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SYMMETRIC DERIVATIVES DEFINED BY WEIGHTED SPHERICAL MEANS

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ABSTRACT. We consider, for functions of several variables, symmetric derivatives defined by taking weighted spherical averages. We apply these derivatives to establish theorems of Lebesgue type for multiple trigonometric series.

1. **Introduction.** Let $f(t)$ be a function defined in a neighborhood of $t_0 \in \mathbf{R}$. We say f has a *first symmetric derivative* at t_0 with value s [9, vol. I, p. 59] if

$$\frac{1}{2}\{f(t_0 + t) - f(t_0 - t)\} = st + o(t) \tag{1.1}$$

as $t \to 0$. This definition has the following applications to formally integrated trigonometric series [9, vol. I, p. 322 and p. 324].

THEROEM A. *Let $T: \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$ be a trigonometric series with $c_n = O(1/n)$. If T converges at θ_0 to finite sum s then*

$$f(\theta) = c_0\theta + \Sigma' \frac{c_n}{in} e^{in\theta} \tag{1.2}$$

has at θ_0 a first symmetric derivative with value s.

THEOREM B. *Suppose the coefficients of $T: \Sigma\, c_n e^{in\theta}$ satisfy $c_n \to 0$ as $n \to \infty$. If T converges at θ_0 to finite sum s, then the function $f(\theta)$ defined by* (1.2) *has at θ_0 a first symmetric approximate derivative equal to s. That is, the limit in* (1.1) *exists as it tends to* 0 *through a set having* 0 *as a point of density.*

A two dimensional version of (1.1) and of Theorems A and B appears in [5] and [6]. In two dimensions let us write $x = (x_1, x_2) = te^{i\theta}$ and $n = (n_1, n_2)$. Let

$$\Omega(\theta) = \cos\theta + \sin\theta. \tag{1.3}$$

Let $L(x)$ be defined in a neighborhood of $x_0 \in E_2$ and integrable over each circle $|x - x_0| = t$, for t small. We say $L(x)$ has at x_0 a *first generalized symmetric derivative* with value s if

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$$(1.4)\qquad \frac{1}{2\pi}\int_0^{2\pi} L(x_0 + te^{i\theta})\Omega(\theta)\,d\theta = \frac{1}{2}st + o(t) \quad \text{as } t \to 0.$$

Observe that for C^2 functions, $s = (\partial L/\partial x_1) + (\partial L/\partial x_2)$. See Theorem 1 below for details. If the limit in (1.4) exists as $t \to 0$ through a set having 0 as a point of density, we say $L(x)$ has at x_0 a *first generalized symmetric approximate derivative* at x_0 with value s.

Let

$$(1.5)\qquad T: \sum_{n\in \mathbf{Z}_2} c_n e^{in\cdot x}$$

be a double trigonometric series. If $\beta \geqq 0$, we will say T is (BR, β) summable at x_0 to s if

$$\lim_{R\to\infty} \sum_{|n|<R} c_n e^{in\cdot x_0}\left(1 - \left(\frac{|n|}{R}\right)^2\right)^\beta = s.$$

We will denote the Fourier series of a function $L(x)$ by $S[L]$.

THEOREM A′. *Suppose the series* (1.5) *is* (BR, β) *summable at* x_0 *to finite sum* s *for some* β *with* $0 \leqq \beta < \frac{1}{2}$. *Suppose the coefficients of* (1.5) *satisfy*

$$\sum_{n_1+n_2=0} |n|^\alpha |c_n|^2 + \sum_{n_1+n_2\neq 0} |n|^\alpha (n_1 + n_2)^{-2}\, z|c_n|^2 < \infty$$

for some number $\alpha > 1$. *Then the series*

$$\sum_{n_1+n_2=0} \frac{1}{2}(x_1 + x_2)c_n e^{in\cdot x} + \sum_{n_1+n_2\neq 0} \frac{-ic_n}{n_1 + n_2} e^{in\cdot x}$$

converges spherically a.e. on T_2 *to a function* $L(x)$ *which has at* x_0 *a first generalized symmetric derivative equal to* s.

THEOREM B′. *Suppose the series* (1.5) *converges spherically at* x_0 *to finite sum* s. *Suppose there are functions* $L_1(x)$ *and* $L_2(x)$ *such that*

$$\sum_{n_1+n_2=0} c_n e^{in\cdot x} = S[L_1]$$

and

$$\sum_{n_1+n_2\neq 0} \frac{-ic_n}{n_1 + n_2} e^{in\cdot x} = S[L_2].$$

Let

$$L(x) = \frac{1}{2}(x_1 + x_2)L_1(x) + L_2(x).$$

Then $L(x)$ has at x_0 a first generalized symmetric approximate derivative with value s.

The purpose of this paper is to establish some p-dimensional analogues of the above results. We begin in § 2 with a natural extension of the definition in (1.4) to p dimensions and analogues of Theorems A′ and B'. We are able to show that, as the dimension p increases, the hypothesis on the order of summability required in the analogue to Theorem A′ becomes weaker, (although the growth conditions on c_n become stronger).

In § 5 we consider definitions of "symmetric derivative" for functions $L(x)$ defined for $x \in E_p$, $p \geqq 2$, based upon some weighted spherical means of L. We are able to establish different theorems of Lebesgue type for multiple trigonometric series by using different weights.

2. In this section we extend the definition of (1.4) of p dimensions and give analogues of Theorems A′ and B′. In p dimensions, $p \geqq 2$, we write $x = (x_1, \cdots, x_p)$ and $n = (n_1, \cdots, n_p)$. We set $n \cdot x = n_1x_1 + \cdots + n_px_p$ and $|x| = (x \cdot x)^{1/2}$. We let $x' = x/|x|$ and $\Sigma = \{x \in E_p |\ |x| = 1\}$. We write $ds(\eta)$ to indicate the surface element in $(p - 1)$-dimensional surface integrals.

Definition. Let

$$\Omega(x) = x_1 + \cdots + x_p. \tag{2.1}$$

Let $L(x)$ be defined in a neighborhood of $x_0 \in E_p$. We will say $L(x)$ has at x_0 a *first generalized symmetric derivative* if L is integrable over each sphere $|x - x_0| = t$, for t small, and if

$$(2\pi)^{-p/2} \int_{\eta \in \Sigma} L(x_0 + t\eta)\Omega(\eta)\, ds(\eta) = \frac{s}{2^{p/2}\Gamma((p+2)/2)} t + o(t) \tag{2.2}$$

as $t \to 0$.

If the limit in (2.2) exists only as t tends to 0 through a set having 0 as a point of density, we will say $L(x)$ has at x_0 a *first generalized symmetric approximate derivative* equal to s.

THEOREM 1. *Suppose that $L(x)$ and all partial derivatives of $L(x)$ of order $\leqq 2$ exist and are continuous in a neighborhood of $x_0 \in E_p$. Then $L(x)$ has at x_0 a first generalized symmetric derivative with value*

$$s = \left(\frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_p} \right) \Big|_{L(x_0)} .$$

THEOREM 2. *Let*

$$T: \sum_{n \in \mathbf{Z}_p} c_n e^{in \cdot x} \tag{2.3}$$

be a trigonometric series in p variables. Let β be a non-negative number with $\beta < (p-1)/2$. Suppose T is summable (BR, β) at a point x_0 to finite sum s. Suppose, in addition,

$$\begin{aligned} &\sum_{n_1+\cdots+n_p \neq 0} |c_n|^2 (n_1 + \cdots + n_p)^{-2} |n|^{p-1+\epsilon} \\ &+ \sum_{n_1+\cdots+n_p = 0} |c_n|^2 |n|^{p-1+\epsilon} < \infty \end{aligned} \tag{2.4}$$

for some $\epsilon > 0$. Then the series

$$\begin{aligned} &\sum_{n_1+\cdots+n_p \neq 0} \frac{-ic_n}{n_1 + \cdots + n_p} e^{in \cdot x} \\ &+ \frac{1}{p} (x_1 + \cdots + x_p) \sum_{n_1+\cdots+n_p = 0} c_n e^{in \cdot x} \end{aligned} \tag{2.5}$$

converges spherically to a function $L(x)$ which has at x_0 a first generalized symmetric derivative equal to s.

THEOREM 3. *Suppose the series (2.3) converges spherically at x_0 to finite sum s. Suppose there are functions $L_1(x)$ and $L_2(x)$ such that*

$$\sum_{n_1+\cdots+n_p \neq 0} \frac{-ic_n}{n_1 + \cdots + n_p} e^{in \cdot x} = S[L_1]$$

and

$$\sum_{n_1+\cdots+n_p = 0} c_n e^{in \cdot x} = S[L_2].$$

Let

$$L(x) = L_1(x) + \frac{1}{p}(x_1 + \cdots + x_p) L_2(x).$$

Then $L(x)$ has at x_0 a first generalized symmetric approximate derivative with value s.

3. Before we give the proofs of the theorems we obtain some preliminary results. We derive some formulae with more generality than immediately needed in order to facilitate the proofs of later results. The author is indebted to Professor Richard Wheeden for a suggestion that has greatly simplified the computations.

By a *surface harmonic of degree ν*, $S_\nu(\eta)$, we will mean the restriction to the unit sphere of a homogeneous harmonic polynomial of degree ν. We will denote the Bessel's function of order ν by $J_\nu(x)$.

LEMMA 1. *Let $S_\nu(\eta)$ be a surface harmonic of degree ν and let ξ be a unit vector in E_p. Then*

$$(3.1)\quad (2\pi)^{-p/2}\int_{\eta\in\Sigma} \xi\cdot\eta\, S_\nu(\eta)\,ds(\eta) = \begin{cases} \dfrac{1}{2^{p/2}\Gamma((p+2)/2)} S_\nu(\xi) & \text{if } \nu = 1 \\ 0 & \text{if } \nu \neq 1. \end{cases}$$

LEMMA 2. *Let $\Omega(x) = x_1 + \cdots + x_p$. For $n \in Z_p$, $|n| \neq 0$, define*

$$(3.2)\quad g_n(x) = \begin{cases} \dfrac{-ie^{in\cdot x}}{n_1 + \cdots + n_p} & \text{if } n_1 \cdots + n_p \neq 0 \\ \dfrac{1}{p}(x_1 + \cdots + x_p)\, e^{in\cdot x} & \text{if } n_1 + \cdots + n_p \doteq 0. \end{cases}$$

Then for $t > 0$,

$$(3.3)\quad (2\pi)^{-p/2}\int_{\eta\in\Sigma} g_n(t\eta)\Omega(\eta)\,ds(\eta) = \frac{J_{p/2}(|n|t)}{|n|^{p/2}t^{(p-2)/2}}\,.$$

PROOF. Suppose $f(h)$ is defined and has ν continuous derivatives for $h \in [-1, 1]$. Let ξ be a unit vector in E_p. The *Funk-Hecke Theorem* [3, p. 181] says

$$(3.4)\quad \int_{\eta\in\Sigma} f(\xi\cdot\eta)S_\nu(\eta)\,ds(\eta) = \frac{2\pi^{(p-1)/2}}{\Gamma((p-1)/2)} S_\nu(\xi)\int_{-1}^{1} f(h)P_\nu(h)(1-h^2)^{(p-3)/2}\,dh,$$

where $P_\nu(h)$ is the Legendre polynomial of degree ν in p dimensions,

$$(3.5)\quad P_\nu(h) = (-2)^{-\nu}\frac{\Gamma((p-1)/2)}{\Gamma(\nu+(p-1)/2)}(1-h^2)^{(3-p)/2}D^\nu(1-h^2)^{\nu+(p-3)/2}.$$

Substituting (3.5) into the right side of (3.4) and integrating by parts ν times we get

$$\int_{\eta\in\Sigma} f(\xi\cdot\eta)S_\nu(\eta)\,ds(\eta)$$

$$= S_\nu(\xi)\,\frac{(-1)^\nu\, 2^{-\nu+1}\pi^{(p-1)/2}}{\Gamma(\nu+(p-1)/2)}$$

$$\int_{-1}^{1} f(h)D^\nu(1-h^2)^{\nu+(p-3)/2}\,dh \tag{3.6}$$

$$= S_\nu(\xi)\,\frac{2^{-\nu+1}\pi^{(p-1)/2}}{\Gamma(\nu+)p\;x\;1)s2}$$

$$\int_{-1}^{1} f^{(\nu)}(h)(1-h^2)^{\nu+(p-3)/2}\,dh.$$

To prove Lemma 1 we let $f(h) = h$. Clearly, if $\nu \neq 1$, the integral on the right of (3.6) vanishes. Thus

$$\int_{\eta\in\Sigma} \xi\cdot\eta\; S_\nu(\eta)\,ds\,(\eta) = 0$$

if $\nu \neq 1$,

If $\nu = 1$, then (3.6) becomes

$$\int_{\eta\in\Sigma} \xi\cdot\eta\; S_1\,(\eta)\,ds\,(\eta)$$

$$= S_1(\xi)\,\frac{\pi^{(p-1)/2}}{\Gamma((p+1)/2)}\int_{-1}^{1}(1-h^2)^{(p-1)/2}\,dh. \tag{3.7}$$

The integral on the right of (3.7) may be computed by reduction formulae (using different formulae for the cases when p is even or odd). We get

$$\int_{-1}^{1}(1-h^2)^{(p-1)/2}\,dh = \pi^{1/2}\,\frac{\Gamma((p+1)/2)}{\Gamma((p+2)/2)}\,.$$

Thus,

$$(2\pi)^{-p/2}\int_{\eta\in\Sigma} \xi\cdot\eta\; S_1(\eta)\,ds\,(\eta) = \frac{2^{-p/2}}{\Gamma((p+2)/2)}\,S_1(\xi).$$

This completes the proof of Lemma 1.

To prove Lemma 2 we again use (3.6). Here we fix $n = (n_1, \cdots, n_p)$ and fix $t > 0$. We let $f(h) = \exp(i|n|t\,h)$ and set $\xi = n' = |n|^{-1}n$. Then (3.6) becomes

$$\int_{\eta\in\Sigma} e^{in\cdot t\eta}\, S_\nu(\eta)\, ds\,(\eta)$$

$$= \int_{\eta\in\Sigma} f(n'\cdot\eta)\, S_\nu(\eta)\, ds\,(\eta)$$

$$= S_\nu(n') \frac{2^{-\nu+1}\pi^{(p-1)/2}}{\Gamma(\nu+(p-1)/2)} \int_{-1}^{1} f^{(\nu)}(h)(1-h^2)^{\nu+(p-3)/2}\, dh$$

$$= S_\nu(n') \frac{2^{-\nu+1}\pi^{(p-1)/2}}{\Gamma(\nu+(p-1)/2)} (i|n|t)^\nu$$

$$\int_{-1}^{1} e^{i|n|th}(1-h^2)^{\nu+(p-3)/2}\, dh$$

$$= S_\nu(n') \frac{2^{-\nu+1}\pi^{(p-1)/2}(i|n|t)^\nu}{\Gamma(\nu+(p-1)/2)}$$

$$\times \frac{\Gamma(\nu+(p-1)/2J)_{\nu+(p-1)/2}(|n|t)}{\pi^{-1/2}(1/2|n|t)^{\nu+(p-2)/2}}$$

by formula 7 from [1, p. 81]. Hence,

$$(3.8)\qquad \begin{aligned}&(2\pi)^{-p/2}\int_{\eta\in\Sigma} e^{in\cdot t\eta}\, s_\nu(\eta)\, ds\,(\eta)\\ &= S_\nu(n')\,\frac{i^\nu J_{\nu+(p-2)/2}(|n|t)}{(|n|t)^{(p-2)/2}}\ .\end{aligned}$$

We now complete the proof of Lemma 2. If $n_1+\cdots+n_p\neq 0$, we apply (3.8) with $\nu=1$, $S_1(x)=\Omega(x)$.

$$(2\pi)^{-p/2}\int_{\eta\in\Sigma} g_n(t\eta)\Omega(\eta)\, ds\,(\eta)$$

$$= \frac{-i}{n_1+\cdots+n_p}(2\pi)^{-p/2}\int_{\eta\in\Sigma} e^{in\cdot t\eta}\Omega(\eta)\, ds\,(\eta)$$

$$= \frac{-i}{n_1+\cdots+n_p}\,\Omega(n')\,\frac{iJ_{p/2}(|n|t)}{(|n|t)^{(p-2)/2}}$$

$$= \frac{-i}{n_1+\cdots+n_p}\,\frac{n_1+\cdots+n_p}{|n|}\,\frac{iJ_{p/2}(|n|t)}{(|n|t(^{(p-2)/2}}$$

$$= |n|^{-p/2}t^{-(p-2)/2}J_{p/2}(|n|t).$$

If $n_1+\cdots+n_p=0$ then

$$(2\pi)^{-p/2} \int_{\eta \in \Sigma} g_n(t\eta)\Omega(\eta)\, ds\,(\eta)$$

$$= (2\pi)^{-p/2} \int_{\eta \in \Sigma} \frac{1}{p}(t\eta_1 + \cdots + t\eta_p)e^{in\cdot t\eta}(\eta_1 + \cdots + \eta_p)\, ds\,(\eta)$$

$$= p^{-1}t(2\pi)^{-p/2} \int_{\eta \in \Sigma} (\eta_1 + \cdots + \eta_p)^2 e^{in\cdot t\eta}\, ds\,(\eta)$$

$$= p^{-1}t\left\{ \sum_{j=1}^{p} (2\pi)^{-p/2} \int_{\eta \in \Sigma} \eta_j^2\, e^{in\cdot t\eta}\, ds\,(\eta) \right.$$

$$\left. + \sum_{j\neq k} (2\pi)^{-p/2} \int_{\eta \in \Sigma} \eta_j\eta_k\, e^{in\cdot t\eta}\, ds\,(\eta) \right\}$$

$$= p^{-1}(A_1 + A_2).$$

To compute A_1 we apply (3.8) with $\nu = 0$, $s_0(\eta) \equiv 1$.

$$A_1 = (2\pi)^{-p/2} \int_{\eta \in \Sigma} \sum_{j=1}^{p} \eta_j^2\, e^{in\cdot t\nu}\, ds\,(\eta)$$

$$= (2\pi)^{-p/2} \int_{\eta \in \Sigma} e^{in\cdot t\eta}\, ds\,(\eta) \tag{3.10}$$

$$= (|n|t)^{-(p-2)/2} J_{(p-2)/2}(|n|t).$$

To compute A_2 we use (3.8) with $\nu = 2$, $s_\nu(\eta) = \eta_j\eta_k$.

$$(2\pi)^{-p/2} \int_{\eta \in \Sigma} \eta_j\eta_k\, e^{in\cdot t\eta}\, ds(\eta)$$

$$= \frac{n_j n_k}{|n|^2} \frac{(-1)J_{(p+2)/2}(|n|t)}{(|n|t)^{(p-2)/2}}.$$

Recall that $n_1 + \cdots + n_p = 0$. Hence

$$0 = (n_1 + \cdots + n_p)^2 = \sum_{k=1}^{p} n_i^2 + \sum_{j\neq k} n_j\, n_k$$

$$= |n|^2 + \sum_{j\neq k} n_j\, n_k.$$

Thus, $\Sigma_{j\neq k}\, n_j\, n_k = -|n|^2$.

$$
\begin{aligned}
A_2 &= \sum_{j\neq k} (2\pi)^{-p/2} \int_{\eta\in\Sigma} \eta_j\,\eta_k\, e^{in\cdot t\eta}\, ds\,(\eta) \\
&= \sum_{j\neq k} \frac{n_j\, n_k}{|n|^2} \; \frac{(-1)J_{(p+2)/2}(|n|t)}{(|n|t)^{(p-2)/2}} \\
&= \frac{J_{(p+2)/2}(|n|t)}{(|n|t)^{(p-2)/2}} \cdot \frac{(-1)}{|n|^2} \sum_{j\neq k} n_j\, n_k \\
&= \frac{J_{(p+2)/2}(|n|t)}{(|n|t)^{(p-2)/2}} .
\end{aligned} \tag{3.11}
$$

We now return to (3.9).

$$
\begin{aligned}
&(2\pi)^{-p/2} \int_{\eta\in\Sigma} g_n(t\eta)\Omega(\eta)\, ds\,(\eta) \\
&= p^{-1}t\,(A_1 + A_2) \\
&= p^{-1}t \left\{ \frac{J_{(p-2)/2}(|n|t)}{(|n|t)^{(p-2)/2}} + \frac{J_{(p+2)/2}(|n|t)}{(|n|t)^{(p-2)/2}} \right\} \\
&= |n|^{-p/2} t^{-(p-2)/2} J_{p/2}(|n|t)
\end{aligned}
$$

by formula 56 from [1,p. 12]. This completes the proof of Lemma 2.

Now suppose we have a numerical series $\Sigma_{n\in Z_p} c_n$. We set

$$S_R = \sum_{|n|<R} c_n$$

and for $\beta > 0$, we set

$$S_R{}^{\beta} = \frac{1}{\Gamma(\beta)} \int_0^R S_u\,(R-u)^{\beta-1}\, du$$

and say Σc_n is (BR, β) summable to s if $\Gamma(\beta+1)R^{-\beta}S_R{}^{\beta} \to s$ as $R \to \infty$. It can be shown [2] that the series Σc_n is (BR, β) summable to finite sum s if and only if

$$\lim_{R\to\infty} \sum_{|n|<R} c_n \left(1 - \frac{|n|}{R}\right)^{\beta} = s.$$

Hence, if Σc_n is (BR, β) summable to 0, then

$$S_R{}^{\beta} = o(R^{\beta}) \tag{3.12}$$

as $R \to \infty$.

LEMMA 3. *Suppose the series* $\sum_{n\in\mathbf{Z}_p} c_n$ *is* $(BR, m+1)$ *summable to* 0. *Suppose, in addition,*

$$\sum_{n_1+\cdots+n_p\neq 0} |c_n|^2(n_1+\cdots+n_p)^{-2}|n|^{p-1+\epsilon} + \sum_{n_1+\cdots+n_p=0} |c_n|^2|n|^{p-1+\epsilon} < \infty \tag{3.13}$$

for some $\epsilon > 0$. *Then*

$$S_R^{\nu} = O(r^{m+1}) \tag{3.14}$$

for $\nu = 0, 1, \cdots, m+1$.

PROOF. Since $|n_1 + \cdots + n_p| < p|n|$,

$$\sum_{n_1+\cdots+n_p\neq 0} |c_n|^2(n_1+\cdots+n_p)^{-2}|n|^{p-1+\epsilon} \geqq k \sum_{n_1+\cdots+n_p\neq 0} |c_n|^2|n|^{p-3+\epsilon}.$$

Thus, from (3.13),

$$\sum_{n\in\mathbf{Z}_p} |c_n|^2|n|^{p-3+\epsilon} < \infty.$$

Applying Hölder's inequality,

$$\begin{aligned}\sum_{|n|<R} |c_n| &= \sum_{|n|<R} \{|c_n|\, |n|^{(p-3+\epsilon)/2}\}\{|n|^{-(p-3+\epsilon)/2}\}\\ &\leqq \left(\sum_{|n|<R} |c_n|^2|n|^{p-3+\epsilon}\right)^{1/2}\left(\sum_{|n|<R} |n|^{-p+3-\epsilon}\right)^{1/2}\\ &\leqq C(R^{-p+3-\epsilon+p})^{1/2}\\ &= o(R^{3/2})\end{aligned}$$

as $R \to \infty$. Having established this, the proof of Lemma 3 is identical to the proof of Lemma 1 of [6].

4. Proof of Theorems 1, 2, and 3.

PROOF OF THEOREM 1. We use Taylor's Formula. Let

$$\xi = \left(\frac{\partial F}{\partial x_1}(x_0), \cdots, \frac{\partial F}{\partial x_p}(x_0) \right).$$

We may assume, by normalizing, if necessary, that $|\xi| = 1$.

$$F(x_0 + t\eta) = F(x_0) + t \sum_{i=1}^{p} \eta_i \frac{\partial F}{\partial x_i}(x_0)$$

$$+ \frac{t^2}{2!} \sum_{i,j=1}^{p} \eta_i \eta_j \frac{\partial^2 F}{\partial x_i \partial x_j}(x_0 + r\eta)$$

for some $r \in (0, t)$. Then,

$$(2\pi)^{-p/2} \int_{\eta \in \Sigma} F(x_0 + t\eta)\Omega(\eta)\, ds(\eta)$$

$$= (2\pi)^{-p/2} \int_{\eta \in \Sigma} F(x_0)\Omega(\eta)\, ds(\eta)$$

$$+ t(2\pi)^{-p/2} \int_{\eta \in \Sigma} \sum_{i=1}^{p} \eta_i \frac{\partial F}{\partial x_i}(x_0)\Omega(\eta)\, ds(\eta)$$

$$\text{(4.1)} \qquad + \frac{t^2}{2!} (2\pi)^{-p/2}$$

$$\int_{\eta \in \Sigma} \sum_{i,j=1}^{p} \eta_i \eta_j \frac{\partial^2 F}{\partial x_i \partial x_j}(x_0 + r\eta)\Omega(\eta)\, ds(\eta)$$

$$= 0 + t \cdot (2\pi)^{-p/2} \int_{\eta \in \Sigma} \eta \cdot \xi\, \Omega(\eta)\, ds(\eta) + R(x_0).$$

Clearly $R(x_0) = t^2 O(1) = o(t)$, as $t \to 0$. Also, by Lemma 1,

$$(2\pi)^{-p/2} \int_{\eta \in \Sigma} \eta \cdot \xi\, \Omega(\eta)\, ds(\eta)$$

$$= \frac{1}{2^{p/2}\Gamma((p+2)/2)} \Omega(\xi)$$

$$\text{(4.2)} \qquad = \frac{1}{2^{p/2}\Gamma((p+2)/2)} (\xi_1 + \cdots + \xi_p)$$

$$= \frac{1}{2^{p/2}\Gamma((p+2)/2)} \left(\frac{\partial F}{\partial x_1}(x_0) + \cdots + \frac{\partial F}{\partial x_p}(x_0) \right).$$

Returning to (4.1),

$$(2\pi)^{-p/2}\int_{\eta\in\Sigma} F(x_0+t\eta)\Omega(\eta)\,ds\,(\eta)$$

$$=\frac{s}{2^{p/2}\Gamma((p+2)/2)}\,t+o(t),$$

where $s = \partial F/\partial x_1\,(x_0) + \cdots + \partial F/\partial x_p\,(x_0)$. This proves Theorem 1.

PROOF OF THEOREM 2. Having established Lemma 2, the proof of Theorem 2 is very similar to the proof of the theorems in [6]. Write $\beta = m + \alpha$ where m is an integer and $0 \leqq \alpha < 1$. We first prove Theorem 2 in the special case $\alpha = 0$. We may assume without loss of generality that $x_0 = 0$, $c_0 = 0$, $s = 0$.

Write $S_R = S_R(0) = \sum_{|n|<R} c_n$. Then

$$S_R{}^1 = \int_0^R S_u\,du,\ \cdots,\ S_R{}^m = \int_0^R S_u{}^{m-1}\,du.$$

By (3.12), we may assume

$$S_R{}^m = o(R^m), \tag{4.3}$$

as $R \to \infty$.

The condition (2.4) in the coefficients $\{c_n\}$ insures that the series defining $L(x)$ converges spherically a.e. on each sphere $|x| = t$ [4, Theorem 1]. Moreover, by Theorem 2 of [4], we may integrate this series by term over each sphere $|x| = t$. Thus

$$(2\pi)^{-p/2}\int_{\eta\in\Sigma} L(t\eta)\Omega(\eta)\,ds(\eta)$$

$$=\lim_{R\to\infty}\sum_{|n|<r}(2\pi)^{-p/2}\int_{\eta\in\Sigma} c_n g_n(t\eta)\Omega(\eta)\,ds(\eta)$$

$$=\lim_{R\to\infty}\sum_{|n|<R} c_n\,\frac{J_{p/2}(|n|t)}{|n|^{p/2}t^{(p-2)/2}}$$

by Lemma 2. We set

$$\gamma(z) = z^{-p/2}J_{p/2}(z). \tag{4.4}$$

Then

$$\begin{aligned}(2\pi)^{-p/2}\int_{\eta\in\Sigma} L(t\eta)\Omega(\eta)\,ds(\eta)\\ = \lim_{R\to\infty} t\sum_{|n|<R} c_n\gamma(|n|t).\end{aligned} \tag{4.5}$$

We change the last sum to an integral and integrate by parts m times.

$$\sum_{|n|<R} c_n\gamma(|n|t) = S_R\gamma(Rt) - \int_0^R S_u \frac{d}{du}\gamma(ut)\,du$$

$$= S_R\gamma(Rt) - S_R{}^1 \frac{d}{dR}\gamma(Rt)$$

$$(4.6) \qquad + \int_0^R S_u{}^1 \frac{d^2}{du^2}\gamma(ut)\,du$$

$$= S_R\gamma(Rt) - S_R{}^1 \frac{d}{dR}\gamma(Rt)$$

$$+ \cdots + (-1)^m S_R{}^m \frac{d^m}{dR^m}\gamma(Rt)$$

$$+ (-1)^{m+1} \int_0^R S_u{}^m \frac{d^{m+1}}{du^{m+1}}\gamma(ut)\,du.$$

By repeatedly using formula (51) of [1, page 11] and the estimate $J_\nu(z) = O(z^{-1/2})$ as $z \to \infty$, it is clear that

$$(4.7) \qquad \frac{d^r}{dz^r}\gamma(z) = O(z^{-(1/2)-(p/2)})$$

as $z \to \infty$, for $r = 0, 1, 2, \cdots$. Hence, with Lemma 3,

$$S_R{}^r \frac{d^r}{dR^r}\gamma(Rt) = O(R^{-(1/2)-(p/2)})O(R^{m+1})$$
$$= o(1)$$

for $r = 0, 1, \cdots, m$, since $m < (p-1)/2$. Returning to (4.5),

$$(2\pi)^{-p/2}\int_{\eta\in\Sigma} L(t\eta)\Omega(\eta)\,ds\,(\eta) = t \lim_{R\to\infty} \sum_{|n|<R} c_n\gamma(|n|t)$$

$$= t(-1)^{m+1}\int_0^\infty S_u{}^m \frac{d^{m+1}}{du^{m+1}}\gamma(ut)\,du$$

$$= 0 \cdot t + t \cdot A(t).$$

To prove Theorem 2, we must show $A(t)$ tends to 0 as t tends to 0.

$$A(t) = (-1)^{m+1}\int_0^{1/t} S_u{}^m \frac{d^{m+1}}{du^{m+1}}\gamma(ut)\,du$$

$$+ (-1)^{m+1}\int_{1/t}^\infty S_u{}^m \frac{d^{m+1}}{du^{m+1}}\gamma(ut)\,du$$

$$= A_1(t) + A_2(t).$$

To estimate $A_1(t)$ we use the fact that $\gamma(z)$ is an entire function, so for $|z| < 1$, $|\gamma^{(m+1)}(z)| \leqq C$.

$$A_1(t) = (-1)^{m+1} \int_0^{1/t} S_u^{\ m} \frac{d^{m+1}}{du^{m+1}} \gamma(ut)\, du$$

$$= (-m)^{m+1} \int_0^{1/t} o(u^m)\, t^{m+1} \cdot C\, du$$

$$= o(1).$$

To estimate $A_2(t)$, we use (4.7), obtaining

$$A_2(t) = (-1)^{m+1} \int_{1/t}^{\infty} S_u^{\ m} \frac{d^{m+1}}{du^{m+1}} \gamma(ut)\, du$$

$$= (-1)^{m+1} \int_{1/t}^{\infty} o(u^m)\, t^{m+1} O((ut))^{-(1/2)-(p/2)})\, du$$

$$= o(t^{m+(1/2)-(p/2)}) \int_{1/t}^{\infty} u^{m-(1/2)-(p/2)}\, du$$

$$= o(1).$$

Note that we needed the hypothesis $m < (p-1)/2$ to compute the last integral. This completes the proof of Theorem 2 when $\beta = m$ is an integer.

We now prove Theorem 2 for the case $\beta = m + \alpha$, $0 < \alpha < 1$. We proceed as in the proof above, but at step (4.6) we integrate by parts once again. After showing that the integrated terms tend to zero, we get

$$(2\pi)^{-p/2} \int_{\eta \in \Sigma} L(t\eta)\Omega(\eta)\, ds(\eta)$$

$$(4.8) \qquad = t(-1)^{m+2} \int_0^{\infty} S_u^{\ m+1} \frac{d^{m+2}}{du^{m+2}} \gamma(ut)\, du$$

$$= O \cdot t + tA(t).$$

If $f(u)$ is a function defined for $u > 0$, and η is a positive number, we denote

$$I^{\eta} f(z) = \frac{1}{\Gamma(\eta)} \int_0^z (z-u)^{\eta-1} f(u)\, du,$$

the fractional integral of order η of f [7]. Then

$$S_u^{\ m+1} = I^{m+1}(S_u) = I^{1-\alpha} I^{m+\alpha}(S_u) = I^{1-\alpha}(S_u^{\ m+\alpha})$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{1-\alpha-1} S_z^{\ m+\alpha}\, dz.$$

Then

$$A(t) = (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \gamma(ut)\, du$$

$$= \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^\infty \int_0^u (u-z)^{-\alpha} S_z^{m+\alpha}\, dz \frac{d^{m+2}}{du^{m+2}} \gamma(ut)\, du$$

$$= \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^R S_z^{m+\alpha} \left\{ \int_z^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \gamma(ut)\, du \right\} dz$$

$$= \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^R S_z^{m+\alpha} H(z, t, R)\, dz$$

$$= \lim_{R\to\infty} \left\{ \int_0^{1/t} + \int_{1/t}^R \right\}$$

$$= A_1(t) + A_2(t).$$

We use the estimates

$$H(z, t, R) = \begin{cases} O\left(\frac{1}{t} - z\right)^{-\alpha} t^{m+1} & \text{if } z \leqq 1/t \\ O(tz)^{-(p+1)/2} t^{m+1+\alpha} & \text{if } z \geqq 1/t. \end{cases}$$

The proofs are similar to proofs of corresponding estimates in [6] and are omitted. We refer the reader to that paper for details.

$$A_1(t) = \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^{1/t} S_z^{m+\alpha} H(z, t, R)\, dz$$

$$= \int_0^{1/t} o(z^{m+\alpha})\, O\left(\frac{1}{t} - z\right)^{-\alpha} t^{m+1}\, dz$$

$$= o(1).$$

$$A_2(t) = \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_{1/t}^R S_z^{m+\alpha} H(z, t, R)\, dz$$

$$= \int_{1/t}^\infty o(z^{m+\alpha})\, O(z^{-(p+1)/2})\, t^{m+\alpha-(p-1)/2}\, dz$$

$$= o(t^{m+\alpha-(p-1)/2}) \int_{1/t}^\infty z^{m+\alpha-(p+1)/2}\, dz$$

$$= o(1).$$

The hypothesis $\beta = m + \alpha < (p-1)/2$ is needed here to make the last integral converge. This completes the proof of Theorem 2.

PROOF OF THEOREM 3. Having established Lemma 2, the proof of Theorem 3 is identical to the proof of Theorem 3 of [5] and is omitted.

5. We now investigate extensions of Definition 1 formed by replacing the $\Omega(x)$ in equation (2.2) by an arbitrary surface harmonic.

Let $\Omega(\eta) = S_\nu(\eta))$ be a surface harmonic of order ν. Let L(x) be defined in a neighborhood of $x_0 \in E_p$ and integrable over each sphere $|x - x_0| = t$ for t small. We will say $L(x)$ has at x_0 a *first Ω-derivative* with value s if

$$(5.1) \qquad (2\pi)^{-p/2} \int_{\eta \in \Sigma} L(x_0 + t\eta)\Omega(\eta)\, ds(\eta) = \frac{s}{2^{p/2}\Gamma((p+2)/2)} t + o(t)$$

as $t \to 0$.

It is clear that if $\nu \neq 1$, equation (5.1) does not have any value for us. For if $\nu \neq 1$ and $L(x)$ is smooth enough, then $s = 0$ by Lemma 1. (However, if $\nu \neq 1$ and r is an integer such that $r \geqq \nu$ and r has the same parity as ν, then we may define an *r-th order Ω-derivative* for $L(x)$ by expanding the left side of (5.1) into a Taylor's series o even or odd powers of t, depending upon the parity of ν. It is reasonable to expect that p-dimensional analogues of the theorem in [9, volume II, page 66] will hold with this definition.)

If $\nu = 1$, then $\Omega(\eta)$ must be of the form $a_1\eta_1 + \cdots + a_p\eta_p$. In this case we are able to derive analogues to Theorems 1, 2 and 3.

Let $\alpha = (a_1, \cdots, a_p)$ be a fixed element of E_p with $|\alpha| \neq 0$. For $\eta \in \Sigma$ let

$$(5.2) \qquad \Omega_\alpha(\eta) = a_1\eta_1 + \cdots + a_p\eta_p.$$

We will say a function $L(x)$ has a first Ω_α-derivative equal to s if equation (5.1) *holds with* $\Omega(\eta) = \Omega_\alpha(\eta)$.

THEOREM 1′. *Suppose that* L(x) *and all partial derivatives of* L(x) *of order at most* 2 *exist and are continuous in a neighborhood of* $x_0 \in E_p$. *Then* L(x) *has at* x_0 *a first* Ω_α*-derivative with value*

$$s = \left(a_1 \frac{\partial}{\partial x_1} + \cdots + a_p \frac{\partial}{\partial x_p} \right) \Big|_{L(x_0)}.$$

THEOREM 2′. *Let*

$$(5.3) \qquad T: \sum_{n \in \mathbf{Z}_p} c_n e^{in\cdot x}$$

be a trigonometric series in p variables. Let β be a non-negative number with $\beta < (p-1)/2$. Suppose T is summable (BR, β) at a point x_0 to a finite sum s. Suppose, in addition,

$$\sum_{a_1n_1+\cdots+a_pn_p\neq 0} |c_n|^2(a_1n_1 + \cdots + a_pn_p)^{-2}|n|^{p-1+\epsilon}$$

$$+ \sum_{a_1n_1+\cdots+a_pn_p=0} |c_n|^2|n|^{p-1+\epsilon} < \infty$$

for some $\epsilon > 0$. *Let*

$$L_\alpha(x) = \sum_{a_1n_1+\cdots+a_pn_p\neq 0} \frac{-ic_n}{a_1n_1 + \cdots + a_pn_p} e^{in\cdot x}$$

$$+ |\alpha|^{-2}(a_1x_1 + \cdots + a_px_p) \sum_{a_1n_1+\cdots+a_pn_p=0} c_ne^{in\cdot x}.$$

Then $L_\alpha(x)$ *has at* x_0 *a first* Ω_α*-derivative with value* s.

THEOREM 3′. *Suppose the series* (5.3) *converges spherically at* x_0 *to finite sum* s. *Suppose there are functions* $L_1(x)$ *and* $L_2(x)$ *such that*

$$\sum_{a_1n_1+\cdots+a_pn_p\neq 0} \frac{-ic_n}{a_1n_1 + \cdots + a_pn_p} e^{in\cdot x} = S[L_1]$$

and

$$\sum_{a_1n_1+\cdots+a_pn_p=0} c_n e^{in\cdot x} = S[L_2].$$

Let $L_\alpha(x) = L_1(x) + |\alpha|^{-2} (a_1x_1 + \cdots + a_px_p)L_2(x)$. *Then* $L_\alpha(x)$ *has at* x_0 *a first* Ω_α*-derivative with value* s.

REMARK. One "disadvantage" of Theorems 2 and 3 is that the hypothesis requires a different standard of behavior for the terms c_n of the series T which correspond to n situated on the hyperplane $x_1 + \cdots + x_p = 0$. In the hypotheses of Theorems 2′ and 3′ we require a different standard of behavior for the terms c_n when n is situated on the hyperplane $a_1x_1 + \cdots + a_px_p = 0$. Perhaps, in applications of Theorems 2′ or 3′ to a specific series T, the hyperplane $a_1x_1 + \cdots + a_px_p = 0$ can be chosen to optimize this situation.

6. The proof of Theorem 1′ is identical to the proof of Theorem 1. To prove Theorems 2′ and 3′ we establish the following lemma. After the lemma is proved, the proofs of Theorem 2′ and 3′ are essentially identical to the proofs of Theorems 2 and 3 and are omitted.

LEMMA 2′. *Let* $\alpha = (a_1, \cdots, a_p)$, $|\alpha| \neq 0$. Let $\Omega_\alpha(x) = a_1x_1 + \cdots + a_px_p$. *For* $n \in Z_p$, $|n| \neq 0$, *define*

$$g_{\alpha,n}(n) = \begin{cases} \dfrac{-i\, e^{in\cdot x}}{a_1n_1 + \cdots + a_pn_p} & \text{if } a_1n_1 + \cdots + a_pn_p \neq 0 \\ \dfrac{1}{|\alpha|^2}(a_1x_1 + \cdots + a_px_p)e^{in\cdot x} & \\ & \text{if } a_1n_1 + \cdots + a_pn_p = 0. \end{cases}$$

Then for $t > 0$,

$$(6.1) \qquad (2\pi)^{-p/2} \int_{\eta\in\Sigma} g_{\alpha,n}(t\eta)\Omega_\alpha(\eta)\, ds\,(\eta) = \frac{J_{p/2}(|n|t)}{|n|^{p/2}t^{(p-2)/2}}.$$

PROOF. (i) Suppose first $a_1n_1 + \cdots + a_pn_p \neq 0$. Then using formula (3.8) with $\nu = 1$,

$$(2\pi)^{-p/2} \int_{\eta\in\Sigma} g_{\alpha,n}(t\eta)\Omega_\alpha(\eta)\, ds\,(\eta)$$

$$= \frac{-i}{a_1n_1 + \cdots + a_pn_p} \cdot (2\pi)^{-p/2} \int_{\eta\in\Sigma} e^{in\cdot t\eta}\Omega_\alpha(\eta)\, ds\,(\eta)$$

$$= \frac{-i}{a_1n_1 + \cdots + a_pn_p} \cdot \frac{a_1n_1 + \cdots + a_pn_p}{|n|} \cdot \frac{i\,J_{p/2}(|n|t)}{(|n|t)^{(p-2)/2}}$$

$$= \frac{J^{p/2}(|n|t)}{|n|^{p/2}t^{(p-2)/2}}.$$

(ii) Suppose now $a_1n_1 + \cdots + a_pn_p = 0$. This situation is more complicated. Consider first the special case $\alpha = (a_1, 0, \cdots, 0)$, with $a_1 \neq 0$. Then $(n_1, \cdots, n_p) = (0, n_2, \cdots, n_p)$. We may assume $n_2 \neq 0$.

$$(2\pi)^{-p/2} \int_{\eta\in\Sigma} g_{\alpha,n}(t\eta)\Omega_\alpha(\eta)\, ds\,(\eta)$$

$$(6.2) \qquad = (2\pi)^{-p/2} \int_{\eta\in\Sigma} \frac{a_1\eta_1 t}{|\alpha|^2} e^{in\cdot t\eta} a_1\eta_1\, ds\,(\eta)$$

$$= t(2\pi)^{-p/2} \int_{\eta\in\Sigma} \eta_1{}^2 e^{in\cdot t\eta}\, ds\,(\eta).$$

We convert the last integral to hyperspherical coordinates (see [1, p. 233], where p has a different meaning).

$$\eta_1 = \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-2} \sin\varphi,$$

$$\eta_2 = \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-2} \cos\varphi,$$

$$\eta_3 = \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-3} \cos\theta_{p-2},$$

$$\eta_4 = \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-4} \cos\theta_{p-3},$$

$$\vdots$$

$$\eta_{p-1} = \sin\theta_1 \cos\theta_2,$$
$$\eta_p = \cos\theta_1,$$
$$ds(\eta) = (\sin\theta_1)^{p-2} \cdots (\sin\theta_{p-2})^1 \, d\theta_1 \cdots d\theta_{p-2} \, d\varphi.$$

Note that since $n_1 = 0$, when we write $e^{in\cdot t\eta}$ in hyperspherical coordinates we can "separate out" the term involving φ:

$$e^{in\cdot t\eta} = e^{it(n_2\eta_2 + \cdots + n_p\eta_p)}$$
$$= e^{itn_2 \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-2} \cos\varphi} \cdot e^{itf(\theta_i, \eta_i)}.$$

Hence,

$$\int_{\eta\in\Sigma} \eta_1^2 \, e^{in\cdot t\eta} \, ds(\eta)$$
$$= \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} (\sin\theta_1 \cdots \sin\theta_{p-2} \sin\varphi)^2$$
$$\cdot\, e^{itn_2 \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-2} \cos\varphi} \cdot e^{itf(\theta_i, \eta_i)}$$
$$\cdot\, (\sin\theta_1)^{p-2} \cdots \sin\theta_{p-2} d\varphi \, d\theta_1 \cdots d\theta_{p-2} \tag{6.3}$$
$$= \int_0^\pi \cdots \int_0^\pi \left\{ \int_0^{2\pi} \sin^2\varphi \right.$$
$$\left. e^{itn_2 \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-2} \cos\varphi} \, d\varphi \right\}$$
$$\cdot\, \sin^2\theta_1 \cdots \sin^2\theta_{p-2} e^{itf(\theta_i, \eta_i)}$$
$$\cdot\, (\sin\theta_1)^{p-2} \cdots \sin\theta_{p-2} \, d\theta_1 \cdots d\theta_{p-2}.$$

We integrate the innermost integral by parts.

$$\int_0^{2\pi} (\sin\varphi)(\sin\Omega \, e^{itn_2 \sin\theta_1 \cdots \sin\theta_{p-2} \cos\varphi}) \, d\varphi$$
$$= \int_0^{2\pi} (\cos\varphi) \left(\frac{e^{itn_2 \sin\theta_1 \cdots \sin\theta_{p-2} \cos\varphi}}{itn_2 \sin\theta_1 \cdots \sin\theta_{p-2}} \right) d\varphi.$$

Returning to (6.3),

$$\int_{\eta\in\Sigma} \eta_1^2 e^{in\cdot t\eta} \, ds(\eta)$$
$$= \frac{1}{itn_2} \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \sin\theta_1 \cdots \sin\theta_{p-2} \cos\varphi$$
$$\cdot\, e^{in\cdot t\eta} (\sin\theta_1)^{p-2} \cdots \sin\theta_{p-2} \, d\varphi \, d\theta_1 \cdots d\theta_{p-2}$$
$$= \frac{1}{itn_2} \int_{\eta\in\Sigma} \eta_2 e^{in\cdot t\eta} s\,)\eta).$$

We can now apply equation (3.8), with $\nu = 1$, $S_\nu(\eta) = \eta_2$.

$$(2\pi)^{-p/2} \int_{\eta\in\Sigma} g_{\alpha,n}(t\eta)\Omega_\alpha(\eta)\, ds\,(\eta)$$

$$= t \cdot \frac{1}{itn_2} \cdot (2\pi)^{-p/2} \int_{\eta\in\Sigma} \eta_2 e^{in\cdot t\eta}\, ds\,(\eta)$$

$$= \frac{1}{in_2} \cdot \frac{n_2}{|n|} \cdot \frac{iJ_{p/2}(|n|t)}{(|n|t)^{(p-2)/2}}$$

$$= \frac{J_{p/2}(|n|t}{|n|^{p/2}t^{(p-2)/2}}\,.$$

This completes the proof of (ii) in the special case when $\alpha = (a_1, 0, \cdots, 0)$.

We now prove (ii) for general $\alpha = (a_1, \cdots, a_p)$.

By a rotation we choose a new orthonormal coordinate system $\{\epsilon_1, \cdots, \epsilon_p\}$ for E_p so that $\epsilon_1 = |\alpha|^{-1}\alpha$. Let $n = (n_1, \cdots, n_p)$ have coordinates $(n_1{}^*, \cdots, n_p{}^*)$ in this new coordinate system. Then $0 = a_1 n_1 + \cdots + a_p n_p = \alpha \cdot n = |\alpha| n_1{}^* + 0 \cdot n_2{}^* + \cdots + 0 \cdot n_p{}^*$. Hence $n_1{}^* = 0$. Thus, in our new coordinate system, $\alpha = (|\alpha|, 0, \cdots, 0)$, and $n = (0, n_2{}^*, \cdots, n_p{}^*)$, which reduces to the special case above. Thus the lemma is proved.

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