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THE N-TH ORDER ELLIPTIC BOUNDARY PROBLEM FOR NONCOMPACT BOUNDARIES

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0. **Introduction.** In this paper we will discuss the boundary problem

(p) $u \in \mathfrak{H}_N(\Omega)$, $\langle a \rangle u = f$ on $\Omega \cup \Gamma$, $\langle b^j \rangle u = \varphi_j$ on Γ, $j = 1, \ldots, r$,

where Ω and Γ are chosen as the halfspace $\mathbf{R}_+^{n+1}$ and its boundary $\partial\mathbf{R}_+^{n+1} = \mathbf{R}^n$, as the simplest domain with noncompact boundary, while $\langle a \rangle$ denotes an even order elliptic differential expression over $\Omega \cup \Gamma$, with C^∞-coefficients. The order of $\langle a \rangle$ is N, and there are $r = N/2$ boundary conditions on Γ, determined by differential expressions $\langle b^j \rangle$ over a neighbourhood of Γ, of order $N_j < N$. The $\langle b^j \rangle$ again have C^∞-coefficients, and the system $\langle a \rangle, \langle b^1 \rangle, \ldots, \langle b^r \rangle$ locally is assumed to be elliptic -or, in other words, the $\langle b^j \rangle$ satisfy the so-called Lopatinskij-Shapiro conditions, locally, at each point of Γ.

Conditions at infinity will have to be added, of course, and we generally assume that $f \in L^2(\Omega)$, $\varphi_j \in \mathfrak{H}_{N-N_j-1/2}(\Gamma)$, with the L^2-Sobolev spaces. In fact our assumptions will restrict enough to imply the generalized Sobolev estimates of Agmon-Douglis-Nirenberg [1], and *F.* Browder [4]. However, since the domain and its boundary are non-conpact, these do not imply finiteness of the nullspace or even normal solvability of the problem.

Our result, below, just asserts this normal solvability of (p), replacing in its proof the apriori estimate by a Banach algebra technique, under the following assumptions on the coefficients.

$$
\begin{aligned}
&\langle a \rangle = a(x, D),\ \langle b^j \rangle = b^j(\vec{x}, D),\\
&a(x, \xi) = \sum_{|\alpha| \leq N} a_\alpha(x)\xi^\alpha,\ b^j(\vec{x}, \xi) = \sum_{|\alpha|+k \leq N_j} b^j_{k,\alpha}(\vec{x})\vec{\xi}^\alpha \xi_0^k,\\
&x = (x_0, \ldots, x_n) = (x_0, \vec{x}),\ D = (D_0, \ldots, D_n) = (D_0, \vec{D}),\\
&\xi = (\xi_0, \ldots, \xi_n) = (\xi_0, \vec{\xi}),\ \Omega = \{x : x_0 > 0\},\ \Gamma = \{x : x_0 = 0\}\\
&\lim_{\rho > 0,\, \rho \to \infty} a_\alpha(\rho x) = a_\alpha(\infty x), \text{ as } |x| = 1, x_0 \geqq 0,\\
&\lim_{\rho > 0,\, \rho \to \infty} b^{j(\beta)}_{k,\alpha}(\rho \vec{x}) = b^{j(\beta)}_{k,\alpha}(\infty \vec{x}), \text{ as } |\vec{x}| = 1,
\end{aligned}
\tag{0.1}
$$

where the convergence of the limits in the last row is uniform in x (or $\vec{x}$), and the limits $a_\alpha(\infty \cdot x)$, $b^{j(\beta)}_{k,\alpha}(\infty \cdot \vec{x})$ are continuous over the half sphere and its boundary, for all β.

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THEOREM 0.1. *Under the assumptions* (0.1) *problem* (p) *will be normally solvable if* $\langle a\rangle$ *and* $\langle b^j\rangle$ *are* md-*elliptic, a sharpening of the above condition of ellipticity* (*Lopatinskij-Shapiro*).

$\langle a\rangle$ is called md-*elliptic* (of order N) if

$$(0.2)\quad 0 < c \leqq |a(x, \xi)|(1 + \xi^2)^{-N/2}, \text{ for } x \in \Omega,\ \xi \in \mathbf{R}^{n+1},\ |x| + |\xi| \geqq \delta_0,$$

with c independent of x, ξ, and $\delta_0 > 0$ sufficiently large.

For md-elliptic $\langle a\rangle$, the system $\langle b^j\rangle$, $j = 1, \dots, r = N/2$ of boundary expressions is called md-elliptic (of orders (N_j)), if for all $x = (0, \vec{x}) \in \Gamma$ and all $\xi = (\eta, \vec{\xi}) \in \mathbf{R}^{n+1}$ with $|\vec{x}| + |\vec{\xi}| \geqq \delta_0$ (sufficiently large), the polynomials

$$(0.3)\quad \begin{aligned} p(\eta) &= p_{\vec{x}, \vec{\xi}}(\eta) = (1 + \vec{\xi}^2)^{-N/2}\, a(x, \eta(1 + \vec{\xi}^2)^{1/2}, \vec{\xi}) \\ q_j(\eta) &= q_{j, \vec{x}, \vec{\xi}}(\eta) = (1 + \vec{\xi}^2)^{-N_j/2}\, b_j(\vec{x}, \eta(1 + \vec{\xi}^2)^{1/2}, \vec{\xi}),\ j = 1, \dots, r, \end{aligned}$$

satisfy the following conditions.

(i) $p(\eta)$ (which cannot have real roots due to (0.2)) has precisely $r = N/2$ roots, counting multiplicities, in the complex upper half plane $\operatorname{Im}\eta > 0$, denoted by $\eta_1^+, \dots, \eta_r^+$.

(ii) Let $p^+(\eta) = \Pi_{l=1}^r(\eta - \eta_l^+)$; then the r polynomials $q_j(\eta)$, $j = 1, \dots, r$, are linearly independent modulo $p^+(\eta)$, and uniformly so, in $\vec{x}$, $\vec{\xi}$. That is, if $r_j(\eta) = \sum_{l=0}^{r-1} r_{jl}\eta^l$ denote the remainders of q_j, modulo p^+, then we have $0 < c \leqq |\det((r_{jl}))|$, with c independent of $\vec{x}$, $\vec{\xi}$.

The proof will use *a* 'comparison' with a simpler problem (p_0), discussed in §2, with the same boundary conditions, but a different (constant coefficients) equation $\langle a_0\rangle\, u = f$. Problem (p_0) is easily reduced to a system of singular integral equations over $\mathbf{R}^n$, to which **[8]** provides necessary and sufficient criteria. This uses more or less standard techniques, involving the tangential Fourier transform. On the other hand, our comparison between (p) and (p_0) reduces Theorem 0.1 to the discussion of a certain operator A (§4). (p) is normally solvable if and only if A is Fredholm. Moreover A belongs to a C^* algebra $\mathfrak{A}$ investigated in **[5]**, in particular in view of its Fredholm theory.

The algebra $\mathfrak{A}$ is formed by means of operators relating to the Dirichlet and Neumann problem of the Laplace operator only, except for multiplications by continuous functions. To construct a Fredholm inverse of A will require the investigation of two symbols, σ and τ, where τ is matrix-valued. The inversion of τ requires a somewhat lengthy calculation (§4).

In §1 we define the basic concepts, and recall some results required from earlier papers. In §3 we study the action of an $(n + 1)$-dimensional singular integral operator on the symbol τ. This perhaps might be mentioned as an interesting addendum to the theory of the algebra $\mathfrak{A}$ in **[5]**.

1. **Preparations and Notations.** It will be convenient to use the double

notation $x = (x_0, x_1, \ldots, x_n) = (x_0, \vec{x}) = (y, \vec{x})$, with $y \in \mathbf{R}$, $\vec{x} \in \mathbf{R}^n$, for vectors $x \in \mathbf{R}^{n+1}$ (Similarly $\xi = (\xi_0, \vec{\xi}) = (\eta, \vec{\xi})$). Let $\mathbf{R}^{n+1}_+ = \{(y, \vec{x}) : y > 0\}$ and $\partial\mathbf{R}^{n+1}_+ = \{(y, \vec{x}) : y = 0\}$.

For $m = 1, 2, \ldots$ we denote by $\mathbf{B}^m$ the smallest compactification of $\mathbf{R}^m$ such that the functions $x_j(1 + x^2)^{-1/2}$ extend continuously, and $\mathbf{H}^{n+1}$ will denote the closure of $\mathbf{R}^{n+1}_+$ in $\mathbf{B}^{n+1}$.

We define the differential operator $D = (D_0, \vec{D})$, with $\vec{D} = (D_1, \ldots, D_n)$, and the multiplication operator $M = (M_0, \vec{M})$, with $\vec{M} = (M_1, \ldots, M_n)$, by setting $D_j u = -i\partial u/\partial x_j$, and $(M_j u)(x) = x_j u(x)$, $j = 0, \ldots, n$. The Fourier transform with respect to the $\vec{x}$-variables will be denoted by $\vec{F}$, (c.f, [5, (3.1)]), over $\mathbf{R}^{n+1}$, $\mathbf{R}^{n+1}_+$, or $\mathbf{R}^n = \partial\mathbf{R}^{n+1}_+$.

We will mainly be working in the following Hilbert spaces: $\mathfrak{H} = L^2(\mathbf{R}^{n+1}_+)$, $\mathfrak{h} = L^2([0, \infty))$, $\mathfrak{k} = L^2(\mathbf{R}^n)$, $\mathfrak{K} = L^2(\mathbf{R}^{n+1})$. For any Hilbert or Banach space X the algebra of continuous linear operators and the ideal of compact operators will be denoted by $\mathscr{L}(X)$, and $\mathfrak{C}(X)$, respectively. In particular we shall use a pair of C^*-operator algebras, denoted by $\mathfrak{A}^m_0, m = 1, 2, \ldots$, and $\mathfrak{A}$, as previously discussed in [8] and in [5], respectively. In particular $\mathfrak{A}^m_0 \subset \mathscr{L}(L^2(\mathbf{R}^m))$ is generated by the multiplications $a(M)$ with $a \in C(\mathbf{B}^m)$, and the convolutions $\Lambda = (1 - \Delta)^{-1/2}$ and $S_j = D_j\Lambda$, $j = 1, \ldots, m$. Also, $\mathfrak{A} \subset \mathscr{L}(\mathfrak{H})$ is generated by the multiplications $a(M)$, with $a \in C(\mathbf{H}^{n+1})$, and the operators $\Lambda_d = (1 - \Delta_d)^{-1/2}$, $\Lambda_n = (1 - \Delta_n),^{-1/2}$ $S_{j,d} = D_j\Lambda_d$, $S_{j,n} = D_j\Lambda_n$, $j = 0, 1, \ldots, n$. In each case we mean the smallest operator norm closed algebra containing the generators. Δ_d, Δ_n mean the Laplace operator with Dirichlet and Neumann conditions, respectively.

In [8] and [5] the structure of $\mathfrak{A}^m_0$ and $\mathfrak{A}$ was discussed. In particular, $\mathfrak{A}^m_0/\mathbf{C}(L^2(\mathbf{R}^m))$ is isometrically isomorphic to the algebra $\mathfrak{C}(\mathbf{M}^m_0)$ of continuous complex-valued functions over the compact Hausdorff space $\mathbf{M}^m_0 = \partial(\mathbf{B}^m \times \mathbf{B}^m) = \mathbf{B}^m \times \mathbf{B}^m - \mathbf{R}^m \times \mathbf{R}^m$. The continuous function assigned to A is denoted by σ^m_A and called the symbol of A. If $\mathfrak{E}$ denotes the commutator ideal of $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{E} \simeq C(\mathbf{M})$, where the compact Hausdorff space $\mathbf{M}$ consists of the 'upper half' $\{(x, \xi) \in \partial(\mathbf{B}^{n+1} \times \mathbf{B}^{n+1}) : y \geqq 0\}$ of the space M^{n+1}_0, augmented as follows: A space $\mathbf{B}^n \times \{-\infty \leqq t \leqq +\infty\}$ is attached in such a way, that the points $(\vec{x}, \pm\infty)$ are identified with the points $((0, \vec{x}), \infty(\pm 1, 0))$. For details c.f. [5]. We get $\mathfrak{E} = \mathfrak{C}$ for $n = 0$, but for $n > 0$ the ideal $\mathfrak{C}$ is properly contained in $\mathfrak{E}$. In fact, then we have

$$\mathfrak{E} = U^*(\mathfrak{C}(\mathfrak{h}) \hat{\otimes} \mathfrak{A}^n_0)U, \tag{1.1}$$

with the unitary operator $U = T\vec{F}$, where

$$(Tu)(y, \vec{x}) = \lambda^{1/2}(\vec{x})u(\lambda(\vec{x})y, \vec{x}), \lambda(\vec{x}) = (1 + \vec{x}^2)^{-1/2}, \tag{1.2}$$

and where the tensor decomposition is understood with respect to the decomposition $\mathfrak{H} = \mathfrak{h} \hat{\otimes} \mathfrak{k}$. In particular this implies that

$$(1.3) \qquad \mathfrak{E}/\mathfrak{C}(\mathfrak{h}) \cong C(\mathbf{M}_0^n, \mathfrak{C}(\mathfrak{h})),$$

with the algebra of continuous functions from $\mathbf{M}_0^n$ to $\mathfrak{C}(\mathfrak{h})$. This provides the operators $E \in \mathfrak{E}$ with a compact operator valued symbol (mod $\mathfrak{C}$), the function $\tau_E: \mathbf{M}_0^n \to \mathfrak{C}(\mathfrak{h})$ associated to the coset $\{E + \mathfrak{C}\}$ by (1.3). Similarly the symbol $\sigma_A: \mathbf{M} \to \mathbf{C}$ is introduced for $A \in \mathfrak{A}$ as the function associated to the coset $\{A + \mathfrak{E}\}$.

An operator $A \in \mathfrak{A}_0^m$ is Fredholm if and only if $\sigma_A^m \neq 0$ on $\mathbf{M}_0^m$. An operator $A \in \mathfrak{A}$ is Fredholm if and only if (i) $\sigma_A \neq 0$ on $\mathbf{M}$, and if then $B \in \mathfrak{A}$ can be found with $1 - AB = E$, $1 - BA = E' \in \mathfrak{E}$ such that (ii) $1 - \tau_E$ and $1 - \tau_{E'}$ are invertible in $\mathscr{L}(\mathfrak{h})$ for all points of $\mathbf{M}_0^n$. The symbols usually can be easily calculated (c.f. (1.9) or (1.14)).

It will be of importance that we have

$$(1.4) \qquad (1 - \vec{\Delta})^{-s} a(\vec{M})(1 - \vec{\Delta})^s - a(\vec{M}) \in \mathfrak{C}(\mathfrak{k}), s \in \mathbf{R},$$

for every bounded $a \in C^\infty(\mathbf{R}^n)$ with all derivatives tending to 0 at infinity, where $\vec{\Delta}$ denotes the tangential Laplace operator, interpreted as self-adjoint operator of $\mathfrak{k}$ (c.f. **[6]**, **[7]**).

Also we shall have to use the L^2-Sobolev spaces (for $k = 1, 2, \ldots$)

$$(1.5) \qquad \mathfrak{H}_k(\mathbf{R}_+^{n+1}) = \{u \in \mathfrak{H}: u = v \mid \mathbf{R}_+^{n+1},\ v \in \mathfrak{H}_k(\mathbf{R}^{n+1})\},$$

and $\mathfrak{H}_s(\partial\mathbf{R}_+^{n+1}) = \mathfrak{H}_s(\mathbf{R}^n)$, $s \in \mathbf{R}$, where $\mathfrak{H}_s(\mathbf{R}^m)$ is defined as usual (**[6]**, **[7]**). Evidently $D^\alpha u \in \mathfrak{H}_{k-|\alpha|}$, for $u \in \mathfrak{H}_k$, $|\alpha| \leqq k$. For $u \in \mathfrak{H}_k(\mathbf{R}_+^{n+1})$ the 'boundary-functions' $D^\alpha u \mid \partial\mathbf{R}_+^{n+1} = u_\alpha$ are meaningfully defined and in $\mathfrak{H}_{k-|\alpha|-1/2}(\partial\mathbf{R}_+^{n+1})$, for $|\alpha| < k$, by the 'trace theorem' (c.f. **[10]**).

By a calculation it is found that

$$(1.6) \qquad \mathfrak{H}_k(\mathbf{R}_+^{n+1}) = \{u \in \mathfrak{H}: \lambda^{-k}(\vec{M})Uu \in \mathfrak{h}_k \hat{\otimes} \mathfrak{k}\}.,$$

with λ and U as in (1.1), (1.2), and $\mathfrak{h}_k = \mathfrak{H}_k(\mathbf{R}_+)$, $\mathbf{R}_+ = \mathbf{R}_+^1$. Again, for $v \in \mathfrak{h}_k \hat{\otimes} \mathfrak{k}$ the restrictions $(D_0^l v) \mid \partial\mathbf{R}_+^{n+1}$ are meaningful and in $\mathfrak{k}$, for $l < k$. For $u \in \mathfrak{H}_k(\mathbf{R}_+^{n+1})$ let $\tilde{u} = \lambda^{-k}(\vec{M})Uu$.

$$(1.7) \quad \vec{F}\{(D^\alpha D_0^l u) \mid \partial\mathbf{R}_+^{n+1}\} = \lambda^{k-1-1/2}(\vec{M})\vec{M}^\alpha\{(D_0^l \tilde{u}) \mid \partial\mathbf{R}_+^{n+1}\}, |\alpha| + l \leqq k,$$

as again follows by a calculation.

A *Fredholm inverse* of an operator A is an operator B such that $1 - AB$ and $1 - BA$ have finite rank. It will exist if and only if A is Fredholm and so will a *special Fredholm inverse*, defined by

$$(1.8) \qquad AB = P_{\mathrm{im}\,A},\ BA = 1 - P_{\ker A}.$$

with $P_{\mathscr{X}}$ denoting a continuous projection onto $\mathscr{X}$. If $A \in \mathfrak{A}_0^m$ even is a finite sum of finite products of $a(M)$, with $a^{(\beta)} \in C(\mathbf{B}^m)$, for all β, and factors Λ, S_j, and has symbol $\neq 0$ on $\mathbf{M}_0^m$ then a *special Green inverse* exists. That is, in addition to (1.8) the operator B maps $\mathfrak{H}_s(\mathbf{R}^m)$ to $\mathfrak{H}_s(\mathbf{R}^m)$, for all $s \in \mathbf{R}$, and $1 - AB$ and $1 - BA$ map $\mathfrak{H}_{-\infty}(\mathbf{R}^n)$ to $\mathfrak{H}_\infty(\mathbf{R}^m)$. Using the Fourier transform $\mathfrak{H}_\infty$ may be replaced $\mathfrak{k}_\infty = \bigcap_s L^2(\mathbf{R}^m, (1 + x^2)^s dx)$. Similar remarks hold for square matrices $((A_{jk}))$, $A_{jk} \in \mathfrak{A}_0^m$

Finally, in this section, we provide a list of symbols of the generators of $\mathfrak{A}_0^m$ and $\mathfrak{E}$, and a few further basic facts about $\mathfrak{A}$ and $\mathfrak{E}$, all directly taken from **[8]** and **[5]**. For $\mathfrak{A}_0^m$ over $\mathbf{R}^m$ we let $\mathbf{M}_0^m = \partial(\mathbf{B}^m \times \mathbf{B}^m) = \{(z, \zeta): z, \zeta \in \mathbf{B}^m, |z| + |\zeta| = \infty\}$. For $a(M)$ with $a \in C(\mathbf{B}^m)$, and $\Lambda = (1 - D^2)^{-1/2}$, $S_j = D_j\Lambda$, we let

$$\sigma^m_{a(M)}(M)(z, \zeta) = a(z), \sigma^m_\Lambda(z, \zeta) = (1 + \zeta^2)^{-1/2}, \quad \sigma^m_{S_j}(z, \zeta) = \zeta_j(1 + \zeta^2)^{-1/2}, \tag{1.9}$$

which uniquely determines the homeomorphism between $\mathbf{M}_0^m$ and $\partial(\mathbf{B}^m \times \mathbf{B}^m)$.

The following facts about the generators of $\mathfrak{A}$ will be used. For multiplications $a(M)$, with $a \in C(\mathbf{H}^{n+1})$, and $E \in \mathfrak{E}$ we have

$$(a(M) - a_0(\vec{M}))E \in \mathfrak{C}(\mathfrak{H}), \text{ with } a_0(\vec{x}) = a(0, \vec{x}). \tag{1.10}$$

Also, for $a_0 \in C(\mathbf{B}^n)$ and $K \in \mathfrak{C}(\mathfrak{h})$ and $U = TF$, as above, we get

$$U(1 \otimes a_0(\vec{M}))U^*(K \otimes 1) - K \otimes a_0(-\vec{D}) \in \mathfrak{C}(\mathfrak{H}), \tag{1.11}$$

(c.f. **[5**, Lemmas 3.7 and 3.8**]**). Furthermore, the operators UAU^*, for all the other generators of $\mathfrak{A}$, are explicitly calculated as follows:

$$U\Lambda_\gamma U^* = Q_\gamma \otimes \lambda(\vec{M}), US_{0,\gamma}U^* = P_\gamma \otimes 1, US_{j,\gamma}U^* = Q_\gamma \otimes M_j\lambda(\vec{M}), \quad j = 1, \ldots, n, \gamma = d, n, \tag{1.12}$$

where Q_γ and P_γ denotes the operators Λ_γ and $S_{0,\gamma}$ for $n = 0$, respectively, acting on the space $\mathfrak{h}$. The symbols of the generators of $\mathfrak{A}$ will not be stated explicitly, since they will not be used. Let it be mentionned that (1.12) may serve for their calculation.

For the calculation of the symbol τ_E of $E \in \mathfrak{E}$ we use (1.1) : For $E = U^*(F \otimes A)U \in \mathfrak{E}'$, $F \in \mathfrak{F}(\mathfrak{h})$, $A \in \mathfrak{A}_0^n$, with

$$\mathfrak{E}' = U^*(\mathfrak{F}(\mathfrak{h}) \otimes \mathfrak{A}_0^n)U \subset \mathfrak{E}, \tag{1.13}$$

with the ideal $\mathfrak{F}(\mathfrak{h})$ of operators $\mathfrak{h} \to \mathfrak{h}$ of finite rank, we define

$$\tau_E(\vec{x}, \vec{\xi}) = F. \sigma^n_A(\vec{\xi}, -\vec{x}) \in C(M^n_o, \mathfrak{C}(\mathfrak{H})), \tag{1.14}$$

and then use the fact that $\mathfrak{E}'$ is dense in $\mathfrak{E}$, under norm topology, (c.f. **[3]**).

2. **General Boundary Conditions, for a Constant Coefficient Equation.** Let $\rho > 0$, and consider the boundary problem

$$(\mathrm{p}_0)\qquad \begin{aligned} &u \in \mathfrak{H}_N(\Omega),\ \langle a_0\rangle\, u = D_0^N u + \rho^N(1 - \vec{\Delta})^{N/2} u = f \text{ on } \Omega \cup \Gamma,\\ &\langle b^j\rangle\, u = 0 \text{ on } \Gamma,\ j = 1, \ldots, r, \end{aligned}$$

where $f \in \mathfrak{H}$, Ω and Γ denote $\mathbf{R}^{n+1}_{+}$ and its boundary, respectively, and where $\langle b^j\rangle$ denote differential expressions of order $N_j < N = 2r$:

$$\langle b^j\rangle = \sum_{k=0}^{N_j} \sum_{|\alpha| \leq N_j - k} b^j_{k,\alpha}(\vec{x}) \vec{D}^\alpha D_0^k, \tag{2.1}$$

with coefficients in $C^\infty(\Gamma)$ satisfying $b^{j(\beta)}_{k,\alpha} \in C(\mathbf{B}^n)$, for all β.

Application of the tangential Fourier transform will convert all tangential derivatives into multiplications, but also the multiplications with $b^j_{k,\alpha}(\vec{x})$ into convolutions. Also, application of T (as in (1.2)) will take D_0 into $\lambda^{-1}(\vec{M})D_0$. Therefore, if we introduce the function $\tilde{u} = \lambda^{-N}(\vec{M})Uu$, problem (p_0) proves to be equivalent to

$$(2.2)\qquad \begin{aligned} &\tilde{u} \in \mathfrak{h}_N \hat{\otimes} \mathfrak{k},\ (D_0^N + \rho^N)\tilde{u} = g \text{ on } \Omega \cup \Gamma,\\ &\sum_{k=0}^{N_j} \tilde{B}_{jk}\tilde{u}_k = 0,\ j = 1, \ldots, r, \end{aligned}$$

with $g = Uf$, and the restrictions $\tilde{u}_k = D_0^k\tilde{u} \mid \Gamma$, and with the linear operators $\tilde{B}_{jk} \in \mathscr{L}(\mathfrak{k})$ defined by

$$\tilde{B}_{jk} = \lambda^{-N-N_j+1/2}(\vec{M}) B_{jk}\, \lambda^{N-N_j-1/2}(\vec{M}) \tag{2.3}$$

with

$$B_{jk} = \sum_{|\alpha| \leq N_j - k} b^j_{k,\alpha}(-\vec{D})\, s^\alpha(\vec{M})\, \lambda^{N_j-k-|\alpha|}(\vec{M}),\ \in \mathfrak{A}_0^n, \tag{2.4}$$

where $s(\vec{\xi}) = \vec{\xi}\,\lambda(\vec{\xi})$. In fact, also the $\tilde{B}_{jk}$ are in $\mathfrak{A}_0^n$, due to (1.4), for $s = N - N_j - 1/2$, and $\sigma^n_{B_{jk}} = \sigma^n_{\tilde{B}_{jk}}$, since the difference $B_{jk} - \tilde{B}_{jk}$ is compact.

Now, for a given $g \in \mathfrak{H}$ the ordinary differential equation $(D_0^N + \rho^N)v = g$ admits precisely a family of solutions $v \in \mathfrak{h}_N \hat{\otimes} \mathfrak{k}$ r arbitrary functions $c_l \in \mathfrak{k}$, explicitly given in the form

$$v = \sum_{l=1}^{r} c_l(\vec{\xi})\, e^{i\kappa_l y} + \int_0^\infty G(y - y')g(y', \vec{\xi})\, dy', \tag{2.5}$$

where

$$G(t) = \sum_{l=1}^{r} i\gamma_l\, e^{i\kappa_l |t|},\ \gamma_l^{-1} = 2\kappa_l \prod_{\nu \neq l} (\kappa_l^2 - \kappa_\nu^2), \tag{2.6}$$

and where κ_l, in any order, denote the r distinct roots of $\kappa^N + \rho^N = 0$ with positive imaginary part. Moreover, since the 'Greens function' G must have (piecewise) continuous derivative up to order $N - 1$, it follows that (2.5) may be differentiated under the integral sign, up to $N - 1$ times.

Accordingly we may assume $\tilde{u}$ of the form (2.5) and substitute into the second equation (2.2), to determine the functions c_l. We get

$$\begin{aligned} \tilde{u}_k &= \sum_{l=1}^{r} \kappa_l^k \{c_l + (-1)^k i\gamma_l g_l\}, k = 0, 1, \ldots, N_j, \\ g_l(\vec{\xi}) &= \int_0^\infty e^{i\kappa_l y'} g(y', \vec{\xi})\, dy' \end{aligned} \tag{2.7}$$

so that the second equation (2.2) takes the form

$$\sum_{l=1}^{r} \tilde{Q}_{jl} c_l = \sum_{l=1}^{r} \tilde{P}_{jl} g_l,\ j = 1, \ldots, r, \tag{2.8}$$

with

$$\begin{aligned} \tilde{Q}_{jl} &= \lambda^{-t_j}(\vec{M}) Q_{jl} \lambda^{t_j}(\vec{M}),\ \tilde{P}_{jl} = \lambda^{-t_j}(\vec{M}) P_{jl} \lambda^{t_j}(\vec{M}), \\ t_j &= N - N_j - 1/2, Q_{jl} = b^j - \vec{D}, (\kappa_l/\lambda(\vec{M}), \vec{M}) \lambda^{N_j}(\vec{M}), \\ P_{jl} &= -\, i\gamma_l b^j(-\vec{D}, (-\kappa_l/\lambda)(\vec{M}), \vec{M})\ \lambda^{N_j}(\vec{M}) \\ b^j(\vec{x}, \xi) &= b^j(\vec{x}, \eta, \vec{\xi}) = \sum_{|\alpha|+k \leq N_j} b^j_{k,\alpha}(\vec{x}) \eta^k \vec{\xi}^\alpha \end{aligned} \tag{2.9}$$

Clearly (2.8) is a (singular integral) equation within the C^*-algebra $\mathfrak{A}_0^n$ -or rather, within the tensor product $\mathscr{L}(\mathbf{C}^r) \otimes \mathfrak{A}_0^n$, an operator algebra over $\mathbf{C}^r \otimes \mathfrak{k}$. There exists a Fredholm inverse (and even a special Green inverse, as described in §1 (c.f. (1.8)) $((\tilde{Z}_{jl}))_{j,l=1,\cdots,r}$ of the matrix $((\tilde{Q}_{jl}))$ if and only if the matrix of symbols

$$((b^j(\vec{x}, \kappa_l/\lambda(\vec{\xi}), \vec{\xi})\lambda^{N_j}(\vec{\xi})))_{j,l=1,\ldots,r} \tag{2.10}$$

is non-singular for every $(\vec{x}, \vec{\xi}) \in \mathbf{M}_0^n$ -that is for $\vec{x}, \vec{\xi} \in \mathbf{B}^n$ with $|\vec{x}| + |\vec{\xi}| = \infty$. The explicit calculation of symbols is easily done using (1.9). For a somewhat informal surveying first discussion, using a special Green inverse $((Z_{jl}))$ one will (normally) solve the boundary problem (p_0) by solving (2.8) for the c_l (modulo finite rank) and substituting the formula obtained into (2.5). It is likely then from the derivation that the boundary problem (p_0) will be normally solvable *if and only if* the determinant of the matrix (2.10) does not vanish for every $\vec{x}, \vec{\xi} \in \mathbf{B}^n$ with $|\vec{x}| + |\vec{\xi}| = \infty$. In that case a Fredholm inverse T: $\mathfrak{H} \to \mathfrak{H}_N(\Omega)$ is given explicitly in the form

$$\begin{aligned} \tilde{u}(y, \vec{\xi}) &= \int_0^\infty G(y - y') g(y', \vec{\xi})\, dy' + \sum_{r,l=1}^{k} \tilde{R}_{kl} e^{i\kappa_l y} g_l(\vec{\xi}), \\ \tilde{R}_{kl} &= \sum_{j=1}^{r} \tilde{Z}_{kj} \tilde{P}_{jl} \in \mathfrak{A}_0^n. \end{aligned} \tag{2.11}$$

In order to formulate a theorem we introduce the polynomials

$$q_j(\eta) = q_{j,\vec{x},\vec{\xi}}(\eta) = b^j(\vec{x}, \eta/\lambda(\vec{\xi}), \vec{\xi})\lambda^{N_j}(\vec{\xi}),\ j = 1, \ldots, r, \tag{2.12}$$

of the complex variable η, and observe that q_j is of degree N_j and has coefficients depending continuously on $\vec{x}$ and $\vec{\xi}$ over the compact space $\mathbf{B}^n \times \mathbf{B}^n$ with boundary $\partial(\mathbf{B}^n \times \mathbf{B}^n) = \mathbf{M}_0^n$, after a continuous extension. Then the matrix (2.10) may be written as $((q_j(\kappa_l)))$, and simlarly $((P_{jl}))$ has symbols $((-i\gamma_l q_j(-\kappa_l)))$.

LEMMA 2.1. *Let the r polynomials* (2.12) *be linearly independent for every* $|\vec{x}| + |\vec{\xi}| = \infty$ *(i.e., for* $(\vec{x}, \vec{\xi}) \in \mathbf{M}_0^n$*). Then we have*

$$\det((q_j(\kappa_l))) \neq 0, (\vec{x}, \vec{\xi}) \in \mathbf{M}_0^n, \tag{2.13}$$

provided that the constant ρ *in the differential equation of* (p_0) *is chosen sufficiently large.*

PROOF. It suffices to consider a single point of $\mathbf{M}_0^n$. If for each such point (2.13) holds for $\rho \geqq \rho_0$, then by compactness of $\mathbf{M}_0^n$ and continuity of coefficients a uniform ρ_0, valid for all of $\mathbf{M}_0^n$, may be found. At a given fixed point we may replace the polynomials by others, applying 'row operations' in such a way that all of them have highest coefficient 1, and any two of them have different orders. Clearly, $|\kappa_l| = \rho \to \infty$, as $\rho \to \infty$, so that for large ρ the determinant (2.13) is approximated by

$$\rho^{N_1+\cdots+N_r} \det((\exp(2\pi i N_j \cdot 1/N))) \exp(i\pi(N_1 + \ldots + N_r)/N) \tag{2.14}$$

The determinant in (2.14) is the van der Monde of the r distinct unit roots $e^{2\pi i N_j/N}$, $j = 1, \ldots, r$, and therefore is $\neq 0$, q.e.d.

THEOREM 2.2. *If and only if* (2.12) *are linearly independent for all* $|\vec{x}| + |\vec{\xi}| = \infty$, *and if* ρ *in* (p_0) *is sufficiently large, then there exists a Fredholm inverse* $T: \mathfrak{H} \to \mathscr{D} \subset \mathfrak{H}_N(\Omega)$ *of the linear operator* $L_0: \mathscr{D} \to \mathfrak{H}$ *associated to the problem* (p_0), *where*

$$\begin{aligned} \mathscr{D} &= \{u \in \mathfrak{H}_N(\Omega): \langle b^j \rangle u = 0 \text{ on } \Gamma, j = 1, \ldots, r\}, \\ L_0 u &= \langle a_0 \rangle u, \; u \in \mathscr{D}, \end{aligned} \tag{2.15}$$

such that $L_0 T = 1 - F$, $TL_0 = 1 - F'$, $F \in \mathfrak{F}(\mathfrak{H})$, $F' \in \mathfrak{F}(\mathscr{D})$, *with* $\mathfrak{F}(\mathscr{X}) \subset \mathfrak{L}(\mathscr{X})$ *denoting the ideal of continuous operators of finite rank in the space* $\mathscr{X}$. *(Note that* $\mathscr{D}$ *is a closed subspace of* $\mathfrak{H}_N(\Omega)$, *and thus a Hilbert space.)*

Moreover we get $D^\alpha T \in \mathfrak{A}$ *for all* $|\alpha| \leqq N$, *if T is properly chosen. In fact such a choice is explicitly given by*

$$\begin{aligned} &T = T_0 + K + C, \; UT_0U^* = G \otimes \lambda^N(\vec{M}), \\ &UKU^* = \sum_{k,l=1}^{r} (e_k\rangle\langle e_l) \otimes (\lambda^N(\vec{M})\tilde{R}_{kl}), \; UCU^* = \sum_{j=1}^{\nu} \psi_j\rangle \langle \varphi_j, \end{aligned} \tag{2.16}$$

with the operators

$$(Gv)(y) = \int_0^\infty G(y - y')v(y')\, dy',\ v \in \mathfrak{h},\ \tilde{R}_{kl} \in \mathfrak{A}_0^n,$$

$$(\varphi\rangle\langle\psi)w = \varphi\cdot\int_X \psi\, wdz,\ \varphi, \psi, w \in L^2(X),\ X \subset \mathbf{R}^m,$$

$$\text{(2.17)}\qquad e_j(y) = e^{i\kappa_j y},\ y \in \mathbf{R}_+,\ \varphi_j(y, \vec{\xi}) = \sum_{k=1}^r e_k(y)\omega_{jk}(\vec{\xi}),$$

$$\psi_j(y, \vec{\xi}) = \sum_{k=1}^r (ye_k(y)\chi_{jk}(\vec{\xi}) + e_k(y)\theta_{jk}(\vec{\xi})),\ j = 1, \dots, \nu,$$

$$\omega_{jk}, \chi_{jk}, \theta_{jk} \in \mathfrak{k}_\infty,$$

$$((\sigma^n_{\tilde{R}kl}(\vec{\xi}, -\vec{x}))) = (q_{k,\vec{x},\vec{\xi}}(\kappa_l)))^{-1}((-i\gamma_l q_{k,\vec{x},\vec{\xi}}(-\kappa_l))),$$

where $\bar{\varphi}_1, \dots, \bar{\varphi}_\nu$ form an orthonormal base in $\mathfrak{H}$ of the (finite dimensional) orthogonal complement of im $UL_0U^* \subset \mathfrak{H}$.

Moreover we have a corresponding representation for $T^{k,\alpha} = D_0^k\vec{D}^\alpha T$:

$$T^{k,\alpha} = T_0^{k,\alpha} + K^{k,\alpha} + C^{k,\alpha},\quad UT_0^{k,\alpha}U^* = (D_0^kG) \otimes (\vec{M}^\alpha\lambda^{N-k}(\vec{M}))$$

$$\text{(2.18)}\qquad UK^{k,\alpha}U^* = \sum_{j,l=1}^r (\kappa_j^k e_j\rangle\langle e_l) \otimes (\vec{M}^\alpha\lambda^{N-k}(\vec{M})\tilde{R}_{jl}),$$

$$UC^{k,\alpha}U^* = \sum_{j=1}^\nu (D_0^k\vec{M}^\alpha\lambda^{-k}(\vec{M})\psi_j)\rangle\langle\varphi_j,\qquad |\alpha| + k \leqq N.$$

PROOF. Suppose (2.17) can be verified, then (2.18) follows by differentiation, using that $D_0^k\vec{D}^\alpha U = UD_0^k\lambda^{-k}(\vec{M})\vec{M}^\alpha$. Now, using the properties of our special Green inverse $((\tilde{Z}_{kl}))$ of the operator matrix $((\tilde{Q}_{kl}))$ we first introduce the operator $\tilde{T}: \mathfrak{H} \to \mathfrak{h}_N \hat{\otimes} \mathfrak{k}$ by

$$\text{(2.19)}\qquad g \to (G \otimes 1)g + \sum_{k,l=1}^r ((e_k\rangle\langle e_l) \otimes \tilde{R}_{kl})g = \tilde{T}g = \tilde{u}.$$

The operator $\tilde{T}$ pertains to the transformed boundary problem (2 2) In particular: (i) if u solves (2.2) then $u = \tilde{T}g + \sum_{l=1}^r d_l e_l$, with $(d) = (d_l) \in \mathbf{C}^r \otimes \mathfrak{k}_\infty$ solving $\tilde{Q}(d) = 0$,

(ii) For general $g \in \mathfrak{H}$ we have $\tilde{u} = \tilde{T}g$ solving

$$\text{(2.20)}\qquad \begin{aligned} &\tilde{u} \in \mathfrak{h}_N \hat{\otimes} \mathfrak{k},\ (D_0^N + \rho^N)\tilde{u} = g \ \text{ on } \Omega \cup \Gamma, \\ &\sum_{k=1}^r \tilde{B}_{jk}\tilde{u}_k = -(1 - P_{\operatorname{im}\tilde{Q}})\tilde{P}(g),\qquad (g) = (g_l). \end{aligned}$$

Therefore $\tilde{u} = \tilde{T}g$ solves (2.2) if and only if $(1 - P_{\operatorname{im}\tilde{Q}})\tilde{P}(g) = 0$, which amounts to a finite set of conditions $(\varphi_j, g) = 0$, $j = 1, \dots, \nu$, with φ_j exactly of the form as in (2.17). If these vectors are orthogonalized, they will preserve that form. From (i) and (ii) together it follows that

(1) $\tilde{u} = \tilde{T}g$ solves (2.2) if and only if g is orthogonal to $\varphi_1, \ldots, \varphi_\nu$ and (2) then all solutions of (2.2) are of the form $\tilde{u} = \tilde{T}g + \sum_{l=1}^{r} e_l \otimes d_l$ with $(d_l) \in \ker \tilde{Q} \cap \mathfrak{k}_\infty$.

Let $\tilde{\mathscr{D}} = \lambda^{-N}(\vec{M})\, U\mathscr{D}$, then $\tilde{T}$ does not map $\mathfrak{H}$ into $\tilde{\mathscr{D}}$, but the operator $\hat{T} = \tilde{T}(1 - F)$, $F = \sum_{j=1}^{\nu} \varphi_j\rangle\langle\varphi_j$ will do this. From (1) and (2) it follows that L_0 is a Fredholm operator. Since $1 - F$ projects onto the orthogonal complement of the φ_j we get $\tilde{L}_0\hat{T} = 1 - F$ (with properly defined $\tilde{L}_0$, as a transformation of L_0). Hence a Fredholm inverse exists and $\hat{T}$ is a right Fredholm inverse, therefore also must be a left Fredholm inverse. All remaining statements of the theorem now follow by transforming back onto the original form, applying U^* and $\lambda^N(\vec{M})$, etc.

COROLLARY 2.3. *In the decompositions* (2.6) *and* (2.7) *we have*

$$(2.21)\quad C^{k,\alpha} \in \mathfrak{F}(\mathfrak{H}),\ K^{k,\alpha} \in \mathfrak{E}' \subset \mathfrak{E},\ T_0^{k,\alpha} = \Pi_+^+((D_0^k \vec{D}^\alpha (D_0^N + \rho^N(1 - \vec{\Delta}^r)^{-1}),$$

with E$'$ *of* (1.13) *and* Π_+^+ of (3.5). *In particular* $T_0^{k,\alpha}$ *depend on* ρ *and* N, *but otherwise are entirely independent of the boundary conditions imposed in* (p_0).

The proof is a matter of (1.13) and our explicit formulas for $UC^{k,\alpha}U^*$ and $UK^{k,\alpha}U^*$.

3. **A Relation between $\mathfrak{A}_0^{n+1}$ and $\mathfrak{A}$, and its Effect on Symbols.** Let us define the isometries E_e, E_o, and E_+, E_- from $\mathfrak{H}$ to $\mathfrak{K}$:

$$(3.1)\quad \begin{aligned} (E_+u)(y, \vec{x}) &= u(y, \vec{x}) \text{ on } \Omega, = 0 \text{ when } y < 0,\ u, \in \mathfrak{H}, \\ (E_-u)(y, \vec{x}) &= u(-y, \vec{x}), \text{ when } y < 0, = 0 \text{ on } \Omega,\ u \in \mathfrak{H}, \end{aligned}$$

and

$$(3.2)\quad E_e = 2^{-1/2}(E_+ + E_-),\ E_o = 2^{-1/2}(E_+ - E_-).$$

Clearly E_e and E_o correspond to even and odd extension from $\mathbf{R}_+^{n+1}$ into $\mathbf{R}^{n+1}$ while E_+ simply extends as zero. All maps are isometries but not unitary maps. Thus the adjoints are only partial isometries, explicitly given as the restrictions

$$(3.3)\quad E_\gamma^* v = 2^{1/2} v_\gamma | \Omega,\ \gamma = e, o,\ E_\pm^*\, v = v(\pm y, x) | \Omega,\ v \in \mathfrak{K},$$

with the even and odd part

$$(3.4)\quad \begin{aligned} v_e(y, \vec{x}) &= (v(y, \vec{x}) + v(-y, \vec{x}))/2,\ v_o(y, \vec{x}) \\ &= (v(y, \vec{x}) - v(-y, \vec{x}))/2. \end{aligned}$$

We define the linear contraction maps (not homomorphisms) $\Pi_\pm^\pm : \mathscr{L}(\mathfrak{K}) \to \mathscr{L}(\mathfrak{H})$ (with $\mathscr{L}(\mathscr{X})$ a Banach space with operator norm) by

$$(3.5)\quad \Pi_\gamma^\kappa L = E_\kappa^* L\, E_\gamma,\ L \in \mathscr{L}(\mathfrak{K}),\ \kappa, \gamma = +, -.$$

PROPOSITION 3.1. $\Pi_\gamma^\kappa: \mathfrak{A}_0^{n+1} \to \mathfrak{A}$, $\Pi_\gamma^\kappa : \mathfrak{C}(\mathfrak{K}) \to \mathfrak{C}(\mathfrak{H})$, *for* $\kappa, \gamma = +, -$.

PROOF. Since $E_\pm$ and $E_\pm^*$ are partial isometries it is clear that compactness is preserved by all the maps Π_γ^κ. Also it suffices to show that Π_γ^κ maps the generators $a(M)$ (with $a \in C(\mathbf{B}^{n+1})$) and Λ, S_0, ..., S_n of $\mathfrak{A}_0^{n+1}$ into $\mathfrak{A}$. For if A, $B \in \mathfrak{A}_0^{n+1}$, and $\Pi_\gamma^\kappa A$, $\Pi_\gamma^\kappa B \in \mathfrak{A}$, then we use the relation, which is easily verified.

$$E_+E_+^* + E_-E_-^* = 1 \tag{3.6}$$

to show that

$$\Pi_+^+(AB) = (\Pi_+^+A)(\Pi_+^+B) + (\Pi_-^+A)(\Pi_+^-B) \in \mathfrak{A}. \tag{3.7}$$

Similarly for all other operators $\Pi_\gamma^k(AB)$, with a slightly different formula (3.7).

But for a multiplication $a(M)$ we get $\Pi_+^+ a(M) = b_+(M)$, $\Pi_-^- a(M) = b_-(M)$, with $b_\pm(y, x) = a(\pm y, x) \mid \Omega$, while $\Pi_+^- a(M) = \Pi_-^+ a(M) = 0$, so that indeed $\Pi_\gamma^\kappa a(M) \in \mathfrak{A}$. Moreover we observe that

$$\begin{aligned} E_o^*\Lambda E_o &= \Lambda_d,\ E_e^*\Lambda E_e = \Lambda_n,\ E_o^*S_jE_o = S_{j,d}, \\ E_e^*S_jE_e &= S_{j,n}, j = 1, \dots, n,\ E_e^*S_0E_o = S_{0,d},\ E_o^*S_0E_e = S_{0,n} \\ E_o^*\Lambda E_e &= E_e^*\Lambda E_o = E_o^*S_jE_e = E_e^*S_jE_o = E_e^*S_0E_e \\ &= E_o^*S_0E_o = 0, j = 1, \dots, n, \end{aligned} \tag{3.8}$$

using the fact that Λ, S_1, ..., S_n preserve the spaces of even and odd functions, while S_0 takes even into odd functions, and vice versa. But (3.2) may be solved for $E_\pm$:

$$E_\pm = 2^{-1/2}(E_e \pm E_o), \tag{3.9}$$

which implies that $\Pi_\gamma^\kappa A$ is a linear combination of $E_\mu^*AE_\nu$, $\mu, \nu = e, O$. This completes the proof, since continuity of Π_γ^κ allows closing.

It is clear from Proposition 3.1. that Π_γ^κ induce four continuous maps $\mathfrak{A}_0^{n+1}/\mathfrak{C}(\mathfrak{K}) \to \mathfrak{A}/\mathfrak{C}(\mathfrak{H})$. The first algebra is equal (isometrically isomorphic) to $C(\mathbf{M}_0^{n+1})$. On the other hand, $\mathfrak{A}/\mathfrak{C}(\mathfrak{H})$ induces algebras of linear operators on its closed two-sided ideal $\mathfrak{E}/\mathfrak{C}(\mathfrak{H}) \simeq C(\mathfrak{C}(\mathfrak{h}), \mathbf{M}_0^n)$, a Banach space, by left and right multiplication. Accordingly, for every $\gamma, \kappa = +, -$, and every symbol $a \in C(\mathbf{M}_0^{n+1})$ there exists an operator $A_\gamma^\kappa: C(\mathfrak{C}(\mathfrak{h}), \mathbf{M}_0^n) \to C(\mathfrak{C}(\mathfrak{h}), \mathbf{M}_0^n)$ such that

$$\begin{gathered} A \in \mathfrak{A}_0^{n+1},\ E,\ F_\gamma^\kappa \in \mathfrak{E},\ \sigma_A^{n+1} = a,\ F_\gamma^\kappa = (\Pi_\gamma^\kappa A)E \\ \text{implies } \tau_{F_\gamma^\kappa} = A_\gamma^\kappa \tau_E, \end{gathered} \tag{3.10}$$

with the symbol τ_E of $E \in \mathfrak{E}$. Similarly for the right multiplication of symbols of $E \in \mathfrak{E}$ with the coset of A modulo $\mathfrak{C}(\mathfrak{K})$.

THEOREM 3.2. *The operators* A^{κ}_{γ} *of* (3.10) *coincide with those defined by right multiplication and are explicitly given by*

$$(3.11)\quad A^{\kappa}_{\gamma}(\vec{x}, \vec{\xi}) = \pi^{\kappa}_{\gamma} a(0, \vec{x}, (1 + \vec{\xi}^2)^{1/2} D_0, \vec{\xi}), a \in C(\mathbf{M}^{n+1}_0), \gamma, \kappa = +, -,$$

where π^{κ}_{γ}: $\mathscr{L}(L^2(\mathbf{R})) \to \mathscr{L}(\mathfrak{h})$ *denote the operators* Π^{κ}_{γ} *for* $n = 0$. *That is, for* $A \in \mathfrak{A}^{n+1}_0$, E, F^{κ}_{γ}, $G^{\kappa}_{\gamma} \in \mathfrak{E}$, *with* $\sigma_A = a$, $F^{\kappa}_{\gamma} = (\Pi^{\kappa}_{\gamma} A)E$, $G^{\kappa}_{\gamma} = E(\Pi^{\kappa}_{\gamma} A)$, *we have* (*for* $(\vec{x}, \vec{\xi}) \in \mathbf{M}^{n}_0$)

$$(3.12)\qquad \tau_{F^{\kappa}_{\gamma}}(\vec{x}, \vec{\xi}) = A^{\kappa}_{\gamma}(\vec{x}, \vec{\xi})\tau_E(\vec{x}, \vec{\xi}), \tau_{G^{\kappa}_{\gamma}}(\vec{x}, \vec{\xi}) = \tau_E(\vec{x}, \vec{\xi})A^{\kappa}_{\gamma}(\vec{x}, \vec{\xi}).$$

PROOF. First, for the operators $E = U^*(C \otimes B)U$, $C \in \mathfrak{E}(\mathfrak{h})$, $B \in \mathfrak{A}^{n}_0$, and a multiplication $a(M)$, $a \in C(\mathbf{B}^{n+1})$, the assertion is a direct consequence of (1.10) and (1.11), by a calculation. Also for E, as above, and one of the other generators Λ, S_j of $\mathfrak{A}^{n+1}_0$, one will use (3.8) and (1.12) to directly calculate the symbols τ_E, $\tau_{F^{\kappa}_{\gamma}}$, $\tau_{G^{\kappa}_{\gamma}}$, thus confirming (3.12). Finally, formula (3.6) is valid for dimension 1 just as well as for dimension $n + 1$, which makes the π^{κ}_{γ} behave just like the Π^{κ}_{γ}. For example we get

$$(3.13)\qquad \pi^{+}_{+}(PQ) = \pi^{+}_{+}(P)\pi^{+}_{+}(Q) + \pi^{-}_{+}(P)\pi^{+}_{-}(Q), P, Q \in \mathscr{L}(L^2(\mathbf{R})),$$

similar as (3.7). This implies that (3.11) and (3.12) are valid for $AB \in \mathfrak{A}^{n+1}_0$, if they hold for A and $B \in \mathfrak{A}^{n+1}_0$. All maps introduced clearly are continuous, so that the proof is completed by taking closures. Also the corollary, below, is evident.

COROLLARY 3.3. *The operators* $A^{\kappa}_{\gamma} = \pi^{\kappa}_{\gamma} A$, $A \in \mathfrak{A}^{n+1}_0$, *take* $\mathfrak{E}' + \mathfrak{E}(\mathfrak{H})$ *to itself.*

4. **The operator $A = \langle a \rangle T$ and its Normal Solvability.** We now approach the boundary problem (p), as posed in the introduction, by substituting $u = Tv$, $v \in \mathfrak{H}$, with the Fredholm inverse $T: \mathfrak{H} \to \mathscr{D}$ constructed for the problem (p_0), in §2. We want the boundary expressions $\langle b^j \rangle$ for (p) and (p_0) to be identical, and also $N = 2r$. Also we assume $\varphi_j = 0, j = 1, \ldots, r$, which is no restriction of generality: For general $\varphi_j \in \mathfrak{H}_{N-N_j-1/2}$ it is possible to determine $w \in \mathfrak{H}_N(\Omega)$ satisfying the boundary conditions $\langle b^j \rangle w = \varphi_j$ on Γ, but not necessarily the differential equation, using standard extension procedures and linear independence of the polynomials q_j. Then $v = u - w$ will satisfy (p), with f replaced by $f - \langle a \rangle w$, and with homogeneous boundary conditions.

We assume $\langle a \rangle$ and $\langle b^j \rangle$ to be md-elliptic, as formulated in the introduction. This will imply linear independence of the q_j, as required for construction of w above, and for Proposition 2.1.

The substitution $u = Tv$ will automatically satisfy $u \in \mathfrak{H}_N(\Omega)$, and the boundary conditions $\langle b^j \rangle u = 0$, since $Tv \in \mathscr{D}$ by construction. Since $T: \mathfrak{H} \to \mathscr{D} \subset \mathfrak{H}_N(\Omega)$ is Fredholm, its image will have finite codimension in $\mathscr{D}$. Therefore we always can write $u = Tv$ if only finitely many conditions are satisfied. Moreover, the second equation of (p) takes the form

$$Av = f, A = \sum_{|\beta|\leq N} a_\beta(M)(D^\beta T) \in \mathfrak{A} \tag{4.1}$$

using that $D^\beta T = T^{\beta_0\vec{\beta}} \in \mathfrak{A}$ by Theorem 2.2. and that $a_\beta(M) \in \mathfrak{A}$ by (0.1), which may be written as $a_\alpha^{(\beta)} \in C(\mathbf{H}^{n+1})$. This makes the proposition below evident.

PROPOSITION 4.1. *Problem (p), with homogeneous boundary conditions, is normally solvable if and if the operator A of* (4.1) *is Fredholm as a map from $\mathfrak{H}$ to $\mathfrak{H}$.*

PROPOSITION 4.2. *Let $\langle a \rangle$ be* md-*elliptic, and assume that $\langle b^j \rangle$ and ρ satisfy the assumptions of Lemma* 2.1, *so that T of Theorem* 2.2 *exists. Then there exists an inverse of A (in* (4.1)) *modulo $\mathfrak{C}$.*

PROOF. From Corollary 2.3 it follows that

$$\begin{aligned} A &= \Pi_+^+ A_0 + K_0 + C_0,\ K_0 = \sum_{|\alpha|+k\leq N} a_{(k,\alpha)}(M)K^{k,\alpha} \in \mathfrak{C}' + \mathfrak{C}(\mathfrak{H}), \\ C_0 &= \sum_{|\alpha|+k\leq N} a_{(k,\alpha)}(M)C^{k,\alpha} \in \mathfrak{F}(\mathfrak{H}), \end{aligned} \tag{4.2}$$

while

$$A_0 = \sum_{|\alpha|\leq N} a_\alpha^0(M)D^\alpha(D_0^N + \rho^N(1 + \vec{D}^2)^r)^{-1},\quad a_\alpha^0 = 2^{1/2}E_e a_\alpha. \tag{4.3}$$

Since $C_0, K_0 \in \mathfrak{C}$, it clearly suffices to construct an inverse mod $\mathfrak{C}$ of $\pi_+^+ A_0$ only. But $A_0 \in \mathfrak{A}_0^{n+1}$ has symbol

$$\begin{aligned} \sigma_{A_0}^{n+1}(x, \xi) &= \sum_{|\alpha|\leq N} a_\alpha^0(x)\xi^\alpha(\xi_0^N + \rho^N \lambda^{-N}(\vec{\xi}))^{-1}, \\ &\text{as } x, \xi \in \mathbf{R}^{n+1}, |x| + |\xi| = \infty. \end{aligned} \tag{4.4}$$

But $\sigma_A^{n+1} \neq 0$ on $\mathbf{M}_0^{n+1}$ since a is md-elliptic. Thus there exists a Fredholm inverse $B_0 \in \mathfrak{A}_0^{n+1}$ such that $1 - A_0B_0$, $1 - B_0A_0 \in \mathfrak{F}(\mathfrak{K})$. Using (3.7) it follows that

$$(\Pi_+^+ B_0)(\Pi_+^+ A_0) \equiv 1 - (\Pi_-^+ B_0)(\Pi_+^- A_0) \pmod{\mathfrak{C}(\mathfrak{H})} \tag{4.5}$$

However, a calculation shows that $\Pi_+^- A_0 \in \mathfrak{C}'$, and also $\Pi_-^+ A_0 \in \mathfrak{C}'$, so that the last term in (4.5) is in $\mathfrak{C}' \subset \mathfrak{C}$, and similarly for the other order of multiplication. This proves the proposition. Moreover, we may even explicitly calculate the perturbation terms, as stated in the Corollary, below.

COROLLARY 4.3. *With $B = \Pi_+^+ B_0$ as constructed, we have*

(4.6) $$BA = 1 + E_1,\ AB = 1 + E_2,$$

with E_1, E_2 explicitly given in the form

(4.7) $$\begin{aligned} E_1 &\equiv (\Pi{\mp}B_0)(K_0 + C_0) - (\Pi{\pm}B_0)(\Pi{\mp}A_0) \pmod{\mathfrak{C}(\mathfrak{H})}, \\ E_2 &\equiv (K_0 + C_0)(\Pi{\mp}B_0) - (\Pi{\pm}A_0)(\Pi{\mp}B_0) \pmod{\mathfrak{C}(\mathfrak{H})}. \end{aligned}$$

PROOF. We have explicitly, using (1.10) and (1.11),

(4.8) $$\begin{aligned} U(\Pi{\mp}A_0)U^* &\equiv \sum_{l=1}^{r} i\,\gamma_l(e_l\rangle\langle e_l) \otimes p_{-\vec{D},\vec{M}}(-\kappa_l) \pmod{\mathfrak{C}(\mathfrak{H})}, \\ U(\Pi{\pm}A_0)U^* &\equiv \sum_{l=1}^{r} i\,\gamma_l(e_l\rangle\langle e_l) \otimes p_{-\vec{D},\vec{M}}(\kappa_l) \pmod{\mathfrak{C}(\mathfrak{H})}, \end{aligned}$$

with γ_l, e_l as in §2, and $p_{-\vec{D},\vec{M}}(\eta) \in \mathfrak{A}_0^n$ defined by

(4.9) $$p_{-\vec{D},\vec{M}}(\eta) = \sum_{j=0}^{N} p_j(-\vec{D}, \vec{M})\,\eta^j,$$

where

(4.10) $$p(\eta) = p_{\vec{x},\vec{\xi}}(\eta) = \sum_{j=0}^{N} p_j(\vec{x}, \vec{\xi})\,\eta^j$$

is the polynomial defined in (0.3). It is clear that $e_l\rangle\langle e_l \in \mathfrak{F}(\mathfrak{h})$, so that we indeed get $\pi{\mp}A_0, \pi{\pm}A_0, \in \mathfrak{C}'$. We also have C_0 and B_0 explicitly as in (4.2), and application of Corollary 3.3 completes the proof.

Note that Theorem 3.2 allows calculation of the symbols τ_{E_1}, τ_{E_2}, from the symbols of A_0, B_0 in $\mathfrak{A}_0^{n+1}$: First we get $\tau_{(\pi{\mp}B_0)C_0} = 0$, since C_0 is compact. Similarly $\tau_{C_0(\pi{\mp}B_0)} = 0$.
Also

(4.11) $$\tau_{K_0}(\vec{x}, \vec{\xi}) = \sum_{j,l=1}^{r} (e_j\rangle\langle e_l)\,p_{\vec{x},\vec{\xi}}(\kappa_j)\,r_{jl}(\vec{x}, \vec{\xi}),$$

where (c.f. (2.7))

(4.12) $$\begin{aligned} ((r_{jl}(\vec{x}, \vec{\xi}))) &= ((\sigma^{n}_{R,jl}(\vec{\xi}, -\vec{x}))) \\ &= ((q_{j,\vec{x},\vec{\xi}}(\kappa_l)))^{-1}((-i\gamma_l q_{j,\vec{x},\vec{\xi}}(-\kappa_l))), \end{aligned}$$

Also (4.8) allows calculation of

(4.13) $$\begin{aligned} \tau_{(\pi{\mp}A_0)}(\vec{x}, \vec{\xi}) &= \sum_{l=1}^{r} i\gamma_l(e_l\rangle\langle e_l)\,p_{\vec{x},\vec{\xi}}(-\kappa_l) \\ \tau_{(\pi{\pm}A_0)}(\vec{x}, \vec{\xi}) &= \sum_{l=1}^{r} i\gamma_l(e_l\rangle\langle e_l)\,p_{\vec{x},\vec{\xi}}(\kappa_l) \end{aligned}$$

Next we may explicitly calculate the operators $A^{\kappa}_{\gamma}(\vec{x}, \vec{\xi}) = B^{\kappa}_{\gamma}(\vec{x}, \vec{\xi})$ of (3.11) for $a = b = \sigma^{n+1}_{B_0}$. We get

(4.14) $$b(0, \vec{x}, \eta/\lambda(\vec{\xi}), \vec{\xi}) = (\eta^N + \rho^N)/p_{\vec{x},\xi}(\eta) = h_{\vec{x},\vec{\xi}}(\eta),$$

and then

(4.15) $$B^{\kappa}_{\gamma}(\vec{x}, \vec{\xi}) = \pi^{\kappa}_{\gamma} h_{\vec{x},\vec{\xi}}(D_0) = \pi^{\kappa}_{\gamma}\{(D_0^N + \rho^N)^{-1} p_{\vec{x},\vec{\xi}}(D_0)\}.$$

From now on we omit explicit notation of $\vec{x}$ and $\vec{\xi}$ in all formulas to follow, but keep in mind that expressions will depend on $\vec{x}$, $\vec{\xi}$, and will be defined for $|\vec{x}| + |\vec{\xi}| = \infty$. Using (4.11), (4.12) (4.13), and (4.15) we get, with $\langle B^{\kappa t}_{\gamma} e, v\rangle = \langle e, B^{\kappa}_{\gamma} y\rangle$, $\langle e, v\rangle = \int_0^\infty evdy$, $e, v \in \mathfrak{h}$,

(4.16) $$\begin{aligned} \tau_{E_1} &= \sum_{l=1}^{r} (\sum_{k=1}^{r} p(\kappa_k) r_{kl} B^{+}_{+} e_k - i\gamma_l p(-\kappa_l) B^{+}_{-} e_l)\rangle\langle e_l = \sum_{l=1}^{r} v_l\rangle\langle e_l \\ \tau_{E_2} &= \sum_{l=1}^{r} e_l\rangle\,\langle(\sum_{k=1}^{r} p(\kappa_l) r_{lk} B^{+t}_{+}\, e_k - i\gamma_l p(\kappa_l) B^{-t}_{+} e_l) = \sum_{l=1}^{r} e_l\rangle\langle w_l. \end{aligned}$$

We must check for invertibility of $1 + \tau_{E_j}$, for $|\vec{x}| + |\vec{\xi}| = \infty$. In each case the corresponding equation is of the form

(4.17) $$v + \sum_{l=1}^{r} \zeta_l \langle \theta_l, v\rangle = w, \qquad v, w, \zeta_l, \theta_l \in \mathfrak{h}.$$

As a matter of linear algebra we note that (4.17) is always uniquely solvable if and only if

(4.18) $$\det((\delta_{jk} + \langle \theta_j, \zeta_k\rangle)) \neq 0.$$

Accordingly we must check whether the two matrices (4.19) below have the eigenvalue -1 or not.

(4.19) $$\begin{aligned} \langle e_j, v_l\rangle &= \sum_{k=1}^{r} p(\kappa_k) r_{kl} \langle e_j, B^{+}_{+} e_k\rangle - i\gamma_l p(-\kappa_l)\,\langle e_j, B^{+}_{-} e_l\rangle \\ \langle w_j, e_l\rangle &= \sum_{k=1}^{r} p(\kappa_j) r_{jk} \langle e_k, B^{+}_{+} e_l\rangle - i\gamma_j p(\kappa_j)\,\langle e_j, B^{-}_{+} e_l\rangle. \end{aligned}$$

By a calculation,

(4.20) $$\begin{aligned} \langle e_j, B^{+}_{+} e_k\rangle &= (1/2\pi)\int_{-\infty}^{+\infty} h(\eta) d\eta \int_0^\infty dx dy\, e^{i\eta(x-y)} e_j(x) e_k(y) \\ &= (1/2\pi)\int_{-\infty}^{+\infty} h(\eta) d\eta/((\eta + \kappa_j)(\eta - \kappa_k)), \\ \langle e_j, B^{+}_{-} e_k\rangle &= -(1/2\pi)\int_{-\infty}^{+\infty} h(\eta) d\eta/((\eta + \kappa_j)(\eta + \kappa_k)), \\ \langle e_j, B^{-}_{+} e_k\rangle &= -(1/2\pi)\int_{-\infty}^{+\infty} h(\eta) d\eta/((\eta - \kappa_j)(\eta - \kappa_k)). \end{aligned}$$

Therefore, with $r_{kl} = -i\tilde{r}_{kl}$,

$$\langle e_j, v_l\rangle = (1/2\pi i)\int_{\mathbf{R}} h(\eta)d\eta/(\eta+\kappa_j)\Big(\sum_{k=1}^{r} p(\kappa_k)\tilde{r}_{kl}/(\eta-\kappa_k)$$

$$-\gamma_l p(-\kappa_l)/(\eta+\kappa_l)\Big) \tag{4.21}$$

$$\langle w_j, e_l\rangle = (1/2\pi i)\int_{\mathbf{R}} h(\eta)d\eta/(\eta-\kappa_l)\Big(\sum_{k=1}^{r} p(\kappa_j)\tilde{r}_{jk}/(\eta+\kappa_k) - \gamma_j p(\kappa)/(\eta-\kappa_j)\Big).$$

First we look at the second matrix. The integrand is of order $0(\eta^{-2})$, at $\eta = \infty$, thus the integral may be converted into $\int_{\Gamma_+} + \int_{\Gamma_+^0}$ with closed positively oriented countours Γ_+, Γ_+^0 containing the upper half plane roots of $p(\eta)$ and of $\eta^N + \rho^N$ in the interior, respectively (and all other roots in the outside). But $\int_{\Gamma_+^0} = -\delta_{jl}$ by a simple calculation of residues, using the fact that the first term of $\langle w_j, e_l\rangle$ has its integrand regular inside Γ_+^0, as ρ is large enough, as we assume from now on. Hence

$$\delta_{jl} + \langle w_j, e_l\rangle = (1/2\pi i)\int_{\Gamma_+} h(\eta)J_j(\eta)d\eta/(\eta-\kappa_l) = \alpha_{jl}. \tag{4.22}$$

with

$$\sum_{l=1}^{r} (q_m/p)(\kappa_l)J_l(\eta) = \sum_{l=1}^{r} (q_m(-\kappa_l)\gamma_l/(\eta+\kappa_l) - \gamma_l q_m(\kappa_l)/(\eta-\kappa_l)). \tag{4.23}$$

A calculation of residues shows that (4.23) is the partial fractions decomposition of the function $-q_m(\eta))/(\eta^N + \rho^N)$. Also we have seen before that the matrix $(((q_j/p)(\kappa_l)))$ is nonsingular. Therefore $1 + \tau_{E_2}$ is Fredholm by (4.22) and (4.23) if and only if

$$\beta_{mk} = \int_{\Gamma_+} q_m(\eta)/p(\eta)d\eta/(\eta-\kappa_k) \tag{4.24}$$

gives a nonsingular matrix (for all $|\vec{x}| + |\vec{\xi}| = \infty$.). A simple calculation shows that this is true if and only if the q_j are linearly independent modulo p^+ (i.e., if the boundary conditions are md-elliptic).

Finally, for E_1, one observes that for a countour encircling the roots of $p(\eta)$ in the lower half-plane, positively oriented and with $\pm\kappa_j$ and all other roots of p in the outside, we get

$$(1/2\pi i)\int_{\Gamma_-} J_{jl}d\eta = -\langle e_j, v_l\rangle - \delta_{jl} = \mu_{jl} \tag{4.25}$$

with the integrand J_{jl} of the first relation (4.21); therefore we must investigate when (4.25) defines a nonsingular matrix. This follows because again $J_{jl}(\eta)$ is of order $0(\eta^{-2})$ at infinity, and by evaluation of residues inside a suitable Γ_-^0 in which the first term of J_{jl} stays regular. Let $\varphi(\eta)$ be a polynomial of degree $<r$, and observe the partial fractions decomposition

$$p^-(\eta)\varphi(\eta)(\eta^N + \rho^N)^{-1} = \sum_{l=1}^{r} \gamma_1((p^-\varphi)(\kappa_1)/(\eta - \kappa_l) \tag{4.26}$$
$$-(p^-\varphi)(-\kappa\ )/(\eta + \kappa_{l'})),$$

with $p^-(\eta) = p(\eta)/p^+(\eta)$ the part of $p(\eta)$ belonging to the roots within Γ. Multiply μ_{jl} of (4.25) with $(\varphi/p^+)(-\kappa_l)$ and sum over l, and observe that the last term in $\int_{\Gamma_-} J_{jl}\, dy$ supplies the term

$$\begin{aligned} &-\int_{\Gamma_-} h(\eta)d\eta/(\eta + \kappa_j) \sum_{l=1}^{r} \gamma_l(\varphi p^-)\,(-\kappa_l)/(\eta + \kappa_l) \\ &= -\int_{\Gamma_-} h(\eta)d\eta/(\eta + \kappa_j) \sum_{l=1}^{r} \gamma_l(\varphi p^-)(\kappa_l)/(\eta - \kappa_l), \end{aligned} \tag{4.27}$$

using the fact that

$$\int_{\Gamma_-} h(\eta)d\eta/(\eta + \kappa_j)(\varphi p^-)(\eta)(\eta^N + \rho^N)^{-1} = 0. \tag{4.28}$$

Therefore

$$\begin{aligned} &2\pi i \sum_{l=1}^{r} \mu_{jl}(\varphi/p^+)(-\kappa_l) = \sum_{k=1}^{r} \Theta_{jk}\Psi_k, \\ &\Theta_{jk} = \int_{\Gamma_-} h(\eta)((\eta + \kappa_j)\,(\eta - \kappa_k))^{-1}d\eta \\ &\Psi_k = p(\kappa_k)(\sum_{r}^{l=1} \tilde{r}_{kl}(\varphi/p^+)(-\kappa_l) - \gamma_k(\varphi/p^+)(\kappa_k)). \end{aligned} \tag{4.29}$$

Letting $\varphi_m(\eta) = \eta^{m-1}$, $m = 1, \ldots, r$, and observing that $(((\varphi_m/p^+)(-\kappa_l)))$ is a nonsingular Van der Monde, and that $((\Theta_{jk}))$ is nonsingular, we focus on the corresponding $\Psi_k = \Psi_{km}$. We also have $((s_{jl}^+)) = (((q_j/p)(\kappa_k)))$ nonsingular, and thus focus on

$$\sum_{k=1}^{r} s_{jk}^+ \Psi_{km} = \sum_{l=1}^{r} \gamma_l((q_j\varphi_m/p^+)(-\kappa_l) - (q_j\varphi_m/p^+)(\kappa_l)). \tag{4.30}$$

The right hand side of (4.30) is the sum of residues of the function $q_j(\eta)\eta^{m-1}/(p^+(\eta)(\eta^N + \rho^N))$ within some annulus $\rho - \varepsilon \leqq |\eta| \leqq \rho + \varepsilon$. This rational function has residue at infinity equal to zero. Therefore (4.30) takes the form (up to a non-vanishing constant)

$$\int_{\Gamma_+} q_j(\eta)\eta^{m-1}/(p^+(\eta)(\eta^N + \rho^N))\, d\eta. \tag{4.31}$$

Again (4.31) defines a non-singular matrix if and only if the boundary conditions $\langle b_j \rangle$ are md-elliptic with respect to $\langle a \rangle$.

This completes the proof of Theorem 0.1.

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