{co}(A)$ ($\overline{\mathrm{co}}(A)$) denote the convex hull (closed convex hull) of a set A.

We will say that the norm is a *Kadec-Klee* (KK–)*norm* provided on the unit sphere sequences converge in norm whenever they converge weakly. (This is property (H) in [**2**].) An equivalent formulation is the following.

$$\text{(KK)}: \left.\begin{array}{l} (x_n)_{n=1}^{\infty} \subset \bar{B}_1(0) \\ x_n \to x \text{ wkly} \\ (x_n)_{n=1}^{\infty} \textit{ not} \text{ norm Cauchy} \end{array}\right\} \Rightarrow \|x\| < 1.$$

For notation, given a sequence (x_n) we let

$$\mathrm{sep}(x_n) = \inf\{\|x_n - x_m\| : m \neq n\}.$$

If (x_n) is not norm-Cauchy, then for some subsequence (y_n) we must have $\mathrm{sep}(y_n) > 0$. The above definition can be reformulated as follows.

$$\text{(KK)}: \left.\begin{array}{r} (x_n) \subset \bar{B}_1(0) \\ x_n \to x \text{ wkly} \\ \mathrm{sep}(x_n) > 0 \end{array}\right\} \Rightarrow \|x\| < 1.$$

This formulation suggests the following two successively stronger notions.

The norm will be called *uniformly Kadec-Klee* (UKK) if for every $\varepsilon > 0$ there exists $\delta < 1$ such that

$$\text{(UKK)}: \left.\begin{array}{l} (x_n) \subset \bar{B}_1(0) \\ x_n \to x \text{ wkly} \\ \mathrm{sep}(x_n) \geqq \varepsilon \end{array}\right\} \Rightarrow x \in B_\delta(0).$$

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The norm will be said to be *nearly uniformly convex* (NUC) if for every $\varepsilon > 0$ there exists $\delta < 1$ such that

$$\text{(NUC):}\quad \left.\begin{matrix}(x_n) \subset \bar{B}_1(0)\\ \\ \operatorname{sep}(x_n) \geqq \varepsilon\end{matrix}\right\} \Rightarrow \operatorname{co}(x_n) \cap B_\delta(0) \neq \varnothing$$

The norm is (NUC) if and oniy if it is (UKK) and the space X is reflexive (Theorem 1 below).

Recall that the norm is said to be *uniformly convex* (UC) provided for every $\varepsilon > 0$ there exists $\delta < 1$ such that

$$\text{(UC):}\quad \left.\begin{matrix}x, y \in \bar{B}_1(0)\\ \\ \|x - y\| \geqq \varepsilon\end{matrix}\right\} \Rightarrow \left(\frac{1}{2}x + \frac{1}{2}y\right) \in B_\delta(0).$$

We have (UC) $\Rightarrow$ (NUC) $\Rightarrow$ (UKK) $\Rightarrow$ (KK). Vacuously, every finite dimensional space is (NUC); hence (NUC) $\not\Rightarrow$ (UC). Similarly, every Shur space (*e.g.*, ℓ_1) is (UKK), and since (NUC) spaces are reflexive, (UKK) $\not\Rightarrow$ (NUC).

[We remark that the direct sum $\ell^2 \oplus \ell^1$ with the norm $\|(x, y)\| = \|x\|_2 + \|y\|_1$ is (UKK), non-(NUC), non-reflexive, and non-Shur.] We shall have an example to show that (KK) $\not\Rightarrow$ (UKK). Since Hilbert space ℓ^2 has an equivalent norm whose unit sphere contains a weakly compact, non-compact convex set (e.g., take

$$\|(x_1, x_2, \ldots)\| = \max\{|x_1|, (|x_2|^2 + |x_3|^2 + \ldots)^{1/2})),$$

none of the above properties are isomorphism invariant. For each pair of properties we shall have an example of a space with the weaker property and which is not isomorphic to a space with the stronger property.

It is well-known that a (UC) space is reflexive, but not every reflexive space has an equivalent (UC) norm. We shall see that this remains true if (UC) is replaced by (NUC).

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2. **Main results.**

THEOREM 1. *A norm* $\|\cdot\|$ *for* X *is (NUC) if and only if* X *is reflexive and the norm is (UKK).*

PROOF. ($\Leftarrow$). This implication follows directly from Eberlein's theorem and the separation theorem.

($\Rightarrow$). Suppose $\|\cdot\|$ is (NUC). Call a sequence (y_n) a *c-subsequence* of

(x_n) provided there is a sequence of integers $1 = p_1 \leqq q_1 < p_2 \leqq q_2 < \ldots$ and coefficients $\alpha_i \geqq 0$ such that for each n

$$\sum_{i=p_n}^{q_n} \alpha_i = 1 \text{ and } y_i = \sum_{i=p_n}^{q_n} \alpha_i x_i.$$

If we choose $\delta < 1$ corresponding to ε in the definition of (NUC), and if $\text{sep}(x_n) \geqq \varepsilon$, then there is an entire c-subsequence (y_n) of (x_n) with $(y_n) \subset \bar{B}_\delta(0)$. If $x_n \to x$ weakly, then $y_n \to x$ weakly, so $\|x\| \leqq \delta$. Hence $\|\cdot\|$ is (UKK).

It remains to see that X is reflexive. An easy proof can be given using Jame's characterization of reflexivity in terms of functionals attaining their norms (see [3]). We give an alternate proof which is possibly more enlightening.

We use a theorem of Eberlein and Smulian [2, p. 51]. Let (K_n) be a decreasing sequence of non-void closed convex subsets of $\bar{B}_1(0)$. We need to show that $\bigcap K_n \neq \emptyset$. For each n, choose $x_n \in K_n$. Then for each $\eta > 0$, there exists a c-subsequence (y_n) with $\|y_n - y_m\| < \eta$ for all m, n. [For suppose this is not the case. Let $\varepsilon = \eta/2$ and choose $\delta < 1$ as in the definition of (NUC). There exists a c-subsequence of (x_n) lying in $\bar{B}_\delta(0)$, and that c-subsequence can be chosen to be separated by ε. By repeating the argument, there is a successive c-subsequence lying in $\bar{B}_{\delta^2}(0)$. We need only repeat the argument a sufficient number of times to obtain a contradiction.]

Next, by a "diagonal" argument, there is a c-subsequence of (x_n) which is norm-Cauchy and hence convergent to some $y \in \bar{B}_1(0)$. Then

$$y \in \bigcap_{m=1}^{\infty} \overline{\text{co}}(x_n)_{n=m}^{\infty} \subset \bigcap_{m=1}^{\infty} K_m.$$

The following theorem says, in some sense, that the property (NUC) ignores finite dimensions.

THEOREM 2. *Let Y be a Banach space with a basis $(e_i)_{i \in I}$ (unconditional if I is uncountable), and with norm such that for every finite $J \subset I$,*

$$0 \leqq \alpha_j \leqq \beta_j, \forall j \in J \Rightarrow \Big\|\sum_{j \in J} \alpha_j e_j\Big\| \leqq \Big\|\sum_{j \in J} \beta_j e_j\Big\|.$$

Let $(X_i)_{i \in I}$ be a family of finite dimensional spaces, let

$$Z = \{x = (x_i)_{i \in I} \in \prod_{i \in I} X_i : \sum_i \|x_i\| e_i \in Y\},$$

and let Z have the norm

$$\|x\| = \Big\|\sum_i \|x_i\| e_i\Big\|.$$

If Y is (NUC), then Z is (NUC).

PROOF. If E is a subset of I, let $P_E: Y \to Y$ be defined by

$$P_E(\sum_{i \in I} \alpha_i e_i) = \sum_{i \in E} \alpha_i e_i.$$

(In case I is countable (say $I = \mathbf{N}$) and (e_i) is not unconditional, we consider only those E's of the form $\{1, \dots, n\}$ and $\{n, n+1, n+2, \dots\}$.) Choose $k > 0$ such that $\|P_E\| \leqq k$ for all E.

Let $\varepsilon > 0$ be given and choose $\delta < 1$ such that if (y_n) is a sequence in the unit ball of Y with $\operatorname{sep}(y_n) \geqq \varepsilon/5k$, then $\operatorname{co}(y_n) \cap B_\delta(0) \neq \emptyset$.

Let $\varphi: Z \to Y$ be defined by $\varphi((x_i)) = \sum \|x_i\| e_i$.

Let $(x^{(n)})_{n=1}^\infty$ be a sequence in the unit ball of Z such that $\operatorname{sep}(x^{(n)}) \geqq \varepsilon$. We claim that there exists a subsequence $(x^{(n_k)})$ such that

$$\operatorname{sep}(\varphi(x^{(n_k)})) \geqq \varepsilon/5k.$$

Once this is proved, there will be a convex combination $\sum \beta_k \varphi(x^{(n_k)})$ with norm less than δ. Since

$$\begin{aligned}\|\sum_k \beta_k \varphi(x^{(n_k)})\| &= \|\sum_k \beta_k \sum_i \|x_i^{(n_k)}\| e_i\| \\ &= \|\sum_i (\sum_k \beta_k \|x_i^{(n_k)}\| e_i\| \\ &\geqq \|\sum_i \|\sum_k \beta_k x_i^{(n_k)}\| e_i\| \\ &= \|\sum_k \beta_k x^{(n_k)}\|,\end{aligned}$$

this will complete the proof.

We now prove the claim. It is sufficient to show that for any finite set $\{x^{(n_1)}, \dots, x^{(n_k)}\}$ there exists $x^{(n)}$ such that $\|\varphi(x^{(n_j)}) - \varphi(x^{(n)})\| \geqq \varepsilon/5k$ for all $j = 1, \dots, k$. Suppose this is not the case for some finite set $\{x^{(n_1)}, \dots, x^{(n_k)}\}$. Since (e_i) is a basis for Y, there exists some finite set $E \subset I$ such that

$$\|P_{I \setminus E}(\varphi(x^{(n_j)}))\| < \frac{\varepsilon}{5}, \forall j = 1, \dots, k.$$

Hence for all n, there exists some j so that

$$\begin{aligned}\|P_{I \setminus E}(\varphi(x^{(n)}))\| &\leqq \|P_{I \setminus E}(\varphi(x^{(n_j)}))\| + \|P_{I \setminus E}[\varphi(x^{(n_j)}) - \varphi(x^{(n)})]\| \\ &< \varepsilon/5 + \varepsilon/5 \\ &= 2\varepsilon/5.\end{aligned}$$

Therefore, for all m and n,

$$\|P_{I \setminus E}[\varphi(x^{(n)} - x^{(m)})]\| = \|\sum_{i \in I \setminus E} \|x_i^{(n)} - x_i^{(m)}\| e_i\|$$

$$\leqq \Big\| \sum_{i \in I \setminus E} (\|x_i^{(n)}\| + \|x_i^{(m)}\|) e_i \Big\|$$

$$= \|P_{I \setminus E}(\varphi(x^{(n)}))\| + \|P_{I \setminus E}(\varphi(x^{(m)}))\|$$

$$< 4\varepsilon/5.$$

Next, for each $i \in E$, $(x_i^{(n)})_{n=1}^{\infty}$ is a bounded sequence in X_i and hence has a Cauchy subsequence. By passing to successive subsequences, we may assume that $(x_i^{(n)})_{n=1}^{\infty}$ is Cauchy for every $i \in E$. Then

$$\lim_{m,n \to \infty} \|P_E(\varphi(x^{(n)} - x^{(m)}))\| = \lim_{m,n \to \infty} \Big\| \sum_{i \in E} \|x_i^{(n)} - x_i^{(m)}\| e_i \Big\| = 0.$$

Choose m and n different and sufficiently large so that

$$\|P_E[\varphi(x^{(n)} - x^{(m)})]\| < \frac{\varepsilon}{5}.$$

Then we have $m \neq n$ and

$$\|x^{(n)} - x^{(m)}\| = \|\varphi(x^{(n)} - x^{(m)})\|$$

$$\leqq \|P_E[\varphi(x^{(n)} - x^{(m)})]\| + \|P_{I \setminus E}[\varphi(x^{(n)} - x^{(m)})]\|$$

$$< \varepsilon/5 + 4\varepsilon/5 = \varepsilon,$$

a contradiction. This contradiction proves the claim.

COROLLARY. *There exist* (*NUC*) *norms that are not equivalent to* (*UC*) *norms.*

PROOF. For $i = 1, 2, \ldots$, let X_i denote $\mathbf{R}^i$ with the ℓ^i-norm (i.e., $\|(x_1, \ldots, x_i)\|_i = (\sum |x_j|^i)^{1/i}$). Let $Y = \ell^2$ and construct Z as in the theorem. Then Z is (NUC), while Day [1] showed that Z is not isomorphic to any (UC) space.

THEOREM 3. *Given a set $K \subset X$ and $\varepsilon < 0$, define the ε-derived set of K to be the set*

$\eta_\varepsilon(K) = \{x$: *there exists* $(x_n)_{n=1}^{\infty} \subset K$ *with* $\operatorname{sep}(x_n) > \varepsilon$ and $x_n \to x$ weakly$\}$.

If X has an equivalent norm which is (*UKK*), *then*

$$(*)\begin{cases} \text{for every } \varepsilon > 0 \text{ there exists } n \text{ such that} \\ \quad \eta_\varepsilon^{(n)}(\bar{B}_1(0)) = \varnothing. \end{cases}$$

PROOF. It is clear that $(*)$ is invariant under isomorphisms, and so we may assume the given norm for X is (UKK). Let $\varepsilon > 0$ be given, and choose $\delta < 1$ corresponding to ε. Then $\eta_\varepsilon(\bar{B}_1(0)) \subset \delta \bar{B}_1(0)$, $\eta_\varepsilon^2(\bar{B}_1(0)) \subset \delta \eta_\varepsilon(\bar{B}_1(0)) \subset \delta^2 \bar{B}_1(0)$, $\ldots$, and thus $\eta_\varepsilon^n(\bar{B}_1(0))$ must eventually have diameter less than ε, and so must eventually be empty.

LEMMA. *For the sequence space* $\ell^p(1 < p < \infty)$, *if* $m \leqq 2^p$, *then* $\eta^m_{1/2}(\bar{B}_1(0)) \neq \phi$.

PROOF. Let $(e_n)_{n=1}^\infty$ be the usual unit vector basis for ℓ^p. For all $n_1 < n_2 < \cdots < n_m$, we have

$$\left\| \frac{1}{2} e_{n_1} + \cdots + \frac{1}{2} e_{n_m} \right\| = (m/2^p)^{1/p} \leqq 1,$$

and if $n_1 < n_2 < \cdots < n_{k-1} < \min(n_k, n_k')$ and $n_k \neq n_k'$, then

$$\left\| \frac{1}{2}(e_{n_1} + \cdots + e_{n_k}) - \frac{1}{2}(e_{n_1} + \cdots + e_{n_k'}) \right\| = \frac{1}{2}\left\| e_{n_k} - e_{n_k'} \right\| > \frac{1}{2}.$$

Since $0 = \text{wk-lim}\, e_n$, we have

$$0 = \operatorname*{wk-lim}_{n_1 \to \infty}\left(\cdots \left(\operatorname*{wk-lim}_{n_m \to \infty} \left(\frac{1}{2} e_{n_1} + \cdots + \frac{1}{2} e_{n_m} \right) \cdots \right)\right),$$

and so 0 is in $\eta^m_{1/2}(\bar{B}_1(0))$.

THEOREM 4. *There exist reflexive Banach spaces which are not isomorphic to any* (*UKK*) *space.*

PROOF. Let $Y = \ell^2$ and $X_i = \ell^i$ $(i = 2, 3, \ldots)$, and construct Z as in Theorem 2. Z is known to be reflexible, but we have $\eta^n_{1/2}(\bar{B}_1(0)) \neq 0$, $\forall n$, and hence Z is not (UKK).

We remark that Z is also known to be (KK) **[2]**. (In fact, it follows from results of Troyanski **[6]** that every reflexive space (indeed, any weakly compactly generated space) is isomorphic to a (KK) space.

3. **Remarks; open questions.** (1) If the notion of derived set is taken to be

$$\Psi_\varepsilon(K) = \left\{ \frac{x + y}{2} : x, y \in K, \|x - y\| \geqq \varepsilon \right\},$$

then the condition

$$(\sharp)\begin{cases} \text{for every } \varepsilon > 0 \text{ there exists } n \\ \text{such that } \Psi_\varepsilon^{(n)}\bar{B}_1(0)) = \varnothing \end{cases}$$

is known to be equivalent to the space X being isomorphic to a (UC) space **[4]**.

It is natural to conjecture that (*) of Theorem 3 is equivalent to X being isomorphic to a (UKK) space. A weaker conjecture is that (*) is equivalent to a *reflexive* space X being isomorphic to a (NUC) space.

(2) Kakutani showed that every (UC)-space satisfies the Banach-Saks property **[5]** (see also **[3]**). It is natural to conjecture that every (NUC)-space does also.

(3) It would be of interest to know what conditions are needed for a Lebesgue-Bochner space $L^p(X)$ to be (NUC).

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