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SOME REMARKS ON THE ORIGINS OF THE THEORY OF FUNCTIONS OF A REAL VARIABLE AND OF THE DESCRIPTIVE SET THEORY[1]

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The descriptive set theory arose from the "theory of functions of a real variable" and this, in turn, had its roots in analysis—more particularly—in some erroneous statements of the calculus which appeared around the beginning of the XIXth century.

The fact that even the greatest analysts were led to false statements is not surprising in view of the way in which the calculus was being developed. It is well known that at the time of Newton and Leibniz the fundamental notions of calculus were not well defined and its development was rather dictated by its striking applications to physics which called the attention of their creators and was not, and could not be, rigourous from the logical point of view.

Nevertheless, the founders of the calculus and their immediate followers, like the Bernoullis, Euler, Lagrange etc., edified a wonderful building without taking care of its foundations. Generally speaking, their results were correct, which was certainly due to their infallible intuition.

However, this pleasant time came to an end. The reasonings, not well founded, even of the most celebrated mathematicians, started to lead to false statements.

May I mention two false theorems of Cauchy (theorems which played an important role in the creation of the Theory of functions of a real variable).

In his fundamental "Cours d'Analyse", edited in 1821, Cauchy claims that:

1. The limit of a convergent sequence of continuous functions is continuous [7, p. 120].

2. If a function f of two variables is continuous relatively to each variable separately, then f is continuous (relatively to both variables simultaneously).

Five years later, N.H. Abel gave a counter-example to the first of these statements.

For a counterexample to the second statement, one had to wait longer. It seems (according to P. Dugac; many details contained in my paper are

[1]Lecture held on May 17, 1977, at the University of Kansas at the Conference in honour of Prof. N. Aronszajn on his 70th birthday.

due to this author, as well as to F.A. Médvédév.) that the first to give the required counterexample was J. Thomae, who did it in 1870. Two years later, H.A. Schwarz defined another and much simpler example, which became classical, namely, the function f defined as follows:

$$f(x, y) = \frac{2xy}{x^2 + y^2} \text{ and } f(0, 0) = 0.$$

The discovery of functions which do not satisfy statement 2 led to a study of functions of two variables which are continuous relative to each of these variables separately and to important results chiefly due to René Baire (1897). For example, a function of that kind always admits points of continuity (without—as we know—being necessarily continuous); such a function is the limit of a convergent sequence of continuous functions. Also this kind of functions leads to the notion (introduced by Baire) of semi-continuity.

This approach of Baire to the fact that there exist some peculiar functions which can serve as counterexamples to some more or less intuitive statements is typical of Baire. He was not so much interested in the discovery of "pathological" functions; rather the theory of arbitrary functions which are not necessarily continuous attracted his attention. This attitude of Baire should be strongly emphasized. At this period of the development of mathematics, the search for pathological functions was very frequent; let us remember, for example, the work of H. Hankel, of P. du Bois-Reymond, etc., the numerous examples of continuous functions without derivatives, starting with the Weierstrass function (presented to the Academy of Berlin in 1872), followed by Dini, Darboux and others.

No wonder that among more conservative mathematicians there was some distrust for such a kind of investigations. So, for instance, Henri Poincaré wrote: "In the past, one invented new functions, for practical reasons; nowadays, it is done specifically for the purpose of finding errors in arguments of our fathers; nothing else will ever come out of it." (Citation from the introduction to the book of Saks.)

This angry opinion of Poincaré could by no means be applied to Baire (nor to Borel or Lebesgue). He was not a hunter of pathological functions; his aim was much more serious: he became one of the creators of a new branch of mathematics: the *Theory of functions of a real variable*. (Besides the above cited authors who largely contributed (in the XIXth century) to the development of the Theory of functions of a real variable, one has to mention Darboux, Dini, and Volterra.)

The investigation of functions of two variables we just spoke about formed a chapter of this new general theory. Even more important was the invention of several notions related to these investigations (like semi-continuity) and the research on functions which are limits of sequences of continuous functions (that is, functions of the first class according to the later terminology of Baire).

This last area of research gave rise to a classification of a vast set of functions. Considering functions of the first class, one was led at once to functions which are limits of a sequence of functions of the first class. It was natural to call them functions of the second class. Automatically, Baire introduced the notion of function of the nth class for arbitrary positive integer n. Presumably functions of class $\alpha < \omega_1$ were not studied by Baire. Nor was it known to Baire whether there exist functions of any class n ($n > 2$) which are not of lower classes. For $n = 2$ the problem had been solved by Lejeune-Dirichlet: the characteristic function of the set of rationals is of the second class without being of class 1, since each function of the first class admits points of continuity (by a theorem shown also by Baire).

It is remarkable that a classification of sets, parallel to that of functions, was proposed almost at the same time by E. Borel. He started with closed (or open sets) and proceeded, using two operations: the countable union and the countable intersection).

The philosophies of these two classifications, of functions and of sets, were analogous. That became clearer on the basis of a fundamental theorem of H. Lebesgue.

Namely, the function f is of class α if and only if the inverse image $f^{-1}(E)$ is a set of class α whenever E is a closed set (in the range of E).

The Theory of functions of a real variable, and in particular the work of Baire (like that of Borel), had been highly influenced by the Cantor set theory. For his purposes Cantor even introduced some new set-theoretical (now we would prefer to say: some topological) notions. Such is the notion of sets of *first category* (also called meager sets). Baire made fundamental use of this notion; let us cite three theorems involving that notion:

(i) the interval is not of first category on itself,

(ii) the set of points of discontinuity of a function of class 1 is of first category; i.e., almost all points are points of continuity,

(iii) every function f which belongs to the Baire classification is continuous neglecting a set of first category; i.e., there is a set E of the first category such that the partial function $f|E^c$ is continuous (E^c is the complement of E); this is called the Baire property of f.

Thus, in order to obtain these fundamental results of the Theory of a real variable, Baire not only had to use set theory (he seems to be one of the first in France to systematically apply set theory) but he also introduced and investigated some new notions in that theory. He did it—as we saw—with great success. Nevertheless, the use of set theory was met at that time with distrust by some mathematicians. Let us cite H. Poincaré once more.

At the International Mathematical Congress in Rome (1908), he said: "Set theory is like a childhood disease; one easily recovers and then one forgets it altogether". Poincaré predicted the same fate for set theory. . .

The opinion of Lebesgue was just the opposite: he attributed the great success of Baire to the joined use of analysis and set theory (see **[20]**).

Still one point seemed suspect even to the most prominent mathematicians: namely the use of transfinite numbers. So, for instance, Borel (in 1903) raised the question of avoiding transfinite numbers in the proof of the theorem on the characterization of function of class 1. A similar question of eliminating from a proof given by Baire (also on functions of class 1) was claimed by de la Vallée Poussin insolvable. This was erroneous: practically every time one applies transfinite numbers (at least this is true of applications made by Baire), these numbers can be eliminated using a general method (see **[17]**).

This controversy was due to the fact that at that time the role of transfinite (countable) numbers was not yet well understood.

The further development of the theory of real variables went in various directions. First of all, it ceased to be the theory of real variables. Several authors, among them Felix Hausdorff, extended it to the case where the range of the independent variable did not need any more to be restricted to the interval (or to the n-dimensional cube or to n-dimensional Euclidean space). One could assume quite generally that f is a mapping defined on a metric space, without affecting the essential results of Baire or Lebesgue.

The next generalization of the theory of functions of a real variable concerned the values of the functions. Instead of restricting them to real or complex values, one can make practically no assumptions on their range (except for the assumption of metrizability or sometimes of completeness or separability). This important step was made essentially by Banach (see, for example, [**18**, §31]).

In this way, the theory of functions of a real variable became a part of a larger chapter of mathematics, namely of the *descriptive theory of sets*.

An important event for the development of descriptive set theory—and in particular for the theory of real variables—was the introduction in topology of two so-called *hyperspaces*, denoted Y^X and 2^X. The first denotes the space of all continuous mappings $f: X \to Y$ (it is essentially due to Fréchet); the second, the space of all closed subsets of X (sometimes supposed non-empty), is essentially due to Hausdorff and Vietoris.

For the sake of simplicity, we shall assume X to be compact metric and Y metric. With these assumptions the hyperspaces are metrizable in a very natural way (the first is complete whenever Y is such, the second is compact).

The introduction of the space Y^X (even E^I, where E denotes the space of reals and I denotes the closed interval [0, 1]) had an immense influence on the proofs of many existence theorems. So, for instance, the set of continuous functions which have a derivative at least at one point is a set of the first category in the space E^I. Since this space is complete, it follows by the Baire theorem that almost every continuous function has no derivative at any point (This is a theorem of Mazurkiewicz and Banach; the problem was raised by H. Steinhaus **[27]**); of course, the Weierstrass function is one of them and—in view of this general theorem—it has lost very much of its interest (except its purely historical interest).

The "pathological" function of Weierstrass was a counterexample to a statement of Ampère **[2]**, who claimed that every function has a derivative except at some isolated points (the term "every" is not clear; let us remember that at that time the notion of a continuous function was not yet defined). That was the time when numerous mathematicians worked on this and related problems. The great and rather conservative mathematician, Ch. Hermite wrote to Stieltjes: "I abhor this deplorable plague of functions without derivatives" (citation from S. Saks). One can wonder what Hermite would say if he learned that this "deplorable plague" contains almost the totality of continuous functions?

The above application of Baire Theorem in the space Y^X is just one of numerous examples. This so-called first category method has been also successfully applied to functions of complex variables and to many other problems (see **[18**, §34], where numerous papers of Banach, Mazurkiewicz, Orlicz, Steinhaus etc. in that connection are cited).

It is worth noticing that the category method may be used not only for defining pathological functions; sometimes it leads to positive results. Let us recall an example. By the Menger-Nöbeling theorem, every n-dimensional metric separable space X can be topologically embedded in the $(2n + 1)$-dimensional cube I^{2n+1}. Now one shows (Hurewicz, in 1933 and, in a more general form, myself in 1938, see **[18]**) that almost every $f \in (I^{2n+1})^X$ is the required homeomorphism.

Much similar phenomena can be observed in investigating the hyperspace 2^X (in particular 2^I). Thus, the theorem about the existence of non-Borel sets on the interval is given using rather artificial sets (whose only interest lies in being non-Borel). The situation in the space 2^I is quite different. As was shown by Hurewicz (in 1930) the totality of all countable closed subsets of the space 2^I is non-Borel. Alike, the family of closed sets containing no rational numbers is again an example of a non-Borel subset of 2^I. Both examples are certainly not artificial.

The first category method has also interesting applications to the hyperspace 2^X. Let us cite the following one.

A continuum is called indecomposable if it cannot be represented as a union of two proper subcontinua. Very striking are hereditarily indecomposable continua, i.e., continua all of whose subcontinua are indecomposable. The first example of this kind of continua was defined (in a very complicated way) by B. Knaster **[16]** in 1922.

Now denote by $C(X)$ the subspace of 2^X composed of continua. Mazurkiewicz **[21]** showed in 1930 that almost all subcontinua of the square (considered as elements of $C(I^2)$) are hereditarily indecomposable. This discovery of S. Mazurkiewicz was all the more striking because at that time the hereditarily indecomposable continua were considered as the most complicated geometrical figures (according to an expression of Steinhaus). On the other hand, they deserved—and still deserve—to be deeply investigated because of the multitude of their important properties (they have been used to solve many problems raised long before they were discovered).

It is worth mentioning in that connection the *Janiszewski curve.* At the International Mathematics Congress in Cambridge in 1912. Janiszewski defined (or rather sketched a definition) of a curve on the plane which contains no arc. At that time, the existence of a curve of that kind seemed quite extraordinary. Now—since every hereditarily indecomposable continuum obviously contains no arc—we know that almost every continuum on the plane is a Janiszewski curve.

These few examples of the application of the *category method* prove its strength beyond a doubt. Morever, contrary to the opinion expressed by some authors (e.g., Médvédév **[23**, p. 220]), this method does not lead to "pure existence" theorems; in fact, it is effective, meaning that the use of the method of category enables us to obtain a well-defined example having the property under consideration (e.g., a continuous function with no derivative at any point; see the Appendix). This is connected with the effectiveness of the Baire theorem asserting the existence of a point outside a set of the first category (this is true of any complete separable metric space where a countable dense subset is well defined).

Set-valued mappings. It is natural to ask whether the elementary operations on sets: union ($A \cup B$), intersection ($A \cap B$) and the closure of the difference $\overline{(A - B)}$ are continuous or Baire functions from $2^X \times 2^X$ to 2^X (here we assume that $\emptyset \in 2^X$).

One shows that the union is a continuous mapping, while the intersection and the closure of the difference are of the first Baire class (which don't need to be continuous). The boundary $F(A) = A \cap \overline{A^c}$ is of the second Baire class. So is also the derived mapping A^d (considered as function of A). On the other hand the mapping A^{cond} (composed of condensation points) is not a B-measurable function of A (which follows easily from the above cited theorem of Hurewicz).

Thus far, we have considered problems of Baire or Borel classification which led to the descriptive set theory.

No less important from this point of view, are analytic (Suslin) sets, their complements and more generally projective sets (connected with the names of Lusin, Sierpiński, Alexandrov and others). However, we thought it reasonable to refrain from considering these kinds of problems in this talk, which was assumed to be rather short; they certainly deserve much attention, they are developing very fast, and they form a separate chapter for themselves.

Appendix. We construct a well-defined continuous function having no derivative, with the help of the category method.

Let ϕ be the space of continuous functions $f: I \to E$ which means that $\phi = E^I$. First, let us note that if a function f has a derivative $f'(a)$ in the point a, then the function g of h defined by the condition

$$g(h) = \frac{f(a+h) - f(a)}{h} \quad \text{for } h \neq 0$$

is bounded.

There is namely, an $\varepsilon > 0$ such that $|g(h) - f'(a)| < 1$ for $|h| < \varepsilon$, and on the other hand g is bounded for $|h| \geqq \varepsilon$.

So denote by M_n the subset of ϕ composed of functions f for which there exists a point a such that $|g| \leqq n$. Put $S = M_1 \cup M_2 \cup \dots$. Then our problem of defining a continuous function without derivative reduces to defining a function $f \in \phi - S$.

The property of M_n being closed in ϕ is almost obvious (see, e.g., [**17**, p. 421]). It is also quite easy to show that M_n is nowhere dense (in view of the fact that the set of polynomials with rational coefficients is dense in ϕ).

It follows that S is of the first category in ϕ. Since ϕ is a complete space and the set S is of the first category, then the set $\phi - S$ is non-empty (by the Baire Theorem). Our problem is to define an element of this set.

We proceed as follows. Let us consider the (countable) set of all "balls" in ϕ of the form $A_{p,k} = \{f : |f - p| \leqq 1/k\}$, where p (the "center" of the ball) ranges over the set P of all polynomials with rational coefficients, and $k = 1, 2, \dots$. Arrange the totality of the balls $A_{p,k}$ in a well defined sequence $B_1, B_2, \dots$, and define (by induction) the following subsequence $B_{n_1}, B_{n_2}, \dots$.

n_1 is the least index such that

$$B_{n_1} \cap M_1 = \varnothing \text{ and } \delta(B_{n_1}) < 1.$$

(Such an index exists since P is dense and M_1 is nowhere dense.)

n_2 is the smallest index such that

$$B_{n_2} \cap M_2 = \varnothing,\ B_{n_2} \subset B_{n_1} \text{ and } \delta(B_{n_2}) < 1/2.$$

Generally, n_k is the smallest index such that

$$B_{n_k} \cap M_k = \emptyset, \; B_{n_k} \subset B_{n_{k-1}} \text{ and } \delta(B_{n_k}) < 1/k.$$

By a classical theorem of Cantor, the intersection

$$B_{n_1} \cap B_{n_2} \cap \cdots$$

is not-empty and hence consists of a single element of ϕ. This element is the required element f of $\phi - S$.

Otherwise stated, the function

$$f = \lim_{k=\infty} p_{n_k}$$

is a continuous function with no derivative (p_{n_k} is the center of B_{n_k}).

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