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CHARACTERIZATIONS OF SOME GENERALIZED COUNTABLY COMPACT SPACES AS IMAGES OF *M*-SPACES

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ABSTRACT. The main theorems of this paper characterize regular $w\Delta$-spaces, wM-spaces, and wN-spaces as almost-open images of regular M-spaces. For example, it is proved that a regular space Y is a wN-space if and only if there exist a regular M-space X, with $w\Delta$-function g, and an almost-open mapping f from X onto Y which is a wP-mapping relative to g. Similar theorems are proved for wM-spaces and for $w\Delta$-spaces. Moreover, for the wN-space case, it is shown that if f is an almost-open mapping from any space X onto another space Y, then for Y to be a wN-space it is necessary that f be a wP-mapping relative to some function g.

1. **Introduction.** Characterizing spaces as images of other spaces with more structure is an old and much investigated problem in topology. For example, there are many interesting theorems characterizing certain spaces as images of metric spaces. One can pursue extensions and analogues to these results in various directions. One such direction is to vary the domain from metric to a more general class of spaces, for example M-spaces. The concept of an M-space has emerged as an important generalization of metric spaces, and in fact has been investigated within this framework. Several classes of spaces have been characterized as images of M-spaces, particularly see Chiba [3] and Nagata [15] and [16].

In this paper we continue this study by characterizing $w\Delta$-spaces, wM-spaces, and wN-spaces as almost-open continuous images of M-spaces. These classes of spaces have been studied by various mathematicians recently. For example, $w\Delta$-spaces are studied in [1], [2], [4], [6], [7], and [8]; wM-spaces in [7], [10], and [11]; and wN-spaces are introduced in [7]. These spaces have useful applications in metrization theory. The class of M-spaces was introduced in [12] and is studied in [9], [13], [14], [15], [16], and [17].

The present work was motivated by Chiba's [3] characterization of a q-space as an almost-open continuous image of a regualr M-space. Indeed, we make use of his basic construction. Also, to some extent, our defini-

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tions of wP-mapping and strong wP-mapping were suggested by the definition of Heath's [5] P-mapping in his characterization of a developable space as the open continuous image of a metric space.

2. **Definitions and preliminary results.** The spaces we are considering are defined in terms of sequences of open covers. A unifying point of view concerning sequences of open covers can be achieved through the use of the concept of a COC-function. We should note that some of the definitions which follow are not the original definitions but are characterizations proved after the concept was introduced. Throughout the paper N equals the natural numbers and the notation $\langle x_n \rangle$ indicates a sequence $\{x_n : n = 1, 2, \ldots\}$.

Let (X, T) be a topological space and let g be a function from $N \times X$ into T. Then g is called a COC-function for X (COC = countable number of open covers) if it satisfies these two conditions:

(1) $x \in \bigcap_{n=1}^{\infty} g(n, x)$ for all $x \in X$

(2) $g(n+1) \subset g(n, x)$ for all $n \in N$ and $x \in X$.

Note that if g is a COC-function for X, we obtain countably many open covers of X by taking $\mathscr{G}_n = \{g(n, x) : x \in X\}$ for each $n \in N$.

Now suppose V is a space with COC-function g, and consider the following conditions on g:

(i) $x_n \in g(n, p)$ for each $n \in N$ implies that $\langle x_n \rangle$ has a cluster point.

(ii) $g(n, p) \cap g(n, x_n) \neq \varnothing$, for each n, implies that $\langle x_n \rangle$ has a cluster point.

(iii) $\{p, x_n\} \subset g(n, y_n)$ for each n implies that $\langle x_n \rangle$ has a cluster point.

(iv) $p \in g(n, z_n)$, $g(n, z_n) \cap g(n, y_n) \neq \varnothing$ and $x_n \in g(n, y_n)$ for each n implies that $\langle x_n \rangle$ has a cluster point.

If g satisfies (i), X is called a q-space and g a q-function for X; if g satisfies (ii), X is called a wN-space and g a wN-function for X; $w\Delta$-spaces can be characterized as spaces having a COC-function satisfying (iii), with g called a $w\Delta$-function for X; and wM-spaces can be characterized as having a COC-function satisfying (iv), with g called a wM-function for X.

Recall that if $f: X \to Y$, then f is called almost-open if given $y \in Y$, there exists an $x \in f^{-1}(y)$ and an open basis $\mathscr{B}$ for x with $f(B)$ open for each $B \in \mathscr{B}$. We now give the definitions of the kinds of mappings we need to characterize the above spaces as images of M-spaces.

DEFINITION 2.1 Let $f: X \to Y$, and let g be a COC-function for X. Then f is a wP-mapping relative to g if given any sequence $\langle y_n \rangle$ in Y having no cluster point, and $y \in Y$, there exists an $n \in N$ with $y_n \notin f(\bigcup\{g(n, x) : x \in f^{-1}(y)\})$. If under the same conditions, there exists an $n \in N$ with

$$f(\bigcup\{g(n, x) : x \in f^{-1}(y_n)\}) \cap f(\bigcup\{g(n, x) : x \in f^{-1}(y)\}) = \varnothing$$

then f is called a strong wP-mapping relative to g.

In our first three theorems we see a similarity between our wP and strong wP-mappings and Heath's P-mapping, in the fact that they induce structure on the range space which need not be present on the domain space.

Recall that if $\mathscr{G}$ is a collection of subsets of a space, x a point in the space, and H any subset of the space, then

$$\mathrm{st}(x, \mathscr{G}) = \bigcup\{G \in \mathscr{G} : x \in G\}$$

and

$$\mathrm{st}(H, \mathscr{G}) = \bigcup\{G \in \mathscr{G} : H \cap G \neq \emptyset\}.$$

THEOREM 2.1. *Let (X, T_X) and (Y, T_Y) be spaces, and let $\langle \mathscr{G}_n \rangle$ be a sequence of open covers of X. Let $g(n, x) = \mathrm{st}(x, \mathscr{G}_n)$ for each $n \in N$ and $x \in X$, and suppose that $f\colon X \to Y$ is an almost-open wP-mapping relative to g. Then Y is a $w\Delta$-space.*

PROOF. Define $g^*\colon N \times X \to T_X$ by $g^*(n, x) =$ any element of $\mathscr{G}_n$ which contains x. Now let $y \in Y$; then there is a $p \in f^{-1}(y)$ such that p has a basis $\mathscr{B}$ with $f(B)$ open for each $B \in \mathscr{B}$. For each $n \in N$, choose a $B_{n,y} \in \mathscr{B}$ with $B_{n,y} \subset g^*(n, p)$. Define $h\colon N \times Y \to T_Y$ by $h(n, y) = f(B_{n,y})$. We claim that h is a $w\Delta$-function for Y.

Suppose $\{y, y_n\} \subset h(n, z_n)$ for all n, and assume that $\langle y_n \rangle$ has no cluster point. Then there exists an n with $y_n \notin f(\bigcup\{g(n, p) : p \in f^{-1}(y)\})$ by hypothesis. Now, $h(n, z_n) = f(B_{n,z_n})$ where $B_{n,z_n} \subset g^*(n, t_n)$ for some $t_n \in f^{-1}(z_n)$. So there exist a $p \in f^{-1}(y)$ and a $s_n \in f^{-1}(y_n)$ with $\{p, s_n\} \subset B_{n,z_n} \subset g^*(n, t_n)$, i.e., $s_n \in \mathrm{st}(p, \mathscr{G}_n) = g(n, p)$. Hence we get $f(s_n) = y_n \in f(g(n, p))$ which is a contradiction. Thus $\langle y_n \rangle$ has a cluster point and h is a $w\Delta$-function for Y.

LEMMA 2.1. *Let X be a q-space with q-function g, and suppose that f: $X \to Y$ is a continuous finite-to-one onto mapping. Then f is a wP-mapping relative to g.*

PROOF. Let $p \in Y$, $\langle y_n \rangle$ a sequence with no cluster point in Y. Let $f^{-1}(p) = \{x_1, \ldots, x_m\}$. Then for each i, $1 \leqq i \leqq m$, we can find an n_i such that $f^{-1}(y_n) \cap g(n, x_i) = \emptyset$ for all $n \geqq n_i$. For if not, we could choose a sequence $\langle s_{n_k} \rangle_{k=1}^{\infty}$ with $s_{n_k} \in f^{-1}(y_{n_k}) \cap g(n_k, x_i)$ for all k. But then $\langle s_{n_k} \rangle$ has a cluster point since g is a q-function and so $\langle f(s_{n_k}) \rangle = \langle y_{n_k} \rangle$ would have a cluster point, which is a contradiction. So, if we let $n_0 = \max\{n_i : i = 1, \ldots, m\}$, we have $g(n_0, x_i) \cap f^{-1}(y_{n_0}) = \emptyset$ for each i, $1 \leqq i \leqq m$, i.e., $y_{n_0} \notin f(\bigcup\{g(n_0, x) : x f^{-1}(p)\})$. Hence f is a wP-mapping relative to g.

COROLLARY 2.1. *The finite-to-one continuous almost-open image of a*

$w\Delta$-space is a $w\Delta$-space.

PROOF. If X is a $w\Delta$-space with $w\Delta$-sequence $\langle \mathscr{G}_n \rangle$, then $g(n, x) = \mathrm{st}(x, \mathscr{G}_n)$ is a $w\Delta$-function for X. Applying Lemma 2.1 and Theorem 2.1 then proves the corollary.

THEOREM 2.2. *Let X be a space with a sequence of open covers $\langle \mathscr{G}_n \rangle$, $g(n, x) = \mathrm{st}(x, \mathscr{G}_n)$, and suppose that $f: X \to Y$ is an almost-open strong wP-mapping relative to g. Then Y is a wM-space.*

PROOF. Define $g^*(n, x)$ and $h(n, y)$ exactly as in the proof of Theorem 2.1. We will show that h is a wM-function for X. Suppose $p \in h(n, z_n)$, $h(n, z_n) \cap h(n, y_n) \neq \varnothing$, and $x_n \in h(n, y_n)$ for each n. Assume that $\langle x_n \rangle$ has no cluster point. Then there is an n such that

$$f(\bigcup \{g(n, r): r \in f^{-1}(p)\}) \cap f(\bigcup \{g(n, u): u \in f^{-1}(x_n)\}) = \varnothing.$$

Or equivalently,

$$f(\mathrm{st}(f^{-1}(p), \mathscr{G}_n)) \cap f(\mathrm{st}(f^{-1}(x_n), \mathscr{G}_n)) = \varnothing.$$

Now by hypothesis, we get a $r_n \in f^{-1}(p)$ with $r_n \in B_{n, z_n} \subset g^*(n, t_n)$ for some $t_n \in f^{-1}(z_n)$; also, there is a $q_{n,1} \in B_{n, z_n}$ and a $q_{n,2} \in B_{n, y_n} \subset g^*(n, s_n)$ for some $s_n \in f^{-1}(y_n)$ such that $f(q_{n,1}) = f(q_{n,2})$. Further, there is a $u_n \in f^{-1}(x_n)$ with $u_n \in B_{n, y_n} \subset g^*(n, s_n)$. So we have $q_{n,1} \in \mathrm{st}(r_n, \mathscr{G}_n)$ and $q_{n,2} \in \mathrm{st}(u_n, \mathscr{G}_n)$. But that means that $f(q_{n,1}) = f(q_{n,2})$ is an element of

$$f(\mathrm{st}(f^{-1}(p), \mathscr{G}_n)) \cap f(\mathrm{st}(f^{-1}(x_n), \mathscr{G}_n))$$

which contradicts the fact that this intersection is empty. Hence $\langle x_n \rangle$ has a cluster point and h is a wM-function.

THEOREM 2.3. *Let X be a space with COC-function g and suppose that $f: X \to Y$ is an almost-open strong wP-mapping relative to g. Then Y is a wN-space.*

PROOF. Define h as in the proof of Theorem 2.1, letting $g^* = g$. Now suppose $h(n, p) \cap h(n, y_n) \neq \varnothing$ for each n, and assume that $\langle y_n \rangle$ does not have a cluster point. Then there is an n such that

$$f(\bigcup \{g(n, x): x \in f^{-1}(p)\}) \cap f(\bigcup \{g(n, s): s \in f^{-1}(y_n)\}) = \varnothing.$$

But

$$h(n, p) \subset f(\bigcup \{g(n, x): x \in f^{-1}(p)\})$$

and

$$h(n, y_n) \subset f(\bigcup \{g(n, s): s \in f^{-1}(y_n)\}).$$

So $h(n, p) \cap h(n, y_n) = \varnothing$, a contradiction.

3. **Main results.** We are now in a position to give the characterizations promised in section 1. As indicated, we will make explicit use of Chiba's construction; and so we reproduce it here for reference. The reader is referred to [3] for details.

CONSTRUCTION (Chiba). Let X be a regular q-space and let g be a q-function for X. Since X is regular, we may assume that for each $n \in N$ and $x \in X$, $\mathrm{cl}(g(n+1, x)) \subset g(n, x)$ (cl=closure). Fix $x \in X$, and define $K = \bigcap_{n=1}^{\infty} \mathrm{cl}(g(n, x))$. Then K is countably compact and $\{g(n, x): n = 1, 2, \ldots\}$ is a countable base for the set K. Hence, doing this for each $x \in X$, we get a cover of X, $\{K_\alpha: \alpha \in A\}$, each K_α countably compact and of countable character.

For each $\alpha \in A$, consider a new topological space, X_α, which consists of the points of X with topology as follows: Each point of $X_\alpha - K_\alpha$ is isolated and the neighborhoods of a point in K_α are the same as in X. Then each X_α is regular. Let $\{g(n, x_\alpha): n = 1, 2, \ldots\}$ be a countable base for the set K_α, where $x_\alpha \in K_\alpha$. Now put $\mathscr{G}_n^\alpha = \{g(n, x_\alpha)\} \cup \{\{x\} : x \in X_\alpha - g(n, x_\alpha)\}$, for each $n \in N$. Then $\langle \mathscr{G}_n^\alpha \rangle$ is a normal $w\Delta$-sequence for X_α, i.e., X_α is a regular M-space. Finally let M be the topological sum of the X_α's. Then M is a regular M-space with normal $w\Delta$-sequence $\langle \mathscr{G}_n \rangle$ where $\mathscr{G}_n = \bigcup_{\alpha \in A} \mathscr{G}_n^\alpha$ for each n. Define $f: M \to X$ to be the identity function as a function of sets. Then f is almost-open, continuous, and onto.

In the next two theorems, the sufficiency of the stated conditions follows from Theorems 2.1 and 2.2 respectively. Consequently, we verify only the necessity.

THEOREM 3.1. *A regular space X is a $w\Delta$-space if and only if there exist a regular M-space M, with normal $w\Delta$-sequence $\langle \mathscr{G}_n \rangle$, and an onto almost-open continuous wP-mapping relative to g, where $g(n, x) = \mathrm{st}(x, \mathscr{G}_n)$, from M to X.*

PROOF. (necessity) Let h be a $w\Delta$-function for X, and construct M, $\langle \mathscr{G}_n \rangle$, and f as in the Construction. It then remains to show that f is a wP-mapping relative to g. Assume that this is not the case. That is, suppose there is a sequence $\langle y_n \rangle$ with no cluster point and a point x so that for each n, we have $y_n \in f(\bigcup \{g(n, p): p \in f^{-1}(x)\})$, or equivalently, $y_n \in f(\mathrm{st}(f^{-1}(x), \mathscr{G}_n))$ for each n. Thus for each n, we get an $r_n \in f^{-1}(x)$, $s_n \in f^{-1}(y_n)$ and a $G \in \mathscr{G}_n$, with $\{r_n, s_n\} \subset G$. Now G can have one of two forms. It may be that $G = \{z\}$ for some $z \in M$. This can happen for only a finite number of n since it implies $r_n = s_n$, i.e., $x = y_n$. If this were the case for an infinite number of n, then x would obviously be a cluster point for $\langle y_n \rangle$.

Thus, for all but a finite number of n, we have $G = h(n, t_n)$ for some t_n. In each of these cases we get

$$f(\{r_n, s_n\}) = \{x, y_n\} \subset f(G) = f(h(n, t_n)) = h(n, f(t_n)),$$

since f is the identity as a function of sets, and by the way we constructed M. So we have for all but a finite number of n, $\{x, y_n\} \subset h(n, f(t_n))$. Since h is *a* $w\Delta$-function, $\langle y_n \rangle$ must have a cluster point, and we have a contradiction.

THEOREM 3.2. *A regular space X is a wM-space if and only if there exist a regular M-space M, with normal $w\Delta$-sequence $\langle \mathcal{G}_n \rangle$, and an onto almost-open continuous strong wP-mapping relative to g, where* $g(n, x) = \mathrm{st}(x, \mathcal{G}_n)$, *from M to X.*

PROOF. (necessity) Let h be a wM-function for X, and using h construct M, $\langle \mathcal{G}_n \rangle$, and f as in the construction. It then suffices to show that f is a strong wP-mapping relative to g. Assume not, so that there is a sequence $\langle y_n \rangle$ with no cluster point and a point $x \in X$ such that for each n, we have

$$f(\bigcup \{g(n, r) : r \in f^{-1}(x)\}) \cap f(\bigcup \{g(n, s) : s \in f^{-1}(y_n)\}) \neq \varnothing;$$

or equivalently that

$$f(\mathrm{st}(f^{-1}(x), \mathcal{G}_n)) \cap f(\mathrm{st}(f^{-1}(y_n), \mathcal{G}_n)) \neq \varnothing.$$

Then for each n, we have an $r_n \in f^{-1}(x)$, $s_n \in f^{-1}(y_n)$, $q_{n,1}$ and $q_{n,2}$ with $f(q_{n,1}) = f(q_{n,2})$, and sets R_n and S_n elements of $\mathcal{G}_n$, such that $\{r_n, q_{n,1}\} \subset R_n$ and $\{s_n, q_{n,2}\} \subset S_n$.

Now again there are two possibilities for the forms of R_n and S_n. Using the fact that f is a wP-mapping relative to g (from Theorem 3.1), we can show that there is an n_0 such that for $n \geqq n_0$, $R_n = h(n, t_n)$ and $S_n = h(n, u_n)$ for some t_n and u_n in M.

Hence for $n \geqq n_0$, we have $r_n \in h(n, t_n)$, $s_n \in h(n, u_n)$ and $f(h(n, t_n)) \cap f(h(n, u_n)) \neq \varnothing$, since $f(q_{n,1}) = f(q_{n,2})$ is in this intersection. Consequently, we get for $n \geqq n_0$ (applying f), $x \in h(n, f(t_n))$, $h(n, f(t_n)) \cap h(n, f(u_n)) \neq \varnothing$ and $y_n \in h(n, f(u_n))$. So, since h is a wM-function, $\langle y_n \rangle$ has a cluster point, which is a contradiction.

In the case of a wN-space, we can show that the relationship between space and mapping is more fundamental.

THEOREM 3.3. *Let Y be a wN-space with wN-function h, and suppose $f: X \to Y$ is a continuous surjection. Define* $g(n, x) = f^{-1}(h(n, f(x)))$. *Then f is a strong wP-mapping relative to g.*

PROOF. Suppose not. Then there is a sequence $\langle y_n \rangle$ with no cluster point and a point $y \in Y$ such that for each n, there exists a

$$p_n \in f(\bigcup \{g(n, x) : x \in f^{-1}(y)\}) \cap f(\bigcup \{g(n, s) : s \in f^{-1}(y_n)\}).$$

But by the definition of g, that means that $p_n \in h(n, y) \cap h(n, y_n)$ for each n. So $\langle y_n \rangle$ has a cluster point since h is a wN-function.

Combining Theorem 2.3 and Theorem 3.3, we get the following corollary.

COROLLARY 3.1. *Let $f: X \to Y$ be an almost-open continuous surjection. Then Y is a wN-space if and only if f is a strong wP-mapping relative to some COC-function for X.*

Our characterization of wN spaces then takes the form of the following corollary.

COROLLARY 3.2. *A regular space X is a wN-space if and only if there exist a regular M-space M, with COC-function g, and an onto almost-open continuous strong wP-mapping relative to g from M to X.*

PROOF. (sufficiency) Follows from Theorem 2.3. (necessity) Using the construction, we get a regular M-space, and an almost-open continuous surjection $f: M \to X$. Applying Theorem 3.3, we know that there is a COC-function g relative to which f is a strong wP-mapping.

We remark that using a direct proof similar to the proofs of Theorems 3.1 and 3.2, we can show that g may be taken to be a $w\Delta$-function for M. Hence Corollary 3.2 could be written accordingly.

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