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SOME TOTALLY REAL SUBMANIFOLDS IN A QUATERNION PROJECTIVE SPACE

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0. **Introduction.** Let HP^m be the (real) $4m$-dimensional quaternion projective space. On totally real submanifolds in HP^m, [1] has established some fundamental concepts and formulas. In this paper we employ some techniques developed in [2] and [4] and prove the following theorem.

THEOREM. *Let HP^m be the (real) $4m$-dimensional quaternion projective space of constant quaternion sectional curvature $c > 0$. Let N be an n-dimensional compact totally real minimal submanifold of HP^m. If the sectional curvature γ of N satisfies $\gamma \geqq (n-1)c/4(2n-1)$, then either N is totally geodesic in HP^m or $n = 2$, $m \geqq 4$ and N is the Veronese surface in HP^m with positive constant curvature $c/12$.*

1. **Preliminaries.** Let HP^m be a quaternion projective space with real dimension 4m. On HP^m there exists a 3-dimensional vector space V of tensors of type (1.1) with local basis of almost Hermitian structure I, J, K such that

$$\text{(a)} \quad IJ = -JI = K,\ JK = -KJ = I,\ KI = -IK = J,$$
$$I^2 = J^2 = K^2 = -1;$$
$$\text{(b)} \quad \tilde{\nabla}_x I = r(x)J - q(x)K,\ \tilde{\nabla}_x J = -r(x)I + p(x)K,$$
$$\tilde{\nabla}_x K = q(x)I - p(x)J$$

for some functions $p(x)$, $q(x)$, $r(x)$ on HP^m, where $\tilde{\nabla}$ is the connection on HP^m.

Let X be a unit vector on HP^m. Then X, IX, JX and KX form an orthonormal frame. Let $Q(X)$ be the 4 plane spanned by them. For X, Y on HP^m, if $Q(X)$ and $Q(Y)$ are orthogonal, the plane $\pi(X, Y)$ spanned by X and Y is called a *totally real plane*. Any 2-plane in some $Q(X)$ is called a *quaternion plane*. The sectional curvature of a quaternion plane π is called the *quaternion sectional curvature* of π. The quaternion sectional curvature of HP^m is a constant $c > 0$. HP^m is thus called a *quaternion-space-form*.

Let g be the Riemann metric on HP^m. Then the curvature tensor $\tilde{R}$ of HP^m is given by [3].

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$$(1.1)\quad \tilde{R}(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(IY, Z)IX - g(IX, Z)IY + 2g(X, IY)IZ + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ + g(KY, Z)KX - g(KX, Z)KY + 2g(X, KY)KZ\}$$

Let N be an n-dimensional Riemannian manifold isometrically immersed in HP^m. We call N a *totally real submanifold* of HP^m if each tangent 2-plane of N is mapped by the immersion onto a totally real plane in HP^m.

Let ∇ be the Riemannian connection on N, σ be the second fundamental form of the immersion. $\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$ for $X, Y \in TN$, the tangent space of N. For a normal vector ξ on N, $\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$, where $-A_\xi X$ and $D_X \xi$ are tangential and normal components of $\tilde{\nabla}_X \xi$.

The *mean curvature vector* H is defined by $H =$ trace σ/n. N is *minimal* if $H = 0$. We define $\bar{\nabla}\sigma$ by

$$(\bar{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z),\ X, Y, Z \in TN.$$

Let R be the curvature tensor of N. Then the equation of Gauss is

$$(1.2)\quad g(R(X, Y)Z, W) = g(\tilde{R}(X, Y)Z, W) + g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W)),\ X, Y, Z, W \in TN.$$

Assume that N is a totally real submanifold of M. Then for any orthogonal vectors X, Y in TN, $Q(X) \perp Q(Y)$. We thus have $g(X, \varphi Y) = g(\psi X, Y) = 0$ for φ, ψ be I, J or K. (1.1) and (1.2) reduce to

$$(1.1)'\quad \tilde{R}(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y\},\ X, Y, Z \in TN,$$

and

$$(1.2)'\quad g(R(X, Y)Z, W) = \frac{c}{4}\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} + g(\sigma(Y, Z), \sigma(X, W)) - g(\sigma(X, Z), \sigma(Y, W)).$$

Since N is totally real, if dim $N = n$, then $n \leqq m$. Let $p = 4m - n$. We choose a local field of orthonormal frames

$$e_1, \dots, e_n, e_{n+1}, \dots, e_m;\ e_{\varphi(1)} = \varphi e_1, \dots, e_{\varphi(n)} = \varphi e_n, \dots, e_{\varphi(m)} = \varphi e_m;\quad \varphi = I, J \text{ or } K.$$

The following range of indices are to be used with φ running through I, J and K.

$$A, B, C, \dots = 1, \dots, m, \varphi(1), \dots, \varphi(m); \qquad i, j, k, \dots = 1, 2, \dots, n;$$
$$\alpha, \beta, \gamma, \dots = n+1, \dots, m, \varphi(1), \dots, \varphi(m); \qquad \lambda, \mu, \nu, \dots = n+1, \dots, m.$$

With respect to this frame field let the dual field be

$$\omega^1, \dots, \omega^n, \omega^{n+1}, \dots, \omega^m, \omega^{\varphi(1)}, \dots, \omega^{\varphi(m)}, \varphi = I, J \text{ or } K.$$

Using $g(\varphi e_i, \psi e_j) = 0$ $(i \neq j)$ and condition (*b*) we have for $\varphi = I, J$ or K,

$$\omega^i_j = \omega^{\varphi(i)}_{\varphi(j)}, \ \omega^{\varphi(i)}_j = \omega^{\varphi(j)}_i, \ \omega^\lambda_\mu = \omega^{\varphi(\lambda)}_{\varphi(\mu)},$$
$$\omega^{\varphi(\lambda)}_\mu = \omega^{\varphi(\mu)}_\lambda, \ \omega^i_\lambda = \omega^{\varphi(i)}_{\varphi(\lambda)}, \ \omega^{\varphi(i)}_\lambda = \omega^{\varphi(\lambda)}_i.$$

If we write $\omega^\alpha_i = \sum h^\alpha_{ij}\omega^j$, then we have

$$h^\alpha_{ij} = h^\alpha_{ji}, \ h^\alpha_{ij} = g(A_\alpha e_i, e_j), \ h^{\varphi(i)}_{jk} = h^{\varphi(j)}_{ik} = h^{\varphi(k)}_{ij}, \tag{1.3}$$

where $A_\alpha = A_{e_\alpha}$. By the equation (1.2)′ the sectional curvature $K(X, Y)$ of N for a plane determined by orthonormal vectors X, Y is given by

$$K(X, Y) = g(R(X, Y)Y, X) = \frac{c}{4} + \sum \{g(A_\alpha X, X)g(A_\alpha Y, Y) - g(A_\alpha X, Y)^2\}.$$

As an immediate consequence of this relation we have the following characterization of totally real, totally geodesic minimal submanifolds.

PROPOSITION. *Let N be an n-dimensional totally real minimal submanifold in HP^m. Then N is totally geodesic if and only if N is of constant curvature $K = c/4$.*

2. **Proof of the theorem.** In **[2]** the Laplacian of $\|\sigma\|^2$ was calculated for a minimal submanifold in a locally symmetric manifold, i.e., the following formula holds:

$$\frac{1}{2}\Delta\|\sigma\|^2 = \|\bar{\nabla}\sigma\|^2 + \sum \operatorname{tr}(A_\alpha A_\beta - A_\beta A_\alpha)^2 - \sum(\operatorname{Tr} A_\alpha A_\beta)^2 + \sum(4\tilde{R}_{\alpha\beta ij}h^\alpha_{jk}h^\beta_{ik} - \tilde{R}_{\alpha k\beta k}h^\alpha_{ij}h^\beta_{ij} + 2\tilde{R}_{ijkj}h^\alpha_{il}h^\alpha_{kl} + 2\tilde{R}_{ijkl}h^\alpha_{il}h^\alpha_{jk}). \tag{2.1}$$

If N is a totally real, minimal submanifold of HP^m, the right side of (2.1) becomes (see **[1]**)

$$\frac{1}{2}\Delta\|\sigma\|^2 = \|\bar{\nabla}\sigma\|^2 + \frac{1-a}{2}\sum \operatorname{tr}(A_\alpha A_\beta - A_\beta A_\alpha)^2 + a\sum(\operatorname{tr} A_\alpha A_\beta)^2 - \frac{nac}{4}\|\sigma\|^2 + \frac{c}{4}\sum_{\varphi,i}\operatorname{tr} A^2_{\varphi(i)} + (1+a)\sum(R_{ijkj}h^\alpha_{il}h^\alpha_{kl} + R_{ijkl}h^\alpha_{il}h^\alpha_{jk}), \tag{2.2}$$

where a may be any real number.

Let $\alpha_1, \dots, \alpha_n$ be the eigenvalues of A_α. Then we have

$$\sum_{i,j,k,l}(R_{ijkj}h^\alpha_{il}h^\alpha_{kl} + R_{ijkl}h^\alpha_{il}h^\alpha_{jk}) = \frac{1}{2}\sum_{i,k}(\alpha_i - \alpha_k)^2 R_{ikik}.$$

Now we assume that the sectional curvature of N is greater than or equal to γ. Then

$$\sum_{i,j,k,l}(R_{ijkj}h^\alpha_{il}h^\alpha_{kl} + R_{ijkl}h^\alpha_{il}h^\alpha_{jk}) \geqq \frac{1}{2}\sum_{i,k}(\alpha_i - \alpha_k)^2\gamma. \tag{2.3}$$

Since N is minimal $\sum_i \alpha_i = 0$. Therefore

$$\sum_{i,k}(\alpha_i - \alpha_k)^2 = 2n \operatorname{tr} A^2_\alpha.$$

Let us take a so that $a \geqq -1$. Then (2.2) yields

$$\frac{1}{2}\Delta\|\sigma\|^2 \geqq \|\bar\nabla\sigma\|^2 + \frac{1-a}{2}\sum \operatorname{tr}(A_\alpha A_\beta - A_\beta A_\alpha)^2 + a\sum(\operatorname{tr}A_\alpha A_\beta)^2$$
$$- \frac{nac}{4}\|\sigma\|^2 + \frac{c}{4}\sum \operatorname{tr} A^2_{\varphi(i)} + (1 + a)n\gamma\|\sigma\|^2. \tag{2.4}$$

Let $S_{\alpha\beta} = \operatorname{tr}(A_\alpha A_\beta) = \sum h^\alpha_{ij}h^\beta_{ji}$. Then $(S_{\alpha\beta})$ is a symmetric $p \times p$ matrix and it can be diagonalized for a suitable choice of $\{e_\alpha\}$. Thus we may assume that $\operatorname{tr} A_\alpha A_\beta = 0$ for $\alpha \neq \beta$. In [2] there is an algebraic lemma which proved that

$$\operatorname{tr}(A_\alpha A_\beta - A_\beta A_\alpha)^2 \geqq -2(\operatorname{tr} A^2_\alpha)(tr\ A^2_\beta)$$

and the equality holds for nonzero matrices A_α and A_β if and only if A_α and A_β can be transformed by an orthogonal matrix simultaneously into scalar multiples of $\bar A$ and $\bar B$ respectively where

$$\bar A = \begin{bmatrix} \begin{matrix}0 & 1\\ 1 & 0\end{matrix} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar B = \begin{bmatrix} \begin{matrix}1 & 0\\ 0 & -1\end{matrix} & 0 \\ 0 & 0 \end{bmatrix}.$$

Moreover, if A_1, A_2, A_3 are symmetric $n \times n$ matrices such that

$$-\operatorname{tr}(A_a A_b - A_b A_a)^2 = 2(\operatorname{tr} A^2_a)(tr\ A^2_b),\ 1 \leqq a, b \leqq 3,\ a \neq b,$$

then at least one of the matrices must be zero.

Now from (2.4) we have

$$\frac{1}{2}\Delta\|\sigma\|^2 \geqq (a - 1)\sum_{\alpha\neq\beta}(\operatorname{tr} A^2_\alpha)(\operatorname{tr} A^2_\beta) + a\sum(\operatorname{tr} A^2_\alpha)^2$$
$$-\frac{nac}{4}\|\sigma\|^2 + \frac{c}{4}\sum \operatorname{tr} A^2_{\varphi(i)} + (1 + a)n\gamma\|\sigma\|^2, \tag{2.5}$$
$$\text{for } -1 \leqq a \leqq 1.$$

Since

$$\sum_{\alpha\neq\beta}(\mathrm{tr}\,A_\alpha^2)(\mathrm{tr}\,A_\beta^2)+\sum_\alpha(\mathrm{tr}\,A_\alpha^2)^2=(\sum_\alpha\mathrm{tr}\,A_\alpha^2)^2=\|\sigma\|^4$$

and

$$\sum(\mathrm{tr}\,A_\alpha^2)^2\geqq\|\sigma\|^4/n,$$

by a straightforward calculation (2.5) yields

$$\frac{1}{2}\Delta\|\sigma\|^2\geqq\left\{\frac{1}{n}-(1-a)\right\}\|\sigma\|^4+\left\{(1+a)n\gamma-\frac{nac}{4}\right\}\|\sigma\|^2 + \frac{c}{4}\sum\mathrm{tr}\,A^2_{\varphi(i)}.$$

In particular putting $a = 1 - 1/n$ we obtain

$$\frac{1}{2}\Delta\|\sigma\|^2\geqq\left\{(2n-1)\gamma-\frac{n-1}{4}c\right\}\|\sigma\|^2+\frac{c}{4}\sum\mathrm{tr}\,A^2_{\varphi(i)}. \tag{2.6}$$

Now we assume that the sectional curvature γ of N satisfies $\gamma \geqq (n-1)\,c/4(2n-1)$. The right-hand side of (2.6) is non-negative. Thus by use of Hopf's lemma we obtain $\Delta\|\sigma\|^2 = 0$, and $\sum\mathrm{tr}\,A^2_{\varphi(i)} = 0$. All the inequality signs in this section turn into equalities. In particular we have

$$-\mathrm{tr}(A_\alpha A_\beta - A_\beta A_\alpha)^2 = 2(\mathrm{tr}\,A_\alpha^2)(\mathrm{tr}\,A_\beta^2),\ \alpha\neq\beta.$$

Thus at most two of the A_α's are non-zero. Without loss of generality we may assume that

$$A_{n+1}=a\begin{pmatrix}0&1&\\1&0&0\\0&&0\end{pmatrix},\quad A_{n+2}=b\begin{pmatrix}1&0&\\0&-1&0\\0&&0\end{pmatrix}, \tag{2.7}$$
$$A_\alpha = 0 \text{ for } \alpha\neq n+1,\,n+2.$$

If N is not totally geodesic in HP^m, then $\|\sigma\|\neq 0$ and $\gamma = (n-1)c/4(2n-1)$. We are going to claim that $n = 2$. Assume that $n > 2$. Then for an $i > 2$, by use of (1.2)′ and (2.7), we have $K(\pi(e_1, e_i)) = R_{1i1i} = c/4$. Since N is not totally geodesic, we may assume that $A_{n+2}\neq 0$. For $A_\alpha = A_{n+2}$, we have $\alpha_1 = b$, $\alpha_2 = -b$, $\alpha_i = 0$ for $i > 2$. Thus $(\alpha_1-\alpha_i)^2\neq 0$ for $i > 2$. Therefore from the equality of (2.3) we find

$$R_{1i1i}=\gamma=(n-1)c/4(2n-1)<c/4.$$

This is a contradiction. Hence $n = 2$.

Since $\sum\mathrm{tr}\,A^2_{\varphi(i)} = 0$, we have $m > 2$ and $K = c/12$.
By (2.7) we have

$$\omega_1^{n+1}=a\omega^2,\ \omega_2^{n+1}=a\omega^1,\ \omega_1^{n+2}=b\omega^1,\ \omega_2^{n+2}$$
$$=-b\omega^2,\ \omega_i^\alpha=0,\ \alpha=3,\ldots,4m,\ i=1,2.$$

Since all the inequalities become equalities in this section when $K = c/12, n = 2$, we have

$$\|\bar{\nabla}\sigma\|^2 = \sum (h^\alpha_{ijk})^2 = 0$$

where h^α_{ijk} are given by

$$\sum h^\alpha_{ijk}\,\omega^k = dh^\alpha_{ij} - \sum h^\alpha_{ik}\,\omega^k_j - \sum h^\alpha_{kj}\,\omega^k_i + \sum h^\beta_{ij}\,\omega^\alpha_\beta.$$

$h^\alpha_{ijk} = 0$ yields

$$dh^\alpha_{ij} = \sum h^\alpha_{ik}\,\omega^k_j + \sum h^\alpha_{kj}\,\omega^k_i - \sum h^\beta_{ij}\,\omega^\alpha_\beta. \tag{2.8}$$

In (2.8) setting $\alpha = 3, i = 1$ and $j = 2$, we have $a =$ const., setting $\alpha = 4$, $i = j = 1$, we have $b =$ const. Setting $\alpha = 3, i = j = 1$ in (2.8), we have $\omega^3_4 = (-2a/b)\omega^1_2$. Setting $\alpha \geqq 5$, $i = 1$, $j = 2$ in (2.8), we obtain $\omega^\alpha_3 = 0, \alpha \geqq 5$. Setting $\alpha \geqq 5, i = j = 1$ in (2.8), we obtain $\omega^\alpha_4 = 0, \alpha \geqq 4$.

Since $\sum (\operatorname{tr} A^2_\alpha)^2 = \|\sigma\|^4/2$, we have $a^2 = b^2 = c/12$. Replacing e_3 by $-e_3$ and e_4 by $-e_4$, if necessary, we may assume that $-a = b = \sqrt{c}\,/\,2\sqrt{3}$. The connection form (ω^B_A) of HP^m restricted to N is given by

$$\begin{pmatrix} 0 & \omega^1_2 & b\omega^2 & -b\omega^1 & 0 \cdots 0 \\ \omega^2_1 & 0 & b\omega^1 & b\omega^2 & 0 \cdots 0 \\ -b\omega^2 & -b\omega^1 & 0 & 2\omega^1_2 & 0 \cdots 0 \\ b\omega^1 & -b\omega^2 & -2\omega^1_2 & 0 & 0 \cdots 0 \\ 0 & \cdots & 0 & & 0 \end{pmatrix}, \quad b = \sqrt{c}/2\sqrt{3}. \tag{2.9}$$

From (2.9) we conclude that $m \geqq 4$. Otherwise, $m = 3$ implies from (1.3) that

$$-b\omega^2 = \omega^4_2 = \omega^{3+1}_2 = \omega^{3+2}_1 = \omega^5_1 = 0,$$

which is not true.

The square length of the second fundamental from of N in HP^4 is

$$\|\sigma\|^2 = 2(a^2 + b^2) = c/3.$$

On the other hand, the real 4-dimensional projective space RP^4 with constant curvature $c/4$ is canonically immersed in HP^4 and, further, in HP^m as a totally real, totally geodesic submanifold. In **[2]** it was proved that the Veronese surface is the only compact minimal immersion in RP^4 (and further canonically in HP^4 and in HP^m) with $\|\sigma\|^2 = c/3$. This immersion of the Veronese surface in HP^m has the connection form (2.9) which was proved in **[2]**. Hence our N is locally a Veronese surface.

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