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GROWTH OF DERIVATIVES AND THE MODULUS OF CONTINUITY OF ANALYTIC FUNCTIONS

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1. **Introduction.** Let G be a bounded complex domain and let $f(\xi)$ be analytic on G and continuous on $\overline{G}$. The modulus of continuity of $f(\xi)$ on $\overline{G}$ is a function $\omega(\delta, f, \overline{G})$ defined for $\delta \geqq 0$ by

$$\omega_f(\delta) = \omega(\delta, f, \overline{G}) \quad \sup_{\substack{\xi_1, \xi_2 \in \overline{G} \\ |\xi_1 - \xi_2| \leqq \delta}} |f(\xi_1) - f(\xi_2)|. \tag{1}$$

If

$$\omega_f(\delta) \leqq C\delta^{\alpha},$$

for some $0 < \alpha \leqq 1$ and some constant $C > 0$, then $f(\xi)$ satisfies a Lipschitz condition of order α on $\overline{G}$.

If $G = D = \Delta(0, 1)$ is the open unit disk, a classical theorem of Hardy and Littlewood [1] shows that $f(\xi)$ satisfies a Lipschitz condition of order α on $\overline{D}$ if and only if

$$|f'(\xi)| \leqq C(1 - |\xi|)^{\alpha - 1}$$

for all $\xi \in D$. The positive constant C is independent of ξ. By conformal mapping, the Hardy-Littlewood theorem can be generalized to the case in which G is replaced by a bounded, simply connected domain G with analytic boundary. In particular, if

$$d(\xi, \partial G) = d_{\xi} = \inf_{z \in \partial G} |\xi - z|$$

denotes the distance from a point $\xi \in G$ to ∂G, then the following result holds [4].

THEOREM 1. *Let G be a bounded, simply connected domain with analytic boundary. A function $f(\xi)$ analytic on G and continuous on $\overline{G}$ satisfies a Lipschitz condition of order α on $\overline{G}$ if and only if*

$$|f'(\xi)| \leqq C\{d_{\xi}\}^{\alpha - 1},$$

for all $\xi \in G$.

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As the following well known result shows, Theorem 1 readily generalizes in the necessary direction.

THEOREM 2. *Let G be a bounded complex domain and let $f(\xi)$ have modulus of continuity $\omega_f(\delta) = \omega(\delta, f, \overline{G})$. Then*

$$|f'(\xi)| \leqq \frac{\omega_f(d_\xi)}{d_\xi},$$

for all $\xi \in G$.

In this paper we show that Theorem 1 also generalizes in the sufficient direction, and hence show that Theorem 1 actually holds with much weaker conditions on ∂G. For example, we show the result holds if G is a domain with minimally smooth boundary [5]. This generalization of Theorem 1 will follow from a more general theorem in which we relate the modulus of continuity of a function $f(\xi)$ on $\overline{G}$ to the smoothness of ∂G and the growth of $|f'(\xi)|$.

2. Definitions and the Main Result.

Before stating our main result, we require some definitions classifying the smoothness of the boundary of a domain. These definitions and the following results involve positive constants, denoted by "C"; subsequent appearance of "C" will denote possibly different positive constants.

DEFINITION 1. A function $\omega(x)$, defined for $x \geqq 0$, is a modulus of continuity if ω is increasing, subadditive and $\lim_{x \to 0^+} \omega(x) = 0$.

Note that $\omega(\delta, f, \overline{G})$, the modulus of continuity of $f(\xi)$ on $\overline{G}$ defined in (1), need not be a modulus of continuity in the sense of Definition 1; in particular, $\omega(\delta, f, \overline{G})$ need not be subadditive.

The next two definitions concern the smoothness of the boundary of a domain G. In special cases, these definitions coincide with the definitions of special Lipschitz domains and domains with minimally smooth boundary [5].

DEFINITION 2. Let λ be a modulus of continuity. A domain G is a λ-domain if there is a function $\phi : R \to R$ and a positive constant M such that

$$G = \{x + iy : y > \phi(x)\},$$

and

$$|\phi(x) - \phi(x')| \leqq M\lambda(|x - x'|), \tag{2}$$

for all $x, x' \in R$. The smallest M for which (2) holds is the bound for G.

A λ-domain as described above is in the standard position. Any rotation of a λ-domain is also a λ-domain.

DEFINITION 3. A bounded, simply conncected domain G is the local λ-domain if there exist positive constants ϵ and M and a sequence $\{U_i : i = 1, 2, \cdots\}$ of open sets such that:

(i) For each $z \in \partial G$, there is a U_i with $\Delta(z, \epsilon) \subseteq U_i$.

(ii) For each U_i, there is a λ-domain G_i with bound not exceeding M such that

$$U_i \cap G_i = U_i \cap G.$$

M is called a bound for G. If $\lambda(x) = Cx^\alpha$ (some $0 < \alpha \leqq 1$), then G is a local Lip(α)-domain.

Definition 3 describes what might be called a cusp-condition on ∂G.

In [2], Lorentz shows that if ω is a modulus of continuity as defined in Definition 1, then there is a concave modulus of continuity λ with

$$\lambda(x) \leqq \omega(x) \leqq 2\lambda(x),$$

for all $x \geqq 0$. In the remainder of this paper all moduli of continuity will be assumed concave unless otherwise stated. This assumption will also hold for those moduli of continuity implicit in Definitions 2 and 3.

We recall that if $\lambda(x)$ $(x \geqq 0)$ is concave, then $\lambda(x)$ is continuous for $x \geqq 0$, has a right hand derivative $D^+\lambda(x)$ at each $x \geqq 0$ (with, possibly, $D^+\lambda(0) = +\infty$), and a left hand derivative $D^-\lambda(x)$ at each $x > 0$. For $0 \leqq x < y$, we have

$$D^+\lambda(x) \geqq D^-\lambda(y) \geqq D^+\lambda(y).$$

Thus $\lambda'(x)$ exists and is continuous for all but at most countably many x. If E is the set on which $\lambda'(x)$ is not continuous, then $\lambda'(x)$ has jump discontinuities at each $x \in E$.

We now state our main result.

THEOREM 3. *Let G be a local λ-domain and let μ be a modulus of continuity. Suppose $f(\xi)$ is analytic on G, continuous on $\overline{G}$, and*

$$|f'(\xi)| \leqq \frac{\mu(d_\xi)}{d_\xi}$$

for each $\xi \in G$. Then there is an $\eta > 0$ such that

$$\omega(\delta, f, \overline{G}) \leqq C \int_0^\delta \frac{\mu(t)\lambda'(t)}{t}\, dt, \tag{3}$$

for all $\delta \leqq \eta$ (In this case, $\omega(\delta, f, \overline{G})$ is not necessarily a modulus of continuity as defined in Definition 1; thus $\omega(\delta, f, \overline{G})$ is not assumed to be concave.) *In* (3), *dt is Lebesgue measure.*

Of course Theorem 3 is of interest only when $\mu(t)\,\lambda'(t)/t$ is integrable on $[0, \delta]$; that is, when the right side of (3) is finite. In Section 4 we give some consequences of Theorem 3 for special choices of μ and λ.

3. **Proof of Theorem 3.** The proof of Theorem 3 depends on the following lemma.

LEMMA 4. *Let G be a local λ-domain with bound M. Let $z_0, z_1 \in \partial G$ with $|z_0 - z_1| < \epsilon/4$. Let U_i be an open set* (see Definition 3) *with $\Delta(z_0, \epsilon) \subseteq U_i$, and let G_i be the λ-domain associated with U_i. Suppose G_i is rotated through angle $\theta (\theta \geqq 0)$ from standard position. There is a positive constant $c = c(\lambda, M, \epsilon) > 0$ such that for $\xi \in G$ with $|\xi - z_1| < \epsilon/4$ and $\arg(\xi - z_1) = \theta + \pi/2$, we have*

$$c\lambda^{-1}\left(\frac{|z_1 - \xi|}{M}\right) \leqq d_\xi \leqq |z_1 - \xi|. \tag{4}$$

PROOF. Since rotation of G does not affect the result, we assume G_i is in standard position; that is, $\theta = 0$. The right-hand inequality in (4) is clear. To establish the left inequality, we first show $d(\xi, \partial G) = d(\xi, \partial G_i)$. Let $z \in \partial G$ with $|z - \xi| = d(\xi, \partial G)$, Then

$$|z_0 - z| \leqq |z_0 - z_1| + |z_1 - \xi| < \epsilon/2.$$

Thus,

$$z \in \overline{U_i \cap G} = \overline{U_i \cap G_i}.$$

It follows that $d(\xi, \partial G) \geqq d(\xi, \partial G_i)$. The opposite inequality is proved in a similar way. Lemma 4 now follows from Lemma 5.

LEMMA 5. *Let G be a λ-domain with bound M and in standard position. Let $\eta > 0$ be given. There exists a constant $c = c(\lambda, M, \eta) > 0$ such that if $z \in \partial G$ and $\xi \in G$ with* $\mathrm{Re}(z) = \mathrm{Re}(\xi)$ *and $|z - \xi| \leqq \eta$, then*

$$d_\xi \geqq c\lambda^{-1}\left(\frac{|z - \xi|}{M}\right).$$

PROOF. Since G is in standard position, we can assume $z = 0$ and $\xi = ia$ with $\eta \geqq a > 0$. Let Γ denote the graph of $y = M\lambda(|x|)$ and let ℓ denote the line through $(\lambda^{-1}(a/M), a)$ with slope $MD^{+}\lambda\{\lambda^{-1}(a/M)\}$. Since $\lambda(x)$ $(x \geqq 0)$ is concave, we have

$$d_\xi \geqq d(\xi, \Gamma) \geqq d(\xi, \ell) = \frac{MD^+\lambda\{\lambda^{-1}(a/M)\}\,\lambda^{-1}(a/M)}{[1 + \{MD^+\lambda\,(\lambda^{-1}(a/M))\}^2]^{1/2}}$$

$$\geqq \left(\frac{MD^+\lambda\{\lambda^{-1}(n/M)\}}{[1 + \{MD^+\lambda(\lambda^{-1}(n/M))\}^2]^{1/2}} \right) \lambda^{-1}\left(\frac{a}{M}\right)$$

$$= c\,\lambda^{-1}\left(\frac{a}{M}\right).$$

We now prove Theorem 3.

PROOF. Let z_0, $z_1 \in \partial G$ with

$$|z_0 - z_1| \leqq \eta = \min\left\{\frac{\epsilon}{6}, \lambda^{-1}\left(\frac{\epsilon}{(6M)}\right)\right\},$$

where M is a bound for G. We will prove our result by writing

$$|f(z_1) - f(z_0)| = \left| \int_{\mathscr{C}} f'(\xi)\, d\xi \right|, \tag{5}$$

where $\mathscr{C}$ is an appropriate path of integration, and then estimating the integral in (5).

Let U_i be an open set (see Definition 3) wih $\Delta(z_0, \epsilon) \subseteq U_i$. Assume the associated λ-domain G_i is the standard position; we can then assume $\mathrm{Re}(z_0) < \mathrm{Re}(z_1)$. Select w_0, $w_1 \in G$ with $\mathrm{Re}(z_0) = \mathrm{Re}(w_0)$, $\mathrm{Re}(z_1) = \mathrm{Re}(w_1)$ and

$$|w_0 - z_0| = M\lambda(|z_0 - z_1|) = |w_1 - z_1|.$$

Let γ_0 be the graph of $y = M\lambda(x)$ $(x \geqq 0)$ translated so its vertex is at w_0, and let γ_1 be the graph of $y = M\lambda(|x|)$ $(x \leqq 0)$ translated so its vertex is at w_1. Let w_2 be the intersection of γ_0 and γ_1. We take $\mathscr{C}$ to be the path from z_0 to w_0, along γ_0 to w_2, along γ_1 to w_1 and from w_1 to z_1 (see Figure 1).

For any $\xi \in \mathscr{C}$, we have

$$|z_0 - \xi| \leqq |z_0 - z_1| + |w_0 - z_0 + M\lambda(|z_0 - z_1|) < \epsilon/2.$$

Thus $\mathscr{C} \subseteq G_i \cap U_i = G \cap U_i$ and

$$\mathrm{d}_\xi = d(\xi, \partial G_i),$$

for any $\xi \in \mathscr{C}$.

We can now write

$$\begin{aligned} |f(z_0) - f(z_1)| &\leqq \int_{\mathscr{C}} |f'(\xi)|\,|d\xi| \\ &= \underset{\gamma_0}{\int_{z_0}^{w_0}} + \underset{\gamma_0}{\int_{w_0}^{w_2}} + \underset{\gamma_1}{\int_{w_1}^{w_2}} + \int_{z_1}^{w_1} |f'(\xi)|\,|d\xi| \end{aligned} \tag{6}$$

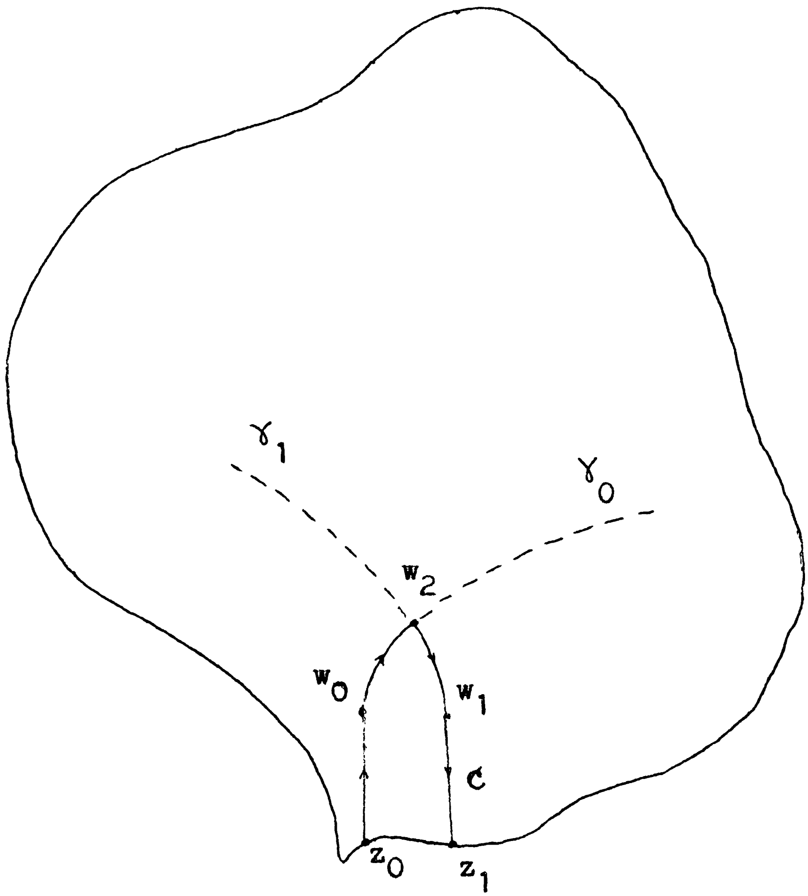

FIGURE I.

The path C is indicated by arrows.

and proceed to estimate these four integrals.

For the first,

$$\int_{z_0}^{w_0} |f'(\xi)|\ |d\xi| \leqq \int_{z_0}^{w_0} \frac{\mu(d_\xi)}{d_\xi} |d\xi|.$$

Since $\mathrm{Re}(\xi) = \mathrm{Re}(z_0)$ and $|z_0 - \xi| < \epsilon/4$ for $\xi \in [z_0, w_0]$, Lemma 4 gives

$$d_\xi \geqq c\lambda^{-1}\left(\frac{|z_0 - \xi|}{M} \right).$$

Since $\mu(x)$ is concave, $\mu(x)/x$ is non-increasing. Hence,

$$\int_{z_0}^{w_0} |f'(\xi)|\,|d\xi| \leqq \frac{c+1}{c}\int_{z_0}^{w_0} \frac{\mu(\lambda^{-1}(|z_0-\xi|/M))}{\lambda^{-1}(|z_0-\xi|/M)}\,|d\xi|$$

$$= c(|z_0 - w_0|)\int_0^1 \frac{\mu(\lambda^{-1}(t|z_0-w_0|/M))}{\lambda^{-1}(t|z_0-w_0|/M)}\,dt.$$

Set $s = \lambda^{-1}(t|z_0 - w_0|/M)$ and recall w_0 was chosen so that

$$\lambda^{-1}\left(\frac{|z_x - w_0|}{M}\right) = |z_0 - z_1|.$$

Then

$$\text{(7)} \qquad \int_{z_0}^{w_0} |f'(\xi)|\,|d\xi| \leqq C\int_0^{|z_0-z_1|} \frac{\mu(s)\,\lambda'(s)}{s}\,ds.$$

The same argument gives this bound for the fourth integral in (6).

For the second (or third) integral in (6), we have

$$\int_{\substack{w_0\\ \gamma_0}}^{w_2} |f'(\xi)|\,|d\xi| \leqq \int_{\substack{w_0\\ \gamma_0}}^{w_2} \frac{\mu(d_\xi)}{d\xi}\,|d\xi|.$$

Let Γ_0 be γ_0 translated so its vertex is at z_0. Then for $\xi \in \gamma_0$,

$$d_\xi \geqq d(\xi, \Gamma_0),$$

giving

$$\int_{\substack{w_0\\ \gamma_0}}^{w_2} |f'(\xi)|\,|d\xi| \leqq \int_{\substack{w_0\\ \gamma_0}}^{w_2} \frac{\mu(d(\xi,\Gamma_0))}{d(\xi,\Gamma_0)}\,|d\xi|.$$

Since γ_0 is concave and is a vertical translate of Γ_0, $d(\xi, \Gamma_0)$ increases as ξ moves along γ_0 away from w_0. Thus,

$$d(\xi, \Gamma_0) \geqq d(w_0, \Gamma_0),$$

for $\xi \in \gamma_0$. By Lemma 4,

$$d(w_0, \Gamma_0) \geqq c\,\lambda^{-1}\left(\frac{|w_0 - z_0|}{M}\right) = c|z_0 - z_1|.$$

Thus,

$$\text{(8)} \qquad \int_{\substack{w_0\\ \gamma_0}}^{w_2} |f'(\xi)|\,|d\xi| \leqq C\,\frac{\mu(|z_0 - z_1|)}{|z_0 - z_1|}\cdot \ell(\gamma_0),$$

where $\ell(\gamma_0)$ is the length of γ_0 from w_0 to w_2. We have

$$\text{(9)} \qquad \ell(\gamma_0) \leqq \int_0^{|z_0-z_1|} (1 + (\lambda'(t))^2)1/2\,dt \leqq C\lambda(|z_0 - z_1|).$$

Since $\mu(t)/t$ is decreasing,

$$\int_0^{\delta} \frac{\mu(t)\,\lambda'(t)}{t}\,dt \geqq \frac{\mu(\delta)\,\lambda(\delta)}{\delta}. \tag{10}$$

Combining (7), (8), (9) and (10) shows

$$|f(z_0) - f(z_1)| \leqq C \int_0^{|z_0 - z_1|} \frac{\lambda'(t)\,\mu(t)}{t}\,dt$$

for $|z_0 - z_1| \leqq \eta$; that is, for $\delta \leqq \eta$,

$$\tilde{\omega}(\delta, f, \partial G) \leqq C \int_0^{\delta} \frac{\lambda'(t)\,\mu(t)}{t}\,dt, \tag{11}$$

where $\tilde{\omega}(\delta, f, \partial G)$ is the modulus of continuity of f on ∂G. The integral on the right in (11) is a positive, non-decreasing, subadditive function of $\delta \geqq 0$. It follows from a theorem of Rubel, Taylor and Shields [3], that

$$\omega(\delta, f, \overline{G}) \leqq C \int_0^{\delta} \frac{\lambda'(t)\,\mu(t)}{t}\,dt,$$

for $\delta \leqq \eta$.

4. **Consequences and Examples.** Several interesting corollaries arise as special cases of Theorem 3.

COROLLARY 6. *If, in addition to the hypotheses of Theorem* 3, *we have*

$$\liminf_{t\to 0} \left\{ \frac{t\,\lambda'(t)}{\lambda(t)} + \frac{t\,\mu'(t)}{\mu(t)} \right\} = \alpha > 1, \tag{12}$$

then there is an $\tilde{\eta} > 0$ *such that*

$$\omega(\delta, f, \overline{G}) \leqq C \frac{\mu(\delta)\,\lambda(\delta)}{\delta},$$

for $\delta \leqq \tilde{\eta}$. *Furthermore,* $\mu(t)\,\lambda(t)/t$ *is a modulus of continuity.*

Before proving Corollary 6 we list some of its immediate consequences.

COROLLARY 7. *Let G be a local* Lip(α) *domain and let* $\beta (0 < \beta \leqq 1)$ *be given with* $\alpha + \beta > 1$. *If* $f(\xi)$ *is continuous on* $\overline{G}$, *analytic on G and*

$$|f'(\xi)| \leqq C\, d_{\xi}^{\beta - 1},$$

for all $\xi \in G$, *then* $f(\xi)$ *satisfies a Lipschitz condition of order* $\alpha + \beta - 1$ *on* $\overline{G}$.

Corollary 8. *Suppose G is local* Lip(1) *domain and* $\mu(t)$ *is a modulus of continuity with*

$$\liminf_{t\to 0} \frac{t\,\mu'(t)}{\mu(t)} > 0.$$

Then a function $f(\xi)$ *analytic on G and continuous on* $\overline{G}$ *has modulus of continuity*

$$\omega(\delta, f, \overline{G}) \leqq C\mu(\delta),$$

for $0 \leqq \delta \leqq \tilde{\eta}$ *if and only if*

$$|f'(\xi)| \leqq C\frac{\mu(d_\xi)}{d_\xi},$$

for all $\xi \in G$.

Corollary 9. *The conclusion of Theorem* 1 *holds if G is a local Lip*(1) *domain.*

We now prove Corollary 6.

Proof. From (12) it follows that given α' with $\alpha > \alpha' > 1$, there exists $\eta' > 0$ so that $0 < t < \eta'$ implies

$$\mu(t)\,\lambda'(t) + \mu'(t)\,\lambda(t) > \alpha'\frac{\mu(t)\,\lambda(t)}{t} > \frac{\mu(t)\,\lambda(t)}{t}, \tag{13}$$

for those t for which $\mu'(t)$ and $\lambda'(t)$ exist. In particular, we find that $\mu(t)\,\lambda(t)/t$ is increasing and

$$\lim_{t\to 0+}\frac{\mu(t)\,\lambda(t)}{t}$$

exists.

If $\delta \leqq \tilde{\eta} = \min(\eta, \eta')$, then Theorem 3 implies

$$\begin{aligned}
\omega(\delta, f, \overline{G}) &\leqq C\lim_{\tau\to 0+}\int_\tau^\delta \frac{\mu(t)\,\lambda'(t)}{t}\,dt \\
&\leqq C\lim_{\tau\to 0+}\int_\tau^\delta \frac{\mu(t)\,\lambda'(t) + \mu'(t)\,\lambda(t)}{t}\,dt \\
&\leqq C\frac{\alpha'}{\alpha'-1}\lim_{\tau\to 0+}\int_\tau^\delta \frac{t\,\mu(t)\,\lambda'(t) + t\,\mu'(t)\,\lambda(t) - \mu(t)\lambda(t)}{t^2}\,dt
\end{aligned}$$

$$= C \left(\frac{\mu(\delta)\,\lambda(\delta)}{\delta} - \lim_{\tau \to 0+} \frac{\mu(\tau)\,\lambda(\tau)}{\tau} \right)$$

$$\leqq C \frac{\mu(\delta)\,\lambda(\delta)}{\delta} .$$

Now $\mu(t)\,\lambda'(t)/t$ is integrable on $[0, \delta]$, so (10) shows

$$\lim_{\delta \to 0^+} \frac{\mu(\delta)\,\lambda(\delta)}{\delta} = 0.$$

Finally, $\mu(\delta)\,\lambda(\delta)/\delta$ is subadditive ([6] p. 97). Thus $\lambda(t)\,\mu(t)/t$ is a modulus of continuity for $t \geqq 0$.

We give an example showing that in one sense Corollary 7 is best possible; we show that if $\alpha + \beta = 1$, then $f(\xi)$ need not satisfy a Lipschitz condition of order γ for any $0 < \gamma \leqq 1$. Let $\overline{G}$ be a closed domain lying in $\{\mathrm{Im}(\xi) > 0\} \cup \{0\}$ with the following property: G is a bounded domain, and in a neighborhood of 0, ∂G is the graph of $y = |x|^{\alpha}$, while outside of this neighborhood, ∂G is smooth (say, analytic.) Then G is a local $\mathrm{Lip}(\alpha)$ domain.

We take

$$f(\xi) = \begin{cases} \dfrac{1}{\log \xi} & (\xi \neq 0) \\ 0 & (\xi = 0) \end{cases}$$

on $\overline{G}$. Then $f(\xi)$ is analytic on G (we take the branch cut in the lower half plane) and continuous on $\overline{G}$. Note that $f(\xi)$ does not satisfy a Lipschitz condition of any order $\beta > 0$ near the origin, and that

$$f'(\xi) = -\frac{1}{\xi \log^2 \xi} ,$$

for $\xi \in G$.

If $\xi \in G$ is close to 0, we have

$$d_{\xi} \leqq d_{i|\xi|} \leqq |\xi|^{1/\alpha}.$$

Thus,

$$|f'(\xi)| = \frac{1}{|\xi \log^2 \xi|}$$

$$\leqq \frac{1}{|\xi|\,\log^2 |\xi|}$$

$$\begin{aligned} &\leqq \frac{1}{\alpha^2 (d_\xi)^\alpha \log^2(d_\xi)} \\ &\leqq C(d_\xi)^{-\alpha} \\ &= C(d_\xi)^{\beta-1}, \end{aligned}$$

with $\beta = 1 - \alpha$. Thus $\alpha + \beta = 1$, but $f(\xi)$ satisfies no Lipschitz condition of positive order on $\overline{G}$.

5. **Further Questions.** In Theorem 6, we obtain no information about $\omega(\delta, f, \overline{G})$ if $\mu(t)\,\lambda'(t)/t$ is not integrable on $[0, \delta]$. Can bounds on $\omega(\delta, f, \overline{G})$ be obtained under weaker conditions? Are there examples showing Corollaries 6 and 7 are best possible, or are stronger results possible? In construction of examples, an answer to the following question would be useful: Let G be a bounded domain and let $\lambda(t)$ be a modulus of continuity. Under what conditions does there exist a function $f(\xi)$ analytic on G, continuous on $\overline{G}$ with

$$c\lambda(t) \leqq \omega(t, f, \overline{G}) \leqq C\,\lambda(t),$$

for some positive constants c and C independent of t?

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