ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 10, Number 2, Spring 1980

UNIFORM FINITE GENERATION OF COMPLEX LIE GROUPS OF DIMENSION TWO AND THREE

RICHARD M. KOCH AND FRANKLIN LOWENTHAL

ABSTRACT. Let G be a connected Lie group with Lie algebra g, $\{X_1, \cdots, X_\ell\}$ a minimal generating set for g. The order of generation of G with respect to $\{X_1, \cdots, X_\ell\}$ is the smallest integer n such that every element of G can be written as a product of n elements taken from $\exp(tX_1), \cdots, \exp(tX_\ell)(t \in C)$; n may equal ∞. We find all possible orders of generation for all complex Lie groups of dimension two and three.

1. **Introduction.** A connected Lie group G is generated by one-parameter subgroups $\exp(tX_1), \cdots, \exp(tX_\ell)$ if every element of G can be written as a finite product of elements chosen from these subgroups. In this case, define the order of generation of G to be the least positive integer n such that every element of G possesses such a representation of length at most n; if no such integer exists, let the order of generation of G be infinity. The order of generation will, of course, depend upon the one-parameter subgroups.

Computation of the order of generation of G for given $X_1, \cdots, X_\ell$ is similar to finding the greatest wordlength needed to write each element of a finite group in terms of generators $g_1, \cdots, g_\ell$. In both cases, it is natural to restrict attention to minimal generating sets. From now on, therefore, suppose that no subset of $\{\exp(tX_1), \cdots, \exp(tX_\ell)\}$ generates G.

It is easy to see that $\exp(tX_1), \cdots, \exp(tX_\ell)$ generate G just in case $X_1, \cdots, X_\ell$ generate the Lie algebra g of G. If σ is an automorphism of G, the order of generation of G with respect to $X_1, \cdots, X_\ell$ is clearly the same as the order of generation of G with respect to $\sigma_*(X_1), \cdots, \sigma_*(X_\ell)$. Call two generating sets $\{X_1, \cdots, X_\ell\}$ and $\{Y_1, \cdots, Y_\ell\}$ *equivalent* if it is possible to find an automorphism σ of G, a permutation τ of $\{1, 2, \cdots, \ell\}$, and non-zero constants $\lambda_1, \cdots, \lambda_\ell$ such that $X_i = \lambda_i\sigma_*(Y_{\tau(i)})$; the order of generation of G depends only on the equivalence class of the generating set.

In a series of previous papers [3, 4, 5, 6, 7, 8], the possible orders of generation for all two and three dimensional real Lie groups were found. In this paper, the same problem is solved for all complex Lie groups of dimension two and three. Moreover, when the order of generation n is finite, we will determine which fixed expressions of length

Received by the editors on August 15, 1977.

n (if any) yield all elements for G (for instance, $XYXY$ means every element of G can be written in the form $\exp(t_1X) \circ \exp(t_2Y) \circ \exp(t_3X) \circ \exp(t_4Y)$ where X and Y are the generators.

2. **Summary of Results.**

NOTATION. $A(2)$ is the complex affine group acting on C^2; $a(2)$ is its Lie algebra.

THEOREM 1. *Let G be a complex connected Lie group of dimension two or three, $[X_1, \cdots, X_k\}$ a minimal generating set for G. Then up to automorphism, we have the table on the following page.*

The proof will be outlined in the next four sections.

3. **Classification of connected two and three-dimensional complex Lie groups.** By methods similar to those described in [7] it is routine to establish the following lemmas.

LEMMA 1. *Every two-dimensional complex Lie algebra is isomorphic to precisely one of the following:*

1. C^2
2. $\alpha z + \beta$*(i.e., the Lie algebra of the complex affine group).*

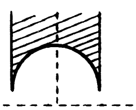
LEMMA 2. *Every three-dimensional complex Lie algebra is isomorphic to precisely one of the following:*

1. C^3
2. $\{\langle A, v\rangle \in a(2) \mid A \in C\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)\}$
3. $\{\langle A, v\rangle \in a(2) \mid A \in C\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)]$
4. $\{\langle A, v\rangle \in a(2) \mid A \in C\left(\begin{smallmatrix}1 & 0\\ 0 & \lambda\end{smallmatrix}\right)\}$ *where λ and $1/\lambda$ give equivalent algebras if $\lambda \neq 0$*
5. $\mathrm{sl}(2, C)$

LEMMA 3. *Let G be a complex Lie group of dimension* 1. *Then*

$$G \cong C,\ C/Z,\ \text{or}\ C/Z + tZ$$

where $t \in$ fundamental region



LEMMA 4. *Let G be a complex Lie group of dimension* 2. *Then G is one of :*

1. *G is abelian (and there exist uncountably many such),*

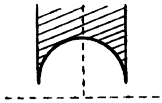
2(a). $$G = \{\langle z_1, z_2\rangle \in C^2 | \langle z_1, z_2\rangle \circ \langle \tilde{z}_1, \tilde{z}_2\rangle = \langle z_1 + \tilde{z}_1, e^{z_1}\tilde{z}_2 + z_2\rangle\}$$

2(b). $$G = \{\langle z_1, z_2\rangle \in C^2 \mid z_1 \neq 0, \langle z_1, z_2\rangle \circ \langle \tilde{z}_1, \tilde{z}_2\rangle = \langle z_1\tilde{z}_1, z_1{}^n\tilde{z}_2 + z_2\rangle\}\ \textit{for}\ n = 1, 2, 3, 4, 5, 6, \cdots.$$

G	X	Y	Z	Order of Generation	Expressions Giving All of g
(1) Abelian, dimension two	Any linearly independent set		—	2	Any
(2) All groups locally isomorphic to $\alpha z + \beta$	$\langle 1, 0\rangle$	$\langle 0, 1\rangle$	—	2	Any
	$\langle 1, 0\rangle$	$\langle 1, \lambda\rangle, \lambda \neq 0$	—	3	Any
(3) Abelian, dimension three	Any linearly independent set			3	Any
(4) $PSL(2, C)$ and $SL(2, C)$	$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$	—	4	Any
	$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	—	4	Any
	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}\lambda & 1\\1 & -\lambda\end{pmatrix} \lambda^2 \neq -1$	—	4	Any
	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 2\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\-2 & -1\end{pmatrix}$	4	None
(5) $\{\langle A, \ell\rangle \in A(2) \mid A = \begin{pmatrix}1 & z\\0 & 1\end{pmatrix}\}/N$	$\langle \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, 0\rangle$	$\langle 0, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	—	4	Any
(6) $\{\langle A, \ell\rangle \in A(2) \mid A = \begin{pmatrix}e^z & ze^z\\0 & e^z\end{pmatrix}\}$	$\langle \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, 0\rangle$	$\langle 0, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	—	4	Any
	$\langle \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, 0\rangle$	$\langle \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	—	3	Any
(7) All groups locally isomorphic to $(\alpha z + \beta) \times C$	$\langle \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0\\\mu\end{pmatrix}\rangle$	$\langle 0, \begin{pmatrix}1\\1\end{pmatrix}\rangle$	—	4	Any
	$\langle \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0\\\mu\end{pmatrix}\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1\\\tau\end{pmatrix}\rangle, \tau \neq \mu$	—	4	Any
	$\langle \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0\\\mu\end{pmatrix}\rangle$	$\langle 0, \begin{pmatrix}1\\0\end{pmatrix}\rangle$	$\langle 0, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	3	Any
	$\langle \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0\\\mu\end{pmatrix}\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0\\\tau\end{pmatrix}\rangle, \tau \neq \mu$	$\langle 0, \begin{pmatrix}1\\0\end{pmatrix}\rangle$	3	Any
	$\langle \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0\\\mu\end{pmatrix}\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1\\\tau\end{pmatrix}\rangle$	$\langle 0, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	4	Any with only one Z
(8) All groups locally isomorphic to $\{\langle A, \ell\rangle \in A(2) \mid A = \begin{pmatrix}e^z & 0\\0 & e^z\end{pmatrix}\}$	$\langle \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, 0\rangle$	$\langle 0, \begin{pmatrix}1\\0\end{pmatrix}\rangle$	$\langle 0, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	3	Any
	$\langle \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, 0\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\rangle$	$\langle 0, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	4	And with only one Z
	$\langle \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, 0\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	4	Any beginning & ending with same letter
(9) All groups locally isomorphic to $\{\langle A, \ell\rangle \in A(2) \mid A = \begin{pmatrix}e^z & 0\\0 & e^{\lambda z}\end{pmatrix}\}$	$\langle \begin{pmatrix}1 & 0\\0 & \lambda\end{pmatrix}, 0\rangle$	$\langle 0, \begin{pmatrix}1\\1\end{pmatrix}\rangle$	—	4	Any
	$\langle \begin{pmatrix}1 & 0\\0 & \lambda\end{pmatrix}, 0\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & \lambda\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\rangle$	—	3	Any
	$\langle \begin{pmatrix}1 & 0\\0 & \lambda\end{pmatrix}, 0\rangle$	$\langle 0, \begin{pmatrix}1\\0\end{pmatrix}\rangle$	$\langle 0, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	3	Any
$\lambda \notin Q$ or if $\lambda \in Q$, $\|\lambda\| \geqq 1$ and $\lambda \notin Z$	$\langle \begin{pmatrix}1 & 0\\0 & \lambda\end{pmatrix}, 0\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & \lambda\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\rangle$	$\langle 0, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	4	Any with only one Z
	$\langle \begin{pmatrix}1 & 0\\0 & \lambda\end{pmatrix}, 0\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & \lambda\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	$\langle 0, \begin{pmatrix}1\\0\end{pmatrix}\rangle$	4	Any with only one Z
(10) All groups locally isomorphic to $\{\langle A, \ell\rangle \in A(2) \mid A = \begin{pmatrix}e^z & 0\\0 & e^{nz}\end{pmatrix}\}$	$\langle \begin{pmatrix}1 & 0\\0 & n\end{pmatrix}, 0\rangle$	$\langle 0, \begin{pmatrix}1\\0\end{pmatrix}\rangle$	$\langle 0, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	3	Any
	$\langle \begin{pmatrix}1 & 0\\0 & n\end{pmatrix}, 0\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & n\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\rangle$	$\langle 0, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	4	Any with only one Z
$n \in Z$, $n \neq 0, 1$	$\langle \begin{pmatrix}1 & 0\\0 & n\end{pmatrix}, 0\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & n\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\rangle$	$\langle 0, \begin{pmatrix}1\\0\end{pmatrix}\rangle$	4	Any with only one Z
	$\langle \begin{pmatrix}1 & 0\\0 & n\end{pmatrix}, 0\rangle$	$\langle 0, \begin{pmatrix}1\\1\end{pmatrix}\rangle$	—	4	Any
$n \neq 2, 3$	$\langle \begin{pmatrix}1 & 0\\0 & n\end{pmatrix}, 0\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & n\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\rangle$	—	4	Any
$n = 2$	$\langle \begin{pmatrix}1 & 0\\0 & n\end{pmatrix}, 0\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & n\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\rangle$	—	4	None
$n = 3$	$\langle \begin{pmatrix}1 & 0\\0 & n\end{pmatrix}, 0\rangle$	$\langle \begin{pmatrix}1 & 0\\0 & n\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\rangle$	—	5	Any

LEMMA 5. *Let G be a complex Lie group of dimension 3. Then G is one of:*

1. *G is abelian (and there exist uncountably many such),*
2. $G = \{\langle (\begin{smallmatrix} 1 & z_1 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} z_2 \\ z_3 \end{smallmatrix}) \rangle\}$ *or* $\{\langle (\begin{smallmatrix} 1 & z_1 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} z_2 \\ z_3 \end{smallmatrix}) \rangle\} / \{\langle I, (\begin{smallmatrix} n \\ 0 \end{smallmatrix}) \rangle \mid n \in Z\}$, *or*
   $G = \{\langle (\begin{smallmatrix} 1 & z_1 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} z_2 \\ z_3 \end{smallmatrix}) \rangle\} / \{\langle I, (\begin{smallmatrix} n_1 + n_2 t \\ o \end{smallmatrix}) \rangle \mid n_1, n_2 \in Z\}$

where $t \in$ fundamental region  ,

3. $$G = \left\{ \left\langle \begin{pmatrix} e^{z_1} & z_1 e^{z_1} \\ 0 & e^{z_1} \end{pmatrix}, \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} \right\rangle \right\},$$

4a. $$G = \left\{ \left\langle \begin{pmatrix} e^{z_1} & 0 \\ 0 & e^{\lambda z} \end{pmatrix}, \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} \right\rangle \right\}, \lambda \notin Q, \lambda \sim 1/\lambda,$$

4b. $G = \{\langle z, v \rangle, z \in C, v \in C^2 \mid \langle z, v \rangle \circ \langle \tilde{z}, \tilde{v} \rangle$
$= \langle z + \tilde{z}, (\begin{smallmatrix} e^z & 0 \\ 0 & e^{\lambda z} \end{smallmatrix}) \tilde{v} + v \rangle)$ *and*
$G = \{\langle z, v \rangle, z \in C, v \in C^2, z \neq 0 \mid \langle z, v \rangle \circ \langle \tilde{z}, \tilde{v} \rangle$
$= \langle z\tilde{z}, (\begin{smallmatrix} z^{qn} & 0 \\ 0 & z^{pn} \end{smallmatrix}) \tilde{v} + v \rangle\}$
for $n = 1, 2, 3, \cdots$ *where* $\lambda = p/q \in Q$, $\lambda \neq 0$, $\lambda \sim 1/\lambda$.

4c. $G_1 \times G_2$ *where G_1 is one of the groups listed in* 2(a) *or* 2(b) *of the two-dimensional theorem and G_2 is one of the groups listed in the one-dimensional theorem.*

5. PSL(2, C) *or* SL(2, C).

LEMMA 6. *Every minimal generating set of a two or three dimensional complex Lie group is equivalent to one listed in Theorem* 1.

4. **Orders of generation for all two-dimensional complex Lie groups.** The abelian case is, of course, trivial. In the non-abelian case, the two-dimensional complex Lie group is locally-isomorphic to the complex affine group and we have:

LEMMA 7. *For all groups G locally-isomorphic to the $\alpha z + \beta$ group we have:*

(a) *if the generators are $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ the order of generation is two and any fixed expression of this length yields all elements of G.*

(b) *if the generators are $\langle 1, 0 \rangle$ and $\langle 1, \lambda \rangle$, $\lambda \neq 0$, the order of generation is three and any fixed expression of this length yields all elements of G.*

PROOF. The details are routine and hence omitted except that it is worth observing that the group element $(-1, -\lambda)$ in group (2b) of Lemma 4 with $n = 1$ cannot be represented as a product of length two in either order for the generators $\langle 1, 0\rangle$ and $\langle 1, \lambda\rangle$. It follows that the order of generation of all groups locally-isomorphic to the complex affine group is at least three in this case.

5. **Orders of generation of PSL(2, C) and SL(2, C).** In the two generator case we obtain exactly one of the canonical forms:

$$\left\{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}; \left\{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\};$$

$$\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}\lambda & 1\\ 1 & -\lambda\end{pmatrix}\right\} \text{ where } \lambda \neq -1;$$

in the three generator case, the only canonical form is:

$$\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\ -2 & -1\end{pmatrix}\right\}$$

Viewing the generators as infinitesimal transformations acting on the Riemann sphere, we can classify the two generator cases as parabolic-parabolic, parabolic-hyperbolic and hyperbolic-hyperbolic while in the three generator case all generators are hyperbolic with each pair having exactly one common fixed point.

In all of the two generator cases, the order of generation is at least four since any Möbius transformation with two fixed points, one in common with each generator, can never be expressed as a product of length three.

LEMMA 8. *The order of generation of both* SL(2, C) *and* PSL(2, C) *in the parabolic-parabolic case is four and any fixed expression of this length yields all elements of G.*

PROOF. Straightforward matrix multiplication shows that the element $\begin{pmatrix}2 & 0\\ 0 & 1/2\end{pmatrix}$ is not achievable as a product of length three (even in PSL(2, C)) while every element in SL(2, C) can be expressed as a product $XYXY$. Since $A \to (-A)^T$ is an automorphism of the algebra, preserving the center, and interchanging the generators, every element in SL(2, C) can also be expressed as a product $YXYX$.

LEMMA 9. *The order of generation of both* SL(2, C) *and* PSL(2, C) *in the hyperbolic-parabolic case is four and any fixed expression of length four yields all elements of G.*

PROOF. First note that as $-I$ is an element of the hyperbolic one-parameter subgroup, the problem for SL(2, C) is completely equivalent to the problem for PSL(2, C); we shall work exclusively in the latter both here and in the proof of the next two lemmas. We have ∞ as the only fixed point of the parabolic generator X and $+1$ and -1 as the fixed points of the hyperbolic generator Y. A Möbius transformation is determined by its action on three points: assume $\alpha \to -1$, $\beta \to 1$, $\gamma \to \infty$. We first seek to take $\alpha \to -1$ and $\beta \to 1$ and then we can apply Y to satisfy the last requirement. Now X takes pairs $(\alpha', \beta') \to (-1, 1)$ if and only if $\beta' - \alpha' = 2$; hence it suffices to prove that we can by a product of length two take $(\alpha, \beta) \to (\alpha', \beta')$ where $\beta' - \alpha' = 2$. Now if $\beta \neq \infty$, first translate β to -1 by X and then apply the hyperbolic generator Y to take $X(\alpha)$ to -3 (if $X(\alpha) = 1$, then it was shown in [3] that there is a product of length two beginning with a translation that takes $(1, -1)$ into (α', β') with $\beta' - \alpha' = 2$; consequently there is a related product using a different translation that takes (α, β) to (α', β').)

If $\beta = \infty$, apply a translation to take $\alpha \to 1$ and then apply the hyperbolic generator to take $\infty \to 3$. This establishes that an arbitrary element of PSL(2, C) can be represented by a fixed expression $YXYX$. Since the order of generation is even, every element can also be represented as $XYXY$ [3].

LEMMA 10. *The order of generation of both* PSL(2, C) *and* SL(2, C) *in the hyperbolic-hyperbolic case is four and any fixed expression of this length yields all elements of* G.

PROOF. We have one parameter subgroups $w = e^t z$ and $(w - \delta_1)/(w - \delta_2) = e^s(z - \delta_1)/(z - \delta_2)$ $(\delta_1 \neq \delta_2;\ \delta_i \neq 0, \infty)$ corresponding to $\exp(tX)$ and $\exp(sY)$ respectively. We proceed as in the proof of Lemma 9. Let M be a Möbius transformation and suppose M maps $\alpha \to 0$, $\beta \to \infty$, $\nu \to \delta_1$. We show that M can be written in the form $XYXY$; it suffices to find a transformation of the form YXY taking $\alpha \to 0$ and $\beta \to \infty$. Applying first a suitable Y, suppose $\alpha \neq 0$ or ∞ and $\beta \neq 0$ or ∞. Let X take (α, β) to (α', β'); in a moment a specific X will be chosen. Let $T(z) = (z - \delta_1)/(z - \delta_2)$; notice that there is an s such that $\exp(sY)$ takes α' to 0 and β' to ∞ if and only if $\beta' = T^{-1}((\delta_2/\delta_1)T(\alpha'))$. But $S(z) = (1/z)T^{-1}((\delta_2/\delta_1)T(z))$ takes every complex value twice counting multiplicity; $S(0) = \infty$ and $S(\infty) = 0$. Suppose $S(\mu) = \beta/\alpha$ where $\mu \neq 0, \infty$. Choose X so $\alpha' = \mu$; then $(1/\alpha')T^{-1}((\delta_2/\delta_1)T(\alpha')) = \beta/\alpha =$ (since X preserves the ratio of any two complex numbers) β'/α', so $\beta' = T^{-1}((\delta_2/\delta_1)T(\alpha'))$ as desired. The proof concludes exactly as that of Lemma 9.

LEMMA 11. *The order of generation of both* PSL(2, *C*) *and* SL(2, *C*) *in the three generator case is four, but no fixed expression of this length yields all elements of G.*

PROOF. The three hyperbolic generators X, Y and Z have respective fixed points 0 and ∞, 1 and ∞ and 0 and 1. Let $T(z)$ be any Möbius transformation, $T(\infty) = \alpha$, $T(0) = \beta$, $T(1) = \gamma$. We shall show below that T can be expressed as a product ZYX if and only if $\alpha \neq 0, 1$ and $\beta \neq 1$. First note that both X and Y leave ∞ fixed; thus Z can be chosen to take $\infty \to \alpha$ provided $\alpha \neq 0, 1$. Since X also leaves 0 fixed, we must choose Y to take $0 \to Z^{-1}(\beta)$; this is possible provided that $Z^{-1}(\beta) \neq 1, \infty$. But since $Z^{-1}(\alpha) = \infty$, we need only require that $Z^{-1}(\beta) \neq 1$, i.e., $\beta \neq 1$. Finally, we require that X takes 1 to $Y^{-1}Z^{-1}(\gamma)$; this is always possible since $Y^{-1}Z^{-1}(\gamma)$ cannot be 0 or ∞. By similar reasoning we can establish the chart below:

ZYX	$\alpha \neq 0, 1,$	$\beta \neq 1$
ZXY	$\alpha \neq 0, 1,$	$\gamma \neq 0$
YZX	$\beta \neq 1, \infty,$	$\alpha \neq 1$
YXZ	$\beta \neq 1, \infty,$	$\gamma \neq \infty$
XYZ	$\gamma \neq 0, \infty,$	$\beta \neq \infty$
XZY	$\gamma \neq 0, \infty,$	$\alpha \neq 0$

It follows that the order of generation is greater than three since if $\alpha = 0$, $\beta = 1$ and $\gamma = \infty$, $T(z)$ cannot be expressed as any of the above products (note any product of length three omitting one of the generators leaves at least one of 0, 1, or ∞ fixed).

For products of length four, we obtain the chart:

$XZYX$	$\alpha \neq 0$
$XYZX$	$\beta \neq \infty$
$YXZY$	$\gamma \neq \infty$
$YZXY$	$\alpha \neq 1$
$ZYXZ$	$\beta \neq 1$
$ZXYZ$	$\gamma \neq 0$

We establish only the first line. Since X leaves 0 fixed and since ZYX could not take ∞ to 0, neither can $XZYX$. To see that $\alpha \neq 0$ is the only restriction on T, note that XZY can represent all Möbius transformations except those for which $\gamma = 0$ or ∞ or $\alpha = 0$. Now first choose X so that $T(X^{-1}(1)) \neq 0, \infty$. Since X leaves ∞ fixed, $T(X^{-1}(\infty)) \neq 0$ so we can express

$$TX^{-1} = XZY$$

or

$$T = XZYX.$$

From the chart it is clear that the order of generation is four. Moreover, no fixed expression in the chart generates all of PSL(2, C); it is routine to verify that neither can any fixed expression in which the same generator is separated by only one other generator, for example $ZXYX$.

6. Groups locally isomorphic to three dimensional subgroups of A(2). All these groups have universal covering group $\tilde{G} = \{\langle z, v\rangle \mid z \in C,\ v \in C^2\}$ with $\langle z, v\rangle \circ \langle w, u\rangle = \langle z + w,\ e^{zA}\, u + v\rangle$ where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, with λ equivalent to $1/\lambda$.

LEMMA 12. *The order of generation of any of the groups* $\{\langle A, \ell\rangle \in A(2) \mid A = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\}/N$ *(the possible discrete subgroups N of the center are listed in* (2) *of lemma* (5)) *with respect to* $X = \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \binom{0}{0}\rangle$, $Y = \langle 0, \binom{0}{1}\rangle$ *is four and any fixed expression of this length yields all elements of G.*

PROOF. The element $\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \binom{\alpha}{0}\rangle$, $\alpha \neq 0$ cannot be expressed as a product of length three (in G/N we have elements of this form not equivalent to any obtainable element). A direct calculation shows that every element of G can be expressed as a product $XYXY$; since 4 is even, the same must be true for the order $YXYX$.

LEMMA 13. *The order of generation of the group* $\{\langle A, \ell\rangle \in A(2) \mid A = \begin{pmatrix} e^z & ze^z \\ 0 & e^z \end{pmatrix}\}$ *with respect to the minimal generating set* $X = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \binom{0}{0}\rangle$, $Y = \langle 0, \binom{0}{1}\rangle$ *is four and any fixed expression of this length yields all elements of G; the order of generation with respect to* $X = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \binom{0}{0}\rangle$, $Y = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \binom{0}{1}\rangle$ *is three and any fixed expression of length three yields all elements of G.*

PROOF. The computation in case (a) is completely routine and hence is omitted. The proof of case (b) is of greater interest because it involves Picard's theorem; here is a sketch. It is routine to show that there is an automophism interchanging the generators, so it suffices to prove that every element can be expressed as a product XYX. The one parameter subgroups are $\exp(tX) = \langle \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}, \binom{0}{0}\rangle$ and $\exp(sY) = \langle \begin{pmatrix} e^s & se^s \\ 0 & e^s \end{pmatrix}, \binom{e^s - se^s - 1}{1 - e^s}\rangle$; a short calculation shows that every element can be written in the form $\exp(tX)\exp(sY)\exp(uX)$ provided the system of equations

$$
(1) \qquad \begin{aligned} ue^u - ve^v &= \alpha \\ e^u - e^v &= \beta \end{aligned}
$$

can always be solved for u and v. The special case $\beta = 0$ is trivial; hence assume $\beta \neq 0$. The second equation in (1) requires that $e^u = \beta + e^v$; substitution in the first equation yields $u = (\alpha + ve^v)/(\beta + e^v)$. Hence we need only establish that there exists $v \in C$ with $e^v \neq -\beta$ and

$$
(2) \qquad e^{(\alpha+ve^v)/(\beta+e^v)} - e^v = \beta.
$$

But $e^v = -\beta$ has discrete solutions and $\alpha + ve^v$ is zero for at most one of these solutions. Hence the function

$$
(3) \qquad f(v) = e^{(\alpha+ve^v)/(\beta+e^v)} - e^v
$$

has a discrete set of essential singularities. By Picard's theorem, in any deleted neighborhood of any one of these essential singularities, say z_0, $f(v)$ omits at most one value. We wish to demonstrate that the one (possible) exceptional value is not β. If it were, then the function

$$
(4) \qquad g(v) = (\beta + e^v)e^{-(\alpha+ve^v)/(\beta+e^v)},
$$

which also has an essential singularity at z_0, would omit in that same deleted neighborhood the value one; however, $g(v)$ clearly already omits the value zero in any sufficiently small deleted neighborhood of z_0 (one which contains none of the zeros of $\beta + e^v$). This proves Lemma 13.

The remaining Lie groups are locally isomorphic to $\{\langle A, \ell\rangle \in A(2) | A = \left(\begin{smallmatrix} e^z & 0 \\ 0 & e^{\lambda z}\end{smallmatrix}\right)\}$ for suitable λ. We must consider separately the cases $\lambda = 0$, $\lambda = 1$, $\lambda = 2$, $\lambda = 3$, $\lambda \in Z - \{0, 1, 2, 3,\}$, $\lambda \in Q - Z$, $\lambda \notin Q$.

LEMMA 14. *For $\lambda = 0$, the orders of generation are as listed in Theorem* 1.

PROOF. If the generators are $X = \langle\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ \mu\end{smallmatrix}\right)\rangle$ and $Y = \langle 0, \left(\begin{smallmatrix} 1 \\ 1\end{smallmatrix}\right)\rangle$, then an element in the center of the universal covering group, $\{\langle 2\pi in, \left(\begin{smallmatrix} 0 \\ t\end{smallmatrix}\right)\rangle \mid n \in Z, t \in C\}$, can be written as a product of length three only if $t = \mu(2\pi in)$. Hence modulo any discrete subgroup of the center there are elements $\langle 0, \left(\begin{smallmatrix} 0 \\ t_0\end{smallmatrix}\right)\rangle$ unobtainable as a product of length three and equivalent only to other unobtainable elements. So the order of generation of $\tilde{G}/N$ is at least four; a routine calculation shows that any fixed expression of length four yields all of $\tilde{G}$. If the generators are $X = \langle\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ \mu\end{smallmatrix}\right)\rangle$ and $Y = \langle\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ \tau\end{smallmatrix}\right)\rangle$ with $\mu \neq \tau$, the same argument works; this time an element of the center can be written as a product of length 3 only if $t = \mu(2\pi im) + \tau(2\pi in)$ for integers m and n.

The three generator cases in which the order of generation is three are easily disposed of. In the remaining three generator case, a simple application of Lemma 7 shows that the order of generation is at most 4; it is exactly 4 because $\langle \pi i, \binom{-1}{0} \rangle$ and all elements equivalent to it cannot be obtained as a product of length three.

Lemma 15. *For $\lambda = 1$, the orders of generation are as listed in Theorem* 1.

Proof. The first case is trivial and the second involves an argument identical to that of Lemma 14. Consider the final case. No element equivalent to $\langle \pi i, \binom{-1}{-1} \rangle$ can be written as a product of length 3, so the order of generation is at least 4. A short calculation shows that every element can be written in the form $XYZX$. Elements equivalent to $\langle \pi i, \binom{-1}{-1} \rangle$ cannot be written in the form $XYXZ$ or $YXZX$. The lemma follows since automorphisms exist permuting X, Y and Z.

Lemma 16. *For $\lambda \neq 0, 1$, when there are three generators, the orders of generation are as listed in Theorem* 1.

Proof. The proof is analogous to the proof of the two previous lemmas and hence is omitted.

Lemma 17. *For $\lambda \neq 0, 1$, the order of generation with respect to $X = \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right), 0 \rangle$ and $Y = \langle 0, \binom{1}{1} \rangle$ is four and any such expression generates.*

Proof. It is routine to check that every element can be written in the form $XYXY$, so the order of generation is at most 4. A short calculation shows that no element of the form $\langle t_n, \binom{1}{0} \rangle$ for $t_n = 2\pi i n/(\lambda - 1)$ can be given by an expression of length three. If $\lambda \notin Q$, the center of the universal covering group is trivial and the order of generation is thus at least 4. If $\lambda = a/b$ in lowest terms, the center of the universal covering group is $\{2\pi i b k \mid k \in Z\}$ and the unobtainable elements $\langle t_n, \binom{1}{0} \rangle$ described above are equivalent only to other unobtainable elements because $(2\pi i n/(\lambda - 1)) + 2\pi i b k = (2\pi i/(\lambda - 1))\,(n + ka - kb)$.

Lemma 18. *For $|\lambda| \geqq 1$, $\lambda \notin Z$, the order of generation with respect to $X = \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right), 0 \rangle$ and $Y = \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right), \binom{1}{1} \rangle$ is three and any such expression generates.*

Proof. We can find an automorphism interchanging X and Y, so it suffices to show that every element of the universal covering group can be written in the form XYX. Thus we must solve the equations

$$(5) \qquad \begin{aligned} e^u - e^v &= \alpha \\ e^{\lambda u} - e^{\lambda v} &= \beta \end{aligned}$$

for u and v. If $\alpha = 0$ or $\beta = 0$ we can explicitly solve, so suppose $\alpha\beta \neq 0$. An easy simplification shows that we need only consider the case $\alpha = 1$. Thus it suffices to find v such that $1 + e^v \neq 0$ and one of the values of $(1 + e^v)^\lambda - e^{\lambda v}$ equals β. Replace v by $v - 2\pi i n$ and $(1 + e^v)^\lambda$ by $(1 + e^v)^\lambda e^{-2\pi i n\lambda}$; then it suffices to find v with $1 + e^v \neq 0$ and $(1 + e^v)^\lambda - e^{\lambda v} = \beta e^{2\pi i \lambda n}$. If the imaginary part of λ is not zero, $|\beta e^{2\pi i\lambda n}|$ can be made arbitrarily small; in this case let $f(v) = (1 + e^v)^\lambda - e^{\lambda v}$ and notice that it suffices by the inverse function theorem to find v_0 with $f(v_0) = 0$ and $f'(v_0) \neq 0$. But a simple calculation shows that $f(v_0) = 0$ implies $f'(v_0) \neq 0$, so it suffices to show that f has a zero, which is the case $\beta = 0$ already disposed of.

Next assume that λ is real but irrational. If there exists a $\beta_0 \neq 0$ for which $f(v) = \beta_0$ has no solution, then we cannot solve $f(v) = \beta$ for β comprising a dense subset of the circle whose radius is $|\beta_0|$. But $f(v)$ is a non-constant holomorphic (and hence open) map from the universal covering of $C - \{w \mid e^w = -1\}$ to the plane so the whole circle $|z| = |\beta_0|$ must be omitted from the range. Since $\beta = 0$ is in the range, it suffices to show that arbitrarily large values of β are in the range to obtain a contradiction.

If $\lambda < 0$, let $v = ix$ for real x and let $x \to \pi$; then $|f(v)| \to \infty$. If $\lambda > 0$, we may assume $\lambda > 1$; let $v = x$ and define $t = e^x$; then $f(v) = (1 + t)^\lambda - t^\lambda$ approaches ∞ as t approaches ∞.

Next consider the case $\lambda \in Q$, $\lambda > 1$, $\lambda \notin Z$. Write $\lambda = m/n$, $m > n > 1$ with $\gcd(m, n) = 1$. Letting $z = e^v$ we wish to prove that for $\beta \neq 0$ we can find $z \neq 0, -1$ such that

$$(1 + z)^{m/n} - z^{m/n} = \beta \tag{6}$$

If w is any n^{th} root of z, this becomes

$$(1 + w^n)^m = (\beta + w^m)^n \tag{7}$$

The w^{nm} terms cancel; the second highest power on the left side is $w^{n(m-1)}$ while on the right side it is $w^{m(n-1)}$. Hence the polynomial equation has (counting multiplicity) $n(m - 1)$ solutions. We must check that some of these solutions satisfy $w \neq 0$, $(-1)^{1/n}$. If $w = 0$ or $(-1)^{1/n}$, $\beta^n = \pm 1$.

Case $\beta^n = -1$: Then $w = 0$ is not a solution. Since $\gcd(m, n) = 1$, there is at most one n^{th} root of -1 whose m^{th} power is $-\beta$. Denote it by w_0. We need only demonstrate that the polynomial $(1 + w^n)^m - (\beta + w^m)^n$ is not a constant times $(w - w_0)^{n(m-1)}$. But $w - w_0$ is a factor of $1 + w^n$ and $\beta + w^m$ exactly once and $n < m$, so it is a factor of $(1 + w^n)^m - (\beta + w^m)^n$ exactly n times. If $n = n(m - 1)$, $m = 2$ and $\lambda = m/n = 2$.

Case $\beta^n = 1$: Zero is an n-fold root; $n < n(m-1)$ unless $m = 2$ and $\lambda = 2$. As before there is at most one n^{th} root w_0 of -1 whose m^{th} power is $-\beta$. We must rule out the possibility that $(1 + w^n)^m - (\beta + w^m)^n$ is a multiple of $w^n(w - w_0)^{n(m-2)}$. But $w - w_0$ is a factor of $1 + w^n$ and $\beta + w^m$ exactly once and $n < m$, so it is a factor of $(1 + w^n)^m - (\beta + w^m)^n$ exactly n times, so $n = n(m-2)$ and $\lambda = 3$.

The case $\lambda \leqq -1$, $\lambda \in Q$, $\lambda \notin Z$ remains. Write $\lambda = m/n$, $1 < n < -m$ with $\gcd(m, n) = 1$. Let $p = -m$; we must solve the polynomial equation

$$w^{pn} - (\beta w^p + 1)^n (1 + w^n)^p = 0 \tag{8}$$

for $w \neq 0$, $w \neq (-1)^{1/n}$. But the polynomial has degree $2pn$ and has constant term -1; if $w_0{}^n = -1$, w_0 is not a root. This completes the proof of the lemma.

LEMMA 19. *For $\lambda \neq 0, 1$ and $\lambda \in Z$, the order of generation with respect to $X = \langle (\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}), 0 \rangle$ and $Y = \langle (\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}), (\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}) \rangle$ is as described in Theorem* 1.

PROOF. The order of generation is at least four because no product of length three yields the elements $\langle 2\pi i n, (\begin{smallmatrix} 0 \\ \beta \end{smallmatrix}) \rangle$, $\beta \neq 0$, in $\tilde{G}$. On the other hand, if $\alpha \neq 0$ $\langle \tau, (\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}) \rangle$ can be written in the form XYX provided equations (5) in Lemma 18 can be solved; the arguments in the proof of Lemma 18 show that these equations can be solved when $\lambda \neq 2, 3$. Hence the order of generation is four when $\lambda \neq 2, 3$ because $\langle t, (\begin{smallmatrix} 0 \\ \beta \end{smallmatrix}) \rangle$ can be written in the form $XYXY$; simply apply Y to obtain $\langle \tilde{t}, (\begin{smallmatrix} \tilde{\alpha} \\ \beta \end{smallmatrix}) \rangle$ with $\tilde{\alpha} \neq 0$.

If $\lambda = 2$, elements $\langle t, (\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}) \rangle$ in $\tilde{G}$ can be written in the form XYX if and only if

$$\begin{aligned} e^u - e^v &= \alpha \\ e^{2u} - e^{2v} &= 2\beta \end{aligned} \tag{9}$$

can be solved; using $e^{2u} - e^{2v} = (e^u - e^v)(e^u + e^v)$, it is easy to see that this is possible if and only if $\alpha = \beta = 0$ or $\alpha \neq 0$ and $\beta \neq \pm\, \alpha^2/2$. In particular $\langle t, (\begin{smallmatrix} -1 \\ -1/2 \end{smallmatrix}) \rangle$ cannot be written in the form XYX; since Y takes this element to $\langle \tilde{t}, (\begin{smallmatrix} -1 \\ -1/2 \end{smallmatrix}) \rangle$ it cannot be written in the form $XYXY$ and its inverse cannot be written in the form $YXYX$. It is easy to see, however, that $\langle t, (\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}) \rangle$ can be written in the form $XYXY$ if $\alpha \neq -1$ or $\beta \neq -1/2$. In the exceptional case, the inverse of $\langle t, (\begin{smallmatrix} -1 \\ -1/2 \end{smallmatrix}) \rangle$ does not have the same form, and so can be written in the form $XYXY$, so $\langle t, (\begin{smallmatrix} -1 \\ -1/2 \end{smallmatrix}) \rangle$ can be written in the form $YXYX$.

If $\lambda = 3$ the elements of $\tilde{G}$ with $\beta = \alpha^3/3$ cannot be obtained as a product XYX. Arguing as before we can show that all elements of $\tilde{G}$ can be expressed as a product $XYXY$ with the exception of those where $\alpha = -1$, $\beta = -1/3$. Now the inverse of the element $\langle t, \binom{-1}{-1/3}\rangle$ is $\langle -t, \binom{e^{-t}}{e^{-3t}/3}\rangle$; all of these inverses can be expressed as a product $XYXY$ with the exception of those for which $e^{-t} = -1$, i.e., $t = (\pi + 2n\pi)i$, $n \in Z$. Hence the element $\langle \begin{pmatrix}-i & 0\\ 0 & -1\end{pmatrix}, \binom{-1}{-1/3}\rangle$ of $\tilde{G}/\text{center}$ cannot be written as a product of length four in either order. Hence the order of generation is at least 5; equality follows from the fact that the unobtainable elements of $\tilde{G}$ are not invariant under either X or Y.

7. **Observations.** In contrast to real Lie groups of dimension three, complex Lie groups of that dimension have finite order of generation (contrast with PSL(2, R)) and none has an unbounded set of orders of generation (contrast with SO(3)). Moreover, the order of generation in the complex case depends only on the local isomorphism class (contrast with sl(2, R)). The applicability of the heavy machinery of function theory to complex Lie groups makes treatments of such groups significantly simpler than the treatment of "comparable" real Lie groups.

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Department of Mathematics, University of Oregon, Eugene, OR 97403

Department of Mathematics, University of Wisconsin-Parkside, Kenosha, WI 53140