ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 10, Number 3, Summer 1980

CONVOLUTIONS AND GROWTH NUMBERS OF ANALYTIC FUNCTIONS

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ABSTRACT. The convolution of two analytic functions $f(z) = \sum_0^\infty a_n z^n$, $|z| < R$, $g(z) = \sum_0^\infty b_n z^n$, $|z| < S$ is defined as $(f*g)(z) = \sum_0^\infty a_n b_n z^n$, $|z| < R^*$. The aim of the paper is to establish a relationship between the growth numbers of f, g and $f*g$. The growth number $\rho(f)$ of an analytic function is defined as lim sup $(\log^+ \log^+ M(r,f)/\log(R/(R-r))$, as $r \to R^-$, where $M(r,f)$ is the maximum modulus function associated with f.

1. **Introduction.** Throughout the paper $A(R)$ will denote the class of functions $f = f(z) = \sum_{n=0}^\infty a_n z^n$ analytic in the disc $|z| < R$, where $R^{-1} = \lim\sup_{n\to\infty} |a_n|^{1/n}$. Also, we shall assume that $0 < R < \infty$ and $\sup_n(|a_n| R^n) = \infty$.

Let $g \in A(S)$ and $g = g(z) = \sum_{n=0}^\infty b_n z^n$. The convolution or Hadamard product of f and g is defined by the power series $(f*g)(z) = \sum_{n=0}^\infty a_n b^n z^n$. This new function is clearly analytic on the disc $|z| < R^*$ for some $R^* \geqq 0$. It is easy to show that $R^* \geqq RS$ and the product is commutative.

The measure of growth for any $f \in A(R)$ is indicated by the real number $\rho(f)$, $0 \leqq \rho(f) \leqq \infty$, which is determined as follows.

$$(*) \qquad \rho(f) = \lim\sup_{r\to R} (\log^+ \log^+ M(r,f)/\log(R/(R-r)),$$

where $\log^+ x = \max(\log x, 0)$ and $M(r,f) = \max_\phi |f(re^{i\phi})|$, the maximum modulus of f. The real number $\rho(f)$, in analogy to entire functions, may be called the "order of f". But we prefer to call it the "growth number of f". Likewise, we define the lower growth number $\lambda(f)$ of f by (*) with limit inferior in place of limit superior.

The object of this note is to relate the growth number of a convolution with the growth numbers of its component functions. To achieve this goal we will establish some allied results which are interesting and are used in proofs of main results.

We shall use the following definitions and notations. We define the functions:

Received by the editors on February 7, 1978, and in revised form on November 6, 1978.

AMS(MOS) subject classifications (1970). Primary 30A10, 30A64.

Key words and phrases: Hadamard product, Convolutions, Growth, orders.

$$u(r, f) = \max_n \{|a_n| r^n\}$$
$$c(r, f) = \max\{n: |a_n| r^n = u(r, f)\}$$

for any r, $0 < r < R$; and the sequences:

$$\{A_n(f)\}_{n=2}^{\infty},\ A_n(f) = \frac{\log^+ \log^+(|a_n| R^n)}{\log n}$$

$$\{B_n(f)\}_{n=2}^{\infty},\ B_n(f) = \frac{\log^+ \log^+ (|a_n| R^n)}{\log n - \log^+ \log^+(|a_n| R^n)}.$$

The function $u(r, f)$ is called the maximum term of f and $c(r, f)$ is called the central index of $u(r, f)$. When there is no chance of confusion, we abbreviate the notations as follows: $\rho(f) = \rho$, $\lambda(f) = \lambda$, $M(r, f) = M(r)$, $u(r, f) = u(r)$, $c(r, f) = c(r)$, $A_n(f) = A_n$ and $B_n(f) = B_n$. The prime (′) on a function will denote its derivative.

2. **Main Results.** Our first result investigates the growth numbers of the derivative of a convolution and the convolution of the derivatives of f and g. This result is without any restrictions as to what classes f and g belong. Again, we assume that the radii of discs of convergence of power series are positive and finite.

THEOREM 1. *Let $f \in A(R)$, $g \in A(S)$, and let $f * g \in A(R^*)$, $R^* \geqq RS$, with growth number ρ^* and lower growth number λ^*. Then both $(f * g)'$ and $(f' * g')$ are in $A(R^*)$ and have growth number ρ^* and lower growth number λ^*.*

Our next theorem gives the relationship between the growth numbers of the convolution and its component functions in a certain subclass.

THEOREM 2. *Let $f \in A(R)$, $g \in A(S)$ with growth numbers, respectively, ρ and ρ'. Let $f * g \in A(R^*)$ such that $R^* = RS$ and be of growth number ρ^*. Then*

$$\rho^*/(\rho^* + 1) \leqq \rho/(\rho + 1) + \rho'/(\rho' + 1).$$

The corresponding result for lower growth number does not hold. However, with more restrictive hypothesis we have the following theorem.

THEOREM 3. *Let f, g, and $f * g$ be as specified in Theorem 2 with lower growth numbers λ, λ' and λ^* respectively. Further, if for $n \geqq 2$ either*

(1) $$\liminf_{n\to\infty} A_n (f * g) = \lambda^*/(\lambda^* + 1)$$

or

(2) $$\liminf_{n\to\infty} B_n (f * g) = \lambda^*,$$

then we have the relation

$$\lambda^*/(\lambda^* + 1) \leqq \begin{cases} \lambda/(\lambda + 1) + \rho'/(\rho' + 1) \\ \lambda'/(\lambda' + 1) + \rho/(\rho + 1). \end{cases}$$

The following theorem indicates a relationship between growth numbers and the maximum terms.

THEOREM 4. *Let* $f * g \in A(R^*)$ *such that* $\sup_n\{|a_n b_n (R^*)^n|\} = \infty$ *and with growth numbers* ρ *and* λ *respectively. Then*

$$(3) \qquad 2(\rho + 1) = \limsup_{r \to R^*} \frac{\log(u(r, f' * g')/u(r, f * g)}{\log (R^*/(R^* - r))}$$

and

$$(4) \qquad 2(\lambda + 1) \geqq \liminf_{r \to R^*} \frac{\log(u(r, f' * g')/u(r, f * g))}{\log(R^*/(R^* - r))}.$$

COROLLARY. *Let* $f * g$ *be as specified in Theorem* 4. *Then, if* $\lambda < \infty$, *we have*

$$(5) \qquad 2(\rho + 1) - \lambda \geqq \limsup_{r \to R^*} \frac{\log(u(r, f' * g')/(u(r, f * g)\log u(r, f * g)))}{\log(R^*/(R^* - r))}$$

3. **Lemmas and Auxiliary Results.** Prior to adding comments on the main results we collect here some results which are needed to prove the proposed results and to quote during the commentary.

LEMMA 1. *If* $f \in A(R)$, *then* $\rho(f) = \rho(f')$ *and* $\lambda(f) = \lambda(f')$.

The proof of this lemma follows the pattern of the proof of a similar result for entire functions [**2**, p. 265] with proper modifications for the finite open disc.

We remark that in view of the Fundamental Theorem of Calculus one can immediately conclude from Lemma 1 that if $F(z) = \int_0^z f(w)dw$, $|z| < R$, then $F \in A(R)$ and $\rho(F) = \rho(f)$, $\lambda(F) = \lambda(f)$. The next two lemmas give interesting relationships between the maximum modulus, the maximum term and the central index of a convolution and that of its component functions.

LEMMA 2. *If* $f * g \in A(R^*)$, *then*

$$\begin{aligned}[M(r, f * g) - |(f * g)(0)|] &\leqq rM(r, f' * g') \\ &\leqq r(r' + r)M(r'' + r' + r, f * g)/(r' r'')\end{aligned}$$

for all r, r' and r'' *such that* $0 < r < r' + r < r'' + r' + r < R^*$.

PROOF. Since

$$(f' * g')(z) = \frac{d}{dz}(z(f * g)'(z)),$$

$$z(f*g)'(z) = \int_0^z (f'*g')(t)dt,$$

where the integral is taken along the straight line joining 0 to z, $0 < |z| < R^*$. Thus

$$M(r, f*g)') \leqq M(r, f'*g').$$

Similarly,

$$(f*g)(z) = \int_0^z (f*g)'(t)dt + (f*g)(0),$$

so

$$M(r, f*g) \leqq rM(r, (f*g)') + |(f*g)(0)|.$$

Therefore

$$[M(r, f*g) - |(f*g)(0)|] \leqq rM(r, f'*g'). \tag{6}$$

Using Cauchy's Integral Theorem for $|z| = r < R^*$,

$$(2\pi i)(f'*g')(z) = \int_c t(f*g)'(t)/[t - z]^2 dt,$$

where c is the circle $\{w: |w - z| = r'\}$ such that $r < r' + r < R^*$. We then have for such r and r'

$$M(r, f'*g') \leqq (r' + r)M(r' + r, (f*g)')/r'.$$

Similarly, for $|z| = r' + r$

$$(2\pi i)(f*g)'(z) = \int_D (f*g)(t)/[t - z]^2 dt,$$

where D is the circle $\{w: |w - z| = r'' > 0\}$ such that $r'' + r' + r < R^*$. Thus,

$$M(r' + r, (f*g)') \leqq M(r'' + r' + r, f*g)/r'',$$

so

$$M(r, f'*g') \leqq (r' + r)M(r'' + r' + r, f*g)/(r'r''). \tag{7}$$

Now the lemma follows from relations (6) and (7).

LEMMA 3. *If* $f*g \in A(R^*)$, *then*

$$(c(r, f*g))^2 \leqq (r \cdot u(r, f'*g'))/u(r, f*g) < (c(r, f'*g') + 1)^2$$

for $0 < r < R^*$.

PROOF. Let $f \in A(R)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let $g \in A(S)$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then the convolutions $(f*g) \in A(R^*)$ and $(f'*g') \in A(R^*)$ for some R^* such that $R^* \geqq RS$; and we have

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n,$$

and

$$(f' * g')(z) = \sum_{n=0}^{\infty} n^2 a_n b_n z^{n-1}.$$

Now for any r, $0 < r < R^*$, let us denote by N_0 the central index $c(r, f * g)$ of the maximum term $u(r, f*g)$. Likewise let $N_1 = c(r, f'*g')$, as the central index of the maximum term $u(r, f' * g')$. Clearly both N_0 and N_1 are non-negative integers. Then

$$\begin{aligned} u(r, f' * g') &= N_1^2 |a_{N_1} b_{N_1}| r^{N_1 - 1} \\ &\geqq N_0^2 a_{N_0} b_{N_0} r^{N_0 - 1} \\ &= N_0^2 u(r, f * g)/r, \end{aligned}$$

since $u(r, f' * g')$ is the maximum term of the series $\sum_{n=0}^{\infty} n^2 a_n b_n z^{n-1}$. Therefore

$$(r \cdot u(r, f' * g'))/u(r, f * g) \geqq (c(r, f * g))^2.$$

On the other hand, we have

$$\begin{aligned} u(r, f' * g') &= N_1^2 |a_{N_1} b_{N_1}| r^{N_1 - 1} \\ &\leqq N_1^2 |a_{N_0} b_{N_0}| r^{N_0}/r \\ &= N_1^2 u(r, f * g)/r \end{aligned}$$

since $u(r, f * g)$ is the maximum term of the series $\sum_{n=0}^{\infty} a_n b_n z^n$. Therefore

$$\begin{aligned} (r \cdot u(r, f' * g'))/u(r, f * g) &\leqq (c(r, f' * g'))^2 \\ &< (c(r, f' * g') + 1)^2. \end{aligned}$$

This proves the lemma completely.

The follwoing theorem appear in **[1]**.

THEOREM A. *Let $f \in A(R)$ with growth number ρ and lower growth number λ.*

Then

$$\limsup_{n\to\infty} A_n(f) = \rho/(\rho + 1) \tag{8}$$

$$\limsup_{n\to\infty} B_n(f) = \rho \tag{9}$$

$$\liminf_{n\to\infty} A_n(f) \leqq \lambda/(\lambda + 1) \tag{10}$$

$$\liminf_{n\to\infty} B_n(f) \leqq \lambda. \tag{11}$$

Equality holds in (10) *if and only if equality holds in* (11).

THEOREM *B*. *Let $f \in A(R)$ with growth number ρ and lower growth number λ be such that* $\sup_n\{|a_n R^n|\} = \infty$. *Then*

$$\left.\begin{matrix}\rho \\ \lambda\end{matrix}\right\} = \lim_{r\to R}\left\{\begin{matrix}\sup \\ \inf\end{matrix}\right\}\frac{\log\log u(r,f)}{\log(R/(R-r))} \tag{12}$$

$$\left.\begin{matrix}1+\rho \\ 1+\lambda\end{matrix}\right\}\begin{matrix}= \\ \geqq\end{matrix} \lim_{r\to R}\left\{\begin{matrix}\sup \\ \inf\end{matrix}\right\}\frac{\log c(r,f)}{\log(R/(R-r))}.$$

4. **Comments.** It should be observed that the denominator in the sequence $\{B_n(f*g)\}$ which appears in (2) is positive for all large n because of the assumption that $0 < R^* < \infty$. Also, it is not difficult to show that the relations (1) and (2) are equivalent. Further, it should also be observed that the left hand sides, both in (1) and (2), could be strictly less than their respective right hand sides. For example, consider the analytic functions

$$f(z) = \sum_{n=0}^{\infty} z^n = (1-z)^{-1}, \quad |z| < 1.$$

$$g(z) = \sum_{n=1}^{\infty} e^n z^{n^2}, \quad |z| < 1.$$

Then $f \in A(1)$, $g \in A(1)$ and their convolution $f * g \equiv g \in A(1)$. Clearly $\rho(f) = \lambda(f) = 0$, and from the definition of growth numbers $\rho(g) = 1 = \lambda(g)$. However, using Theorem B it would be easier to show that $\rho(g) = 1 = \lambda(g)$ since as r tends to R,

$$\log(R/(R-r)) \sim -\log\log(R/r),$$

and for each positive r satisfying

$$\exp(-1)/(2n-1)) \leqq r < \exp(-1/(2n+1)),$$

we have $u(r, g) = e^n r_{\text{I}}^{n^2}$ and $c(r, f) = n^2$. But the left hand side of (1) is equal to zero, which is less than the right hand side, whose value is equal to half. The assertion about (2) follows in view of Theorem A.

Finally, we remark that the assertions of Theorem 2 and Theorem 3 are precise. The functions discussed in the above paragraph show that the equality could hold in both theorems. To show that strict inequality can hold in Theorems 2 and 3, consider the convolution of the function g as described above and of the analytic function h defined by

$$h(z) = \sum_{n=0}^{\infty} (e^n z^{n^2+1} + e^{-n} z^{n^2}), \; |z| < 1.$$

A similar argument as for g proves that $h \in A(1)$ and, by Theorem A, $\rho(h) = 1$. However,

$$(g * h)(z) = ez + \sum_{n=1}^{\infty} z^{n^2}$$

is in $A(1)$ and, by Theorem A, $\rho(g * h) = 0 = \lambda(g * h)$. Therefore, strict inequality can hold both in Theorem 2 and Theorem 3 respectively.

5. **Proofs.**

PROOF OF THEOREM 1. Clearly $(f * g)'$ is in $A(R^*)$ and, since

$$(f' * g')(z) = \sum_{n=0}^{\infty} n^2 a_n b_n z^{n-1},$$

so $f' * g'$ is also in $A(R^*)$. By Lemma 1, $\rho^*((f*g)') = \rho^*(f*g)$ and $\lambda^*((f*g)') = \lambda^*(f * g)$. By Theorem A

$$\rho^*/(\rho^* + 1) = \limsup_{n\to\infty} \frac{\log^+ \log^+|a_n b_n (R^*)^n|}{\log n}.$$

Let $\rho' = \rho(f' * g')$, then by Theorem A

$$\begin{aligned}\rho'/(\rho' + 1) &= \limsup_{n\to\infty} \frac{\log^+ \log^+|n^2 a_n b_n (R^*)^{n-1}|}{\log(n-1)} \\ &\leqq \limsup_{n\to\infty} \frac{\log^+(\log^+ n^2 + \log^+|a_n b_n (R^*)^{n-1}|)}{\log(n-1)} \\ &\leqq \limsup_{n\to\infty} \frac{\log\log n^2}{\log(n-1)} \\ &\quad + \limsup_{n\to\infty} \frac{\log\log(|a_n b_n (R^*)^{n-1}| + O(1))}{\log(n-1)} \\ &= \limsup_{n\to\infty} \frac{\log\log(|a_n b_n (R^*)^{n-1}| + O(1))}{\log(n-1)} \\ &= c\end{aligned}$$

for some constant c. If the sequence $\{|a_n b_n (R^*)^n|\}_{n=0}^{\infty}$ is bounded, then the sequence $\{|a_n b_n R^{*n-1}|\}_{n=1}^{\infty}$ is also bounded and $c = 0 = \rho^*/(\rho^* + 1)$. Therefore $\rho^* = \rho' = 0$. If the sequence $\{|a_n b_n (R^*)^n|\}_{n=0}^{\infty}$ is unbounded, then there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ of positive integers such that

$$\lim_{j\to\infty} |a_{n_j} b_{n_j} (R^*)^{n_j}| = \lim_{j\to\infty} |a_{n_j} b_{n_j} (R^*)^{n_j - 1}| = \infty.$$

Therefore,

$$\rho^*/(\rho^* + 1) = c = \limsup_{n\to\infty} \frac{\log^+ \log^+|a_n b_n (R^*)^n|}{\log n}$$

as R^* is positive and as $n \to \infty$, $\log(n-1) \sim \log n$. Therefore, $\rho'(\rho' + 1) \leqq \rho^*/(\rho^* + 1)$. On the other hand $\rho^*/(\rho^* + 1) \leqq \rho'/(\rho' + 1)$ as

$$\log^+\log^+|a_nb_n(R^*)^n| \leqq \log^+\log^+|n^2a_nb_n(R^*)^{n-1}|$$

for all integers $n \geqq (R^*)^{1/2}$, so $\rho'/(\rho'+1) = \rho^*/(\rho^*+1)$ and $\rho^* = \rho'$.

To prove the assertion for the lower growth number we proceed as follows. Choose r, $0 < r < R^*$, $r' = (R^*-r)/2$ and $r'' = (R^*-(r'+r))/2 = (R^*-r)/4$. From Lemma 2 with r, r' and r'' as prescribed above, we get

$$\begin{aligned}[M(r, f*g) - |(f*g)(0)|]/r &\leqq M(r, f'*g') \\ &\leqq 8R^*\cdot M((3R^*+r)/4, f*g)/(R^*-r)^2.\end{aligned}$$

Now using arguments similar to that in the proof of Lemma 1 [**2**, p. 265], we get $\lambda(f*g) = \lambda(f'*g')$. This proves the theorem.

PROOF OF THEOREM 2. By Theorem A

$$\begin{aligned}\rho^*/(\rho^*+1) &= \limsup_{n\to\infty}\frac{\log^+\log^+|a_nR^nb_nS^n|}{\log n} \\ &\leqq \limsup_{n\to\infty}\frac{\log^+(\log^+|a_nR^n| + \log^+|b_nS^n|)}{\log n} \\ &\leqq \limsup_{n\to\infty}\frac{\log[\log(|a_nR^n| + O(1)) + \log(|b_nS^n| + O(1))]}{\log n} \\ &\leqq \limsup_{n\to\infty}\frac{\log\log(|a_nR^n| + O(1))}{\log n} \\ &+ \limsup_{n\to\infty}\frac{\log\log(|b_nS^n| + O(1))}{\log n} \\ &= \rho/(\rho+1) + \rho'/(\rho'+1).\end{aligned}$$

This proves the theorem.

PROOF OF THEOREM 3. By Theorem A

$$\lambda^*/(\lambda^*+1) = \liminf_{n\to\infty}\frac{\log^+\log^+|a_nR^nb_nS^n|}{\log n}.$$

Now proceeding as in the proof of Theorem 2 we get

$$\begin{aligned}\lambda^*/(\lambda^*+1) \leqq \lim_{n\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log\log(|a_nR^n| + O(1))}{\log n} \\ + \lim_{n\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log\log(|b_nS^n| + O(1))}{\log n}.\end{aligned}$$

That is

$$\lambda^*/(\lambda^*+1) \leqq \begin{cases}\lambda/(\lambda+1) + \rho'/(\rho'+1) \\ \rho/(\rho+1) + \lambda'/(\lambda'+1).\end{cases}$$

PROOF OF THEOREM 4. By Theorem 1, $\rho(f'*g') = \rho$ and $\lambda(f'*g') = \lambda$; so

the results follow from Lemma 3 and relations (12) and (13) of Theorem B.
To prove (5) observe that

$$\log(u(r, f'*g')/(u(r, f*g)) \log u(r, f*g)))$$
$$= \log(u(r, f'*g')/u(r, f*g)) - \log\log u(r, f*g)$$

and the result follows by using (3) and Theorem B. This proves Theorem 4 completely.

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