ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 10, Number 2, Spring 1980

COMPOSITION OPERATORS ON A SPACE OF LIPSCHITZ FUNCTIONS

RAYMOND C. ROAN

ABSTRACT. For $0 < \alpha \leqq 1$, let Lip(α) denote the space of functions f which are analytic on the open unit disk, continuous on the closed unit disk, and whose boundary values satisfy a Lipschitz condition of order $\alpha : |f(z) - f(w)| \leqq K|z - w|^\alpha$, for $|z| = |w| = 1$. For $0 < \alpha < 1$, let lip(α) denote the space of functions f in Lip(α) such that $|f(z) - f(w)| = o(|z - w|^\alpha)$, as $w \rightarrow z$, $|z| = |w| = 1$. We prove that a function φ in Lip(α) (resp., lip(α)), with $|\varphi(z)| \leqq 1$ for $|z| \leqq 1$, induces a composition operator on Lip(α) (resp., lip(α)) if and only if there exists a finite number M and a number $r < 1$ such that $|\varphi(z)| \geqq r$ implies $|\varphi'(z)| \leqq M$. We also prove that a composition operator C_φ on either Lip(α) or lip(α) is compact if and only if for each $\epsilon > 0$ there exists an $r < 1$ such that $|\varphi(z)| \geqq r$ implies $|\varphi'(z)| \leqq \epsilon$.

1. **Introduction.** We shall denote the unit disk $\{|z| < 1\}$ by **U**. For $0 < \alpha \leqq 1$, we let Lip(α) denote the space of functions f which are analytic in **U**, continuous on $\mathbf{U}^-$ (the closure of **U**), and whose boundary values satisfy a Lipschitz condition of order α:

$$\frac{|f(z) - f(w)|}{|z - w|^\alpha} = o(1), \qquad |z| = |w| = 1.$$

For $0 < \alpha < 1$, we let lip(α) denote those functions f in Lip(α) for which

$$\frac{|f(z) - f(w)|}{|z - w|^\alpha} = o(1) \qquad \text{as } w \rightarrow z,\ |z| = |w| = 1.$$

Each of the spaces Lip(α) and lip(α) is a Banach algebra when the norm of an element is defined as

$$\|f\|_\alpha = \|f\|_\infty + \sup_{\substack{z \neq w \\ |z|=|w|=1}} \frac{|f(z) - f(w)|}{|z - w|^\alpha},$$

where $\|f\|_\infty = \sup|f(z)|$ ($|z| < 1$).

Received by the editors on October 31, 1977, and in revised form on February 22, 1978.

AMS (MOS) Subject Classification (1970). Primary 46E15, 46J15; Secondary 30A98, 47B05.

We say that a function $\varphi : \mathrm{U} \to \mathrm{U}$ *induces the composition operator* C_φ on Lip(α) (respectively, lip(α)) if

$$C_\varphi(f) = f \circ \varphi$$

is in Lip(α) (resp. lip(α)) for every function f in Lip(α) (resp. lip(α)). We shall characterize those functions which induce composition operators on both Lip(α) and lip(α). We shall also characterize those functions which induce compact composition operators on both Lip(α) and lip(α). Both characterizations will follow from the estimates proved in Theorems 1 and 2.

2. **Main Theorems.** THEOREM 1. *Suppose* $0 < \alpha \leqq 1$, φ *and* $f_n (n = 1, 2, 3, \cdots)$ *are functions in* Lip(α), $\|\varphi\|_\infty \leqq 1$, *and there exist finite positive numbers* K_1, K_2, M, *and* r (*with* $r < 1$), *such that the following conditions are satisfied*:

(*a*) $|\varphi(z)| \geqq r$ *implies* $|\varphi'(z)| \leqq M$

(b) $\|f_n\|_\alpha \leqq K_1$ *and* $\|f_n\|_\infty \leqq K_2$, *for* $n = 1, 2, 3, \cdots$

(c) $|z| \leqq r$ *implies* $|f_n'(z)| \leqq M^\alpha$.

Then, for $K = 2K_1 + \|\varphi\|_\alpha$,

$$\|f_n \circ \varphi\|_\alpha < K_2 + KM^\alpha.$$

PROOF. Let α, φ, $\{f_n\}$, K_1, K_2, M, and r be as in the statement of the theorem.

Let $|z| = |w| = 1$, $z \neq w$; and let L be the line segment joining z and w. If $|\varphi(z)| \leqq r$, let $z_1 = z$; similarly, if $|\varphi(w)| \leqq r$, let $w_1 = w$. Otherwise, let z_1 (respectively, w_1) be the point of L closest to z (resp., w) such that $|\varphi(z_1)| \leqq r$ (resp., $|\varphi(w_1)| \leqq r$). Such values z_1 and w_1 can be uniquely determined by minimizing the continuous function $d_z(\zeta) = |z - \zeta|$ (resp., $d_w(\zeta) = |w - \zeta|$) on the compact set $L \cap \varphi^{-1}\{\zeta \mid |\zeta| \leqq r\}$. We can assume, with no loss of generality, that $z_1 \neq z$ and $w_1 \neq w$.

By our choice of z_1 and w_1, we see that

$$|z - w|^{-\alpha} \leqq \min\{|z - z_1|^{-\alpha}, |z_1 - w_1|^{-\alpha}, |w_1 - w|^{-\alpha}\}.$$

Consequently,

$$\frac{|f_n(\varphi(z)) - f_n(\varphi(w))|}{|z - w|^\alpha} \leqq \frac{|f_n(\varphi(z)) - f_n(\varphi(z_1))|}{|z - z_1|^\alpha}$$
$$+ \frac{|f_n(\varphi(z_1)) - f_n(\varphi(w_1))|}{|z_1 - w_1|^\alpha} + \frac{|f_n(\varphi(w_1)) - f_n(\varphi(w))|}{|w_1 - w|^\alpha}$$

We shall estimate each term separately.

$$\begin{aligned}
&\frac{|f_n(\varphi(z)) - f_n(\varphi(z_1))|}{|z - z_1|^\alpha} \\
&= \frac{|f_n(\varphi(z)) - f_n(\varphi(z_1))|}{|\varphi(z) - \varphi(z_1)|^\alpha} \left| \frac{\varphi(z) - \varphi(z_1)}{z - z_1} \right|^\alpha \\
&\leqq K_1 \left\{ \frac{1}{|z - z_1|} \int_{z_1}^{z} |\varphi'(\zeta)|\, d\zeta \right\}^\alpha \\
&\leqq K_1 M^\alpha.
\end{aligned}$$

We have used the fact that if $\zeta = \lambda z_1 + (1 - \lambda)z$, $0 \leqq \lambda \leqq 1$, then $|\varphi(\zeta)| \geqq r$, so $|\varphi'(\zeta)| \leqq M$ (by (a)).

Similarly,

$$\frac{|f_n(\varphi(w_1)) - f_n(\varphi(w))|}{|w_1 - w|^\alpha} \leqq K_1 M^\alpha.$$

Finally,

$$\begin{aligned}
&\frac{|f_n(\varphi(z_1)) - f_n(\varphi(w_1))|}{|z_1 - w_1|^\alpha} \\
&= \frac{|f_n(\varphi(z_1)) - f_n(\varphi(w_1))|}{|\varphi(z_1) - \varphi(w_1)|} \frac{|\varphi(z_1) - \varphi(w_1)|}{|z_1 - w_1|^\alpha} \\
&\leqq ||\varphi||_\alpha |\varphi(z_1) - \varphi(w_1)|^{-1} \int_{\varphi(z_1)}^{\varphi(w_1)} |f_n'(\zeta)|\, d\zeta \\
&\leqq ||\varphi||_\alpha M^\alpha.
\end{aligned}$$

We have used the fact that $|\varphi(z_1)| \leqq r$ and $|\varphi(w_1)| \leqq r$ implies that for $\zeta = \lambda\varphi(z_1) + (1 - \lambda)\varphi(w_1)$, $0 \leqq \lambda \leqq 1$, we have $|\zeta| \leqq r$; so that $|f_n'(\zeta)| \leqq M^\alpha$ (by (c)).

Combining these estimates, we see that

$$\frac{|f_n(\varphi(z)) - f_n(\varphi(w))|}{|z - w|^\alpha} \leqq (2K_1 + ||\varphi||_\alpha) M^\alpha.$$

Consequently, if $K = 2K_1 + ||\varphi||_\alpha$, then

$$||f_n \circ \varphi||_\alpha \leqq K_2 + KM^\alpha.$$

THEOREM 2. *Suppose* $0 < \alpha \leqq 1$ *and* φ *is in* Lip(α), $||\varphi||_\infty \leqq 1$. *Suppose* $\{k_n\}$ *is a sequence of positive numbers and there exists a sequence* $\{z_n\}$ *of points in* U *such that* $|\varphi(z_n) - \zeta| < 1/n$ *for some* ζ *with* $|\zeta| = 1$ *and* $|\varphi'(z_n)| > k_n$ *for* $n = 1, 2, 3, \cdots$. *Then, there exists a sequence of* *functions* $\{f_n\}$ *in* Lip(α) *and a constant* $K < \infty$ *such that*

(a) $\{\|f_n\|_\alpha\}$ *is bounded in* n

(b) $f_n(z) \to 0$ *uniformly on* U^-

(c) $\|f_n \circ \varphi\|_\alpha > K(k_n)^\alpha$ *for* $n = 1, 2, 3, \cdots$.

In addition, for $0 < \alpha < 1$, *we can choose the functions* f_n, $n = 1, 2, 3, \cdots$, *to be in* lip(α).

PROOF. Without loss of generality, we may assume that $\zeta = 1$. For $n = 1, 2, 3, \cdots$, let

$$f_n(z) = n^{-\alpha}(z + 1 - \varphi(z_n))^n.$$

(a) Fix n and let $a = 1 - \varphi(z_n)$. If $|z| = |w| = 1$, $z \neq w$, then

$$(1) \qquad \begin{aligned} \frac{|f_n(z) - f_n(w)|}{|z - w|^\alpha} &= \frac{|(z+a)^n - (w+a)^n|}{n^\alpha |z - w|^\alpha} \\ &= \left\{ \frac{|(z+a)^n - (w+a)^n}{n|z-w|} \right\}^\alpha |(z+a)^n - (w+a)' \end{aligned}$$

We will estimate each factor separately. First,

$$(2) \qquad \begin{aligned} \frac{|(z+a)^n - (w+a)^n|}{n|z-w|} &\leqq \frac{1}{n} \sum_{k=0}^{n-1} \left| \frac{z+a}{w+a} \right|^k \\ &\leqq \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{n+1}{n-1} \right)^k \\ &\leqq \left(\frac{n+1}{n-1} \right)^{n-1} \\ &\leqq e^2. \end{aligned}$$

We will make two estimates on the second factor, both of which will make use of the fact that $|z + a| \leqq 1 + (1/n)$ and $|w + a| \leqq 1 + (1/n)$.

$$(3) \qquad |(z+a)^n - (w+a)^n| \leqq |z+a|^n + |w+a|^n \leqq 2\left(1 + \frac{1}{n}\right)^n \leqq 2e$$

$$(4) \qquad \begin{aligned} |(z+a)^n - (w+a)^n| &\leqq |z - w| \sum_{k=0}^{n-1} |z+a|^k |w+a|^{n-k} \\ &\leqq ne|z - w|. \end{aligned}$$

Also, for $|z| \leqq 1$,

$$(5) \qquad |f_n(z)| \leqq en^{-\alpha} \leqq e.$$

If we combine estimates (2) and (3) with (1) and (5), we see that

$$\|f_n\|_\alpha \leqq e + 2^{1-\alpha}e^{\alpha+1}, \quad \text{for } n = 1, 2, 3, \cdots,$$

so $\{\|f_n\|_\alpha\}$ is bounded. Furthermore, if we combine estimates (2) and (4) with (1), we see that if $0 < \alpha < 1$, then each f_n is in lip(α).

(b) The inequality (5) shows that $f_n(z) \to 0$ uniformly on U^-.

(c) We know that both φ and φ' are continuous on U. Therefore, for each $n = 1, 2, 3, \cdots$, there exists a $\delta_n > 0$ such that $|z_n| + \delta_n < 1$ and $|z - z_n| < \delta_n$ implies that

(i) $|\varphi'(z)| > k_n$

(ii) $|\varphi(z) - \varphi(z_n)| < \dfrac{1}{n}$.

Fix n, and suppose $|z - z_n| < \delta_n$. Then

$$\text{(6)} \quad \begin{aligned} &\frac{|f_n(\varphi(z)) - f_n(\varphi(z_n))|}{|z - z_n|^\alpha} \\ &= \left\{ \frac{|f_n(\varphi(z)) - f_n(\varphi(z_n))|}{|z - z_n|} \right\}^\alpha |f_n(\varphi(z)) - f_n(\varphi(z_n))|^{1-\alpha}. \end{aligned}$$

From [4], there exists a ζ, $|\zeta - z_n| \leqq |z - z_n| < \delta_n$, such that

$$f_n(\varphi(z)) - f_n(\varphi(z_n)) = (z - z_n)f_n'(\varphi(\zeta))\varphi'(\zeta).$$

Consequently,

$$\text{(7)} \quad \begin{aligned} \left\{ \frac{|f_n(\varphi(z)) - f_n(\varphi(z_n))|}{|z - z_n|} \right\}^\alpha &= \{|f_n'(\varphi(\zeta))|\ |\varphi'(\zeta)|\}^\alpha \\ &= \{n^{1-\alpha}|1 - (\varphi(z_n) - \varphi(\zeta))|^{n-1}|\varphi'(\zeta)|\}^\alpha \\ &> n^{\alpha(1-\alpha)}(k_n)^\alpha \left| 1 - \frac{1}{n} \right|^{(n-1)\alpha} \\ &> n^{\alpha(1-\alpha)}(k_n)^\alpha e^{-\alpha}. \end{aligned}$$

Similarly,

$$\text{(8)} \quad \begin{aligned} &|f_n(\varphi(z)) - f_n(\varphi(z_n))|^{1-\alpha} \\ &= n^{\alpha(\alpha-1)}|(1 - \varphi(z_n) + \varphi(z))^n - 1|^{1-\alpha} \\ &\geqq n^{\alpha(\alpha-1)}\,|(1 - |\,1 - |\varphi(z_n) - \varphi(z)|\,|^n|^{\,1-\alpha} \\ &\geqq n^{\alpha(\alpha-1)} \left| 1 - \left(1 - \frac{1}{n}\right)^n \right|^{1-\alpha} \\ &\geqq n^{\alpha(\alpha-1)} \left(1 - \frac{1}{e} \right)^{1-\alpha}. \end{aligned}$$

Let $K = e^{-\alpha}(1 - 1/e)^{1-\alpha}$ and use the estimates from (7) and (8) in (6) to get

$$\frac{|f_n(\varphi(z)) - f_n(\varphi(z_n))|}{|z - z_n|^\alpha} > K(k_n)^\alpha \quad \text{for } n = 1, 2, 3, \cdots.$$

Hence, by Theorem 2.2 of [5],

$$\|f_n \circ \varphi\|_\alpha > K(k_n)^\alpha, \quad \text{for } n = 1, 2, 3, \cdots.$$

3. **Applications.** Observe that since the identity function is in Lip(α), if $\varphi : \mathbf{U} \to \mathbf{U}$ is to induce a composition operator on Lip(α), then φ must be in Lip(α). For $\alpha = 1$, this necessary condition is sufficient, as we shall see; but, for $0 < \alpha < 1$, it is not sufficient. For example, consider $\alpha = 1/2$. Define functions φ and f by

$$\varphi(z) = [(1 - z)/2]^{1/2} - 1$$

$$f(z) = (1 + z)^{1/2}.$$

A simple calculation shows that φ and f are in Lip(1/2) and that $\|\varphi\|_\infty = 1$. However, $(f \circ \varphi)'(z) = c(1 - z)^{-3/4}$, for some constant c. Consequently, $(f \circ \varphi)'(z) \neq O((1 - |z|)^{-1/2})$, so $f \circ \varphi$ is not in Lip(1/2) (see [1], Theorem 5.1).

Definition. A function $\varphi : \mathbf{U} \to \mathbf{U}$ is called a U-*primary function* if there exist numbers $M < \infty$ and $r < 1$ such that $|\varphi'(z)| \leqq M$ whenever $|\varphi(z)| \geqq r$.

Remark. Without loss of generality, we could require $r = 1 - 1/M$.

Corollary 1. *Let* $0 < \alpha \leqq 1$. *A function* φ *in* Lip(α) (*resp.*, lip(α)) *induces a composition operator on* Lip(α) (*resp.*, lip(α)) *if and only if* φ *is a* U-*primary function.*

Proof. Let φ be in Lip(α) (resp., lip(α)), $0 < \alpha \leqq 1$, and suppose φ induces a composition operator on Lip(α) (resp., lip(α)). If φ is not a U-primary function, then for every $M = 1, 2, 3, \cdots$ there exists a point z_M in U such that $|\varphi(z_M)| \geqq 1 - 1/M$ and $|\varphi'(z_M)| > M$.

By choosing a subsequence, if necessary, we may assume that $|\varphi(z_M) - \zeta| < 1/M$ for some ζ with $|\zeta| = 1$. Let $k_M = M$, $M = 1, 2, 3, \cdots$. By Theorem 2, there exists a uniformly bounded sequence $\{f_M\}$ in lip(α) $\subseteq$ Lip(α) and a constant K such that

$$\|C_\varphi(f_M)\|_\alpha > KM^\alpha,$$

contradicting the continuity of C_φ (see Proposition 3 of [3]).

Conversely, suppose φ is a U-primary function in Lip(α) and f is in Lip(α). For $n = 1, 2, 3, \cdots$, let $f_n = f$. By Theorem 1.

$$||f \circ \varphi||_\alpha < \infty,$$

so $f \circ \varphi$ is in Lip(α). A simple continuity argument shows that if φ and f are in lip(α), $0 < \alpha < 1$, then $f \circ \varphi$ is actually in lip(α).

Notice that if φ' is in H^∞, the set of bounded analytic functions on U, and if $||\varphi||_\infty \leqq 1$, then φ is a U-primary function. But

$$\text{Lip}(1) = \{h \mid h' \text{ is in } H^\infty\}$$

(see Theorem 5.1 of [1]). Consequently, *every* Lip(1) function which maps U into itself induces a composition operator on Lip(1).

An operator L on a Banach space $\mathscr{B}$ is said to be *compact* if every bounded sequence $\{x_n\}$ in $\mathscr{B}$ contains a subsequence $\{x_{n_k}\}$ such that $\{Lx_{n_k}\}$ converges to a point of $\mathscr{B}$.

LEMMA. *Let* $0 < \alpha \leqq 1$. *The operator* $C_\varphi : \text{Lip}(\alpha) \to \text{Lip}(\alpha)$ *is compact if and only if for each bounded sequence* $\{f_n\}$ *in* Lip(α) *which converges to zero uniformly on* U^-, *we have* $||C_\varphi f_n||_\alpha \to 0$ as $n \to \infty$.

PROOF. Suppose that for each bounded sequence $\{f_n\}$ in Lip(α) which converges to zero uniformly U^- we have $||C_\varphi f_n||_\alpha \to 0$ as $n \to \infty$; and suppose $\{f_n\}$ is a bounded sequence in Lip(α). From the work of P. Duren, B. Romberg, and A. Shields ([2], Theorem 2), we know that Lip(α) is equivalent to the dual of an H^p-space with $1/2 \leqq p < 1$ ($p = (1 + \alpha)^{-1}$). By the Banach-Alaoglu Theorem ([6], Theorem 3.15), there exists a function f in Lip(α) and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ in the weak * topology on Lip(α). With no loss of generality, we may assume that $f = 0$ and $f_n \to 0$ (weak *). Using Theorem 1 of [2] and the fact that $h_\zeta(z) = (1 - \zeta z)^{-1}$ is in H^p for $0 < p < 1$ and $|\zeta| \leqq 1$, one can show that evaluation at a point of U^- is a weak * continuous linear functional on Lip(α). Therefore, $f_n(z) \to 0$ for each z in U^-. But the sequence $\{f_n\}$ is a normal family; hence, equicontinuous. By Ascoli's theorem, $f_n \to 0$ uniformly on U^-. Our hypothesis then shows that $||C_\varphi f_n||_\alpha \to 0$; hence, C_φ is compact.

The proof of the converse is easy and we omit it.

COROLLARY 2. *Let* $0 < \alpha \leqq 1$. *The composition operator* C_φ *is compact on* Lip(α) (*resp.*, lip(α)) *if and only if for each* $\epsilon > 0$ *there exists an* $r < 1$ *such that* $|\varphi'(z)| \leqq \epsilon$ *whenever* $|\varphi(z)| \geqq r$.

PROOF. Suppose $0 < \alpha \leqq 1$ and C_φ is compact on Lip(α) (resp., lip(α)). Suppose there is an $\epsilon > 0$ such that for each $n = 1, 2, 3, \cdots$

there exists a point z_n in U with $|\varphi(z_n)| > 1 - 1/n$ and $|\varphi'(z_n)| > \epsilon$. With no loss of generality, we can assume that $|\varphi(z_n) - \zeta| < 1/n$ for some ζ with $|\zeta| = 1$. By Theorem 2 (with $k_n = \epsilon$), there exists a uniformly bounded sequence $\{f_n\}$ in lip(α) $\subseteq$ Lip(α) which converges to zero uniformly on U^- such that

$$||f_n \circ \varphi||_\alpha > K\epsilon^\alpha > 0 \text{ for } n = 1, 2, 2, \cdots.$$

Thus $\{||f_n \circ \varphi||_\alpha\}$ is bounded away from zero, contrary to the compactness of C_φ.

Conversely, suppose $\epsilon > 0$ and $r < 1$ is such that $|\varphi'(z)| \leqq \epsilon$ whenever $|\varphi(z)| \geqq r$. Suppose the sequence $\{f_n\}$ is bounded in Lip(α) and converges to zero uniformly on U^-. Then $f_n' \to 0$ uniformly on compact subsets of U. In particular, there exists a number N so that $n \geqq N$ implies

$$||f_n||_\infty \leqq \epsilon^\alpha \quad \text{and} \quad \sup|f_n'(z)| < \epsilon^\alpha \quad (|z| \leqq r).$$

By Theorem 1, there exists a constant K (which is independent of n) such that

$$||C_\varphi(f_n)||_\alpha \leqq K\epsilon^\alpha.$$

Therefore, $||C_\varphi(f_n)||_\alpha \to 0$ and C_φ is compact on Lip(α). Finally, lip(α) is a closed subspace of Lip(α), so C_φ is also compact on lip(α), provided φ is in lip(α).

REMARK 1. Although we did not use the full strength of either of Theorems 1 or 2 to prove Corollary 1, we did use the full strength of both to prove Corollary 2.

REMARK 2. One can easily verify the following lemma.

LEMMA. *The composition operator* C_φ *is compact on* Lip(1) *if and only if for each sequence* $\{f_n\}$ *in* H^∞ *which is bounded and converges to zero uniformly on compact subsets of* U *we have* $\lim_{n\to\infty}||f_n(\varphi)\varphi'||_\infty = 0$.

Using this lemma, we obtain the following alternate proof of Corollary 2 for the special case $\alpha = 1$.

A simple estimate proves that if φ is in Lip(1), $||\varphi||_\infty \leqq 1$, and for each $\epsilon > 0$ there exists an $r < 1$ such that $|\varphi'(z)| < \epsilon$ whenever $|\varphi(z)| > r$, then C_φ is compact on Lip(1). To prove the converse, suppose for $n = 1, 2, 3, \cdots$, there exists a point z_n in U such that $|\varphi'(z_n)| \geqq \epsilon$ and $|1 - \varphi(z_n)| < 1/n$. Let $f_n(z) = [z + 1 - \varphi(z_n)]^n$; then the sequence $\{f_n\}$ is bounded in H^∞ and converges to zero uniformly on compact subsets of U. However,

$$\|f_n(\varphi)\varphi'\|_\infty \geqq |f_n(\varphi(z_n)\varphi'(z_n)| \geqq \epsilon, \; n = 1, 2, 3, \cdots$$

so C_φ is not compact on Lip(1).

References

1. P. Duren, *The Theory of H^p Spaces,* Academic Press, New York, 1970.

2. P. Duren, B. Romberg, and A. Shields, *Linear Functionals on H^p spaces with* $0 < p < 1$, J. Reine Angew. Math. **238** (1969) 32–60.

3. R. Roan, *Composition Operators on the Space of Functions with H^p Derivative*, Houston J. Math. **4** (1978) 423-438.

4. J. Robertson, *A Local Mean Value Theorem for C,* Edinburgh Math. Soc. Proc. **16** (1968–1969) 329–331.

5. L. A Rubel, A. L. Shields, B. A. Taylor, *Mergelyan Sets and the Modulus of Conttinuity of Analytic Functions,* J. of Approx. Theory **15** (1975), 23–40.

6. W. Rudin, *Functional Analysis,* McGraw-Hill, New York, 1973.

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506