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ON SOLVING THE EQUATION Aut $(X) = G$

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ABSTRACT. Given a finite group G, are there at most a finite number of finite groups X such that Aut $(X) \approx G$? If so, how does one determine all of them?

The first question has an affirmative answer. In this paper we consider the second question when G is a finite nonabelian simple group or a natural extension of it, a dihedral group, a dicyclic group or a quasidihedral group.

1. **Introduction.** G. A. Miller, in 1900, considered the problem of finding all finite groups having S_3 as their group of automorphisms. He proved that $C_2 \times C_2$ and S_3 are the only such groups. He also determined all finite groups having S_4 as their group of automorphisms. The reader is referred to [10]. De Vries and De Miranda [13] have investigated groups, finite and infinite, with a small number of automorphisms. Heineken and Liebeck [6] and Hallett and Hirsch [4] have worked on similar problems. Finite groups with abelian automorphism groups have been studied by B. E. Earnley [2]. Baer [14] proved in 1955 that a torsion group whose automorphism group is finite must itself be finite. Alperin [15] in 1961 characterized finitely generated groups with finite automorphism groups. Recently D. J. S. Robinson [16] has studied the consequences for a group of the finiteness of its automorphism group.

In (3.1) we prove that given a finite group G there are at most a finite number of finite groups X such that Aut $(X) \approx G$. It is to be expected that the problem of finding all these groups in any given instance would be difficult in general. However, a knowledge of Schur multipliers of various groups and some results concerning the group of central automorphisms of a group combined with elementary group theoretical arguments enables one to solve the above problem in certain instances.

2.1. **Terminology.** All groups mentioned in this paper are finite. Suppose G is a group. The following notation will be used:

$|G|$ = The order of the group G.
$|G|_p$ = The order of a Sylow p-subgroup of G.
π = The set of all primes.
$\pi(G) = \{p \in \pi | p | \ |G|\}$.
(m, n) = The greatest common divisor of the integers m and n.

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$\operatorname{Aut}(G) =$ The automorphism group of G.
$\operatorname{Inn}(G) =$ The inner automorphism group of G.
$A_c(G) = C_{\operatorname{Aut}(G)}(\operatorname{Inn} G) =$ The group of central automorphisms of G.
$C_n =$ The cyclic group of order n.
$Q =$ The quaternion group of order 8.

A PN-group (Purely Nonabelian) is a group having no nontrivial abelian direct factors.

2.2. **Covering groups of a group G.** For an arbitrary group G, we let $C(G)$ be the set of all ordered pairs (L, λ) such that L is a group and $\lambda : L \longrightarrow G$ is an epimorphsim with $\operatorname{Ker} \lambda \subseteq L' \cap Z(L)$. If in addition, $|\operatorname{Ker} \lambda| = |H^2(G, C^\times)|$ (where $C^\times$ is the multiplicative group of non-zero complex numbers) then we say that L is a covering group of G and λ is the associated epimorphism, or for short, (L, λ) is a covering group of G.

The Schur multiplier $M(G)$ of a group G is an abelian group uniquely determined by G such that whenever (L, λ) is a covering group of G, $\operatorname{Ker} \lambda \approx M(G)$.

Two elements (L, λ) and (M, μ) of $C(G)$ are said to be equivalent if there exists an isomorphism $\alpha : L \longrightarrow M$ of L onto M such that $\mu \circ \alpha = \lambda$ and we write $(L, \lambda) \sim (M, \mu)$. Clearly $\sim$ is an equivalence relation on $C(G)$.

If the covering groups of G are all equivalent, then we say that G has a unique covering group and any one of these will be denoted by $\hat{G}$. We say that G is centrally closed if $C(G)$ has precisely one $\sim$-equivalence class for which, of course, $(G, 1_G)$ is a representative.

PROPOSITION 2.3. *If G has a unique covering group $(\hat{G}, \alpha)$ then for every $\tau \in \operatorname{Aut}(G)$ there exists $\hat{\tau} \in \operatorname{Aut}(\hat{G})$ such that $\alpha \circ \hat{\tau} = \tau \circ \alpha$.*

PROOF. See Corollary 2.1, [5].

PROPOSITION 2.4. *Let G be a group such that $|G/G'|$ and $|M(G)|$ are relatively prime. Then G has a unique covering group and any covering group of G is centrally closed.*

PROOF. See Theorem (3), [5].

PROPOSITION 2.5. *Suppose A is an abelian group and $Z(\operatorname{Aut} A) = Z$.*
(a) *If $|Z| = 1$, then A is an elementary abelian 2-group.*
(b) *If $Z \approx C_2$, then $\pi(A) \subseteq \{2, 3\}$. The Sylow 3-subgroup of A has exponent at most 3 and the Sylow 2-subgroup of A has exponent at most 4.*

(c) *If* $Z \approx C_4$, *then* $\pi(A) \subseteq \{2, 5\}$. *If* $5 \in \pi(A)$, *then the exponent of the Sylow* 5-*subgroup of* A *is* 5 *and that of the Sylow* 2-*subgroup of* A *is at most* 2. *If* $5 \notin \pi(A)$ *then the Sylow* 2-*subgroup of* A *has exponent at most* 4.

PROOF. (a) and (b) are trivial and we proceed to prove (c). Clearly the only primes that may divide $|A|$ are 2, 3 and 5. Thus $\pi(A) \subseteq \{2, 3, 5\}$. Let A_p denote the Sylow p-subgroup of A for any prime p.

If $5 \in \pi(A)$, then $Z(\mathrm{Aut}\,(A_5))$ is at least of order 4. Hence $3 \notin \pi(A)$ and $Z(\mathrm{Aut}\,(A_2))$ must be trivial. So $\exp\,(A_2) \leqq 2$.

If $5 \notin \pi(A)$, suppose $3 \in \pi(A)$. If $\exp\,(A_3) > 3$, then $Z(\mathrm{Aut}\,(A_3))$ has order greater than 4. Hence $\exp\,(A_3) = 3$. Then $Z(\mathrm{Aut}\,(A_3)) \approx C_2$. Since $Z(\mathrm{Aut}\,A) \approx Z(\mathrm{Aut}\,A_2) \times Z(\mathrm{Aut}\,A_3)$ and since C_4 is indecomposable, we conclude $3 \notin \pi(A)$. Thus A is a 2-group. If $\exp\,(A) \geqq 8$, then $Z(\mathrm{Aut}\,(A))$ would contain a subgroup isomorphic to $C_2 \times C_2$. So $\exp\,(A) \leqq 4$.

This proves the result.

PROPOSITION 2.6. *If* X *is a finite group and* $\mathrm{Aut}\,(X) \approx S_3$, *then* $X \approx C_2 \times C_2$ *or* S_3, *while if* $\mathrm{Aut}\,(X) \approx S_4$ *then* $X \approx Q$, A_4, $A_4 \times C_2$, $SL(2, 3)$ *or* S_4.

PROOF. See page 39, [10].

PROPOSITION 2.7. *Suppose* $\mathrm{Aut}\,(X) \approx C_n$ *for some* n. *Then*
(a) $n = 1$ *and* $X \approx \{1\}$ or C_2
(b) $n = 2$ *and* $X \approx C_3$, C_4 *or* C_6 *or*
(c) $n = p^{\alpha-1}(p - 1)$ *for some odd prime* p *and* $X \approx C_{p^\alpha}$.

PROOF. See Theorem IV, [10].

PROPOSITION 2.8. *If* $Z(G) = 1$, *then* $A_c(G) = 1$.

PROOF. This is a well-known result.

PROPOSITION 2.9. *If* $p^2 \mid |G|$ *for some prime* p, *then* $p \mid |\mathrm{Aut}\,(G)|$.

PROOF. See [7].

THEOREM 2.10. *If* G *is a finite group whose order is divisible by* $p^{(h^2-h+6)/2}$ *for some prime* p, *then* p^h *divides the order of* $\mathrm{Aut}\,(G)$.

PROOF. See Theorem 4.7, [9].

PROPOSITION 2.11. *Let* $\beta : N \longrightarrow N$ *be defined by* $\beta(H) = (h^2 + h + 6)/2$. *Then for any finite group* G, $|\mathrm{Aut}\,(G)|_p = p^h$ *implies that* $|G|_p < p^{\beta(h)}$.

PROOF. This is a direct consequence of (2.10).

THEOREM 2.12. *If G is a PN-group, then* $|A_c(G)| = |\mathrm{Hom}\,(G/G', Z(G)|$. *In particular, if G is a PN-group, then G has a nontrivial central automorphism iff* $(|G/G'|, |Z(G)|) > 1$.

PROOF. See Theorem 1 and Corollary 1, [1].

PROPOSITION 2.13. *If A, B are abelian p-groups for some prime p, then* $|\mathrm{Hom}\,(A, B)| \geqq \min\,(|A|, |B|)$.

PROOF. See Lemma 2.3, [9].

PROPOSITION 2.14. *If G is a p-group of class 2 for some prime p, then* $|G|$ *divides* $|\mathrm{Aut}\,(G)|$.

PROOF. See [3].

PROPOSITION 2.15. *Suppose G is a finite group and H is a characteristic subgroup of G with $C_G(H) = 1$. Let $\theta : G \to \mathrm{Aut}\, H$ be defined by $\theta(g)(h) = ghg^{-1}$ where $g \in G$, $h \in H$. Then* $\mathrm{Aut}\,(G) \simeq N_{\mathrm{Aut}(H)}(\theta(G))$.

PROOF. θ is clearly a monomorphism. The map $\alpha : \mathrm{Aut}\,(G) \to \mathrm{Aut}\,(H)$ defined by $\alpha(f) = f|_H$, where $f \in \mathrm{Aut}\,(G)$, is a group homomorphism. Suppose $K = \mathrm{Ker}\, \alpha$. Clearly $K \cap \mathrm{Inn}\,(G) = 1$. So $K \subseteq A_c(G)$. However, since $Z(G) = 1$, $A_c(G) = 1$ and so $K = 1$. Thus α is a monomorphism. It is easily verified that $\alpha(\mathrm{Aut}\,(G)) = N_{\mathrm{Aut}(H)}(\theta(G))$. Hence the result.

PROPOSITION 2.16. *Suppose A is a characteristic subgroup of G contained in $Z(G)$ and that $(|G/G'|, |A|) = 1$. Let $\eta : G \to H$ be an epimorphism with $\mathrm{Ker}\, \eta = A$. Then there exists a monomorphsim $\theta : \mathrm{Aut}\, G \to \mathrm{Aut}\,(H)$ such that $\eta \circ \alpha = \theta(\alpha) \circ \eta$ for $\alpha \in \mathrm{Aut}\,(G)$.*

PROOF. Define $\theta : \mathrm{Aut}\, G \to \mathrm{Aut}\,(H)$ by $\theta(\alpha)(\eta(g)) = \eta(\alpha(g))$ for $\alpha \in \mathrm{Aut}\, G$, $g \in G$. Clearly θ is a well-defined homomorphism and $\eta \circ \alpha = \theta(\alpha) \circ \eta$.

Suppose $\alpha \in \mathrm{Ker}\, \theta$. Then $\theta(\alpha)\ (\eta(g)) = \eta(g)$ for every $g \in G$. Thus $\eta(\alpha(g)) = \eta(g)$ for all $g \in G$ and so $g^{-1}\alpha(g) \in \mathrm{Ker}\, \eta = A$ for every $g \in G$.

Define $\tau : G \to A$ by $\tau(g) = g^{-1}\alpha(g)$ for $g \in E$. Clearly τ is a group homomorphism. Since A is abelian, $\mathrm{Ker}\, \tau \supseteq G'$. But $(|G/G'|, |A|) = 1$, so $\mathrm{Ker}\, \tau = G$. Hence $\alpha(g) = g$ for all $g \in G$. So $\mathrm{Ker}\, \theta = 1$ and the result follows.

PROPOSITION 2.17. *Let H be a finite group and A be a cyclic group of order 2. Then $\mathrm{Aut}\,(H \times A) \simeq \mathrm{Aut}\,(H)$ iff $|H/H'|$ and $|Z(H)|$ are both odd.*

PROOF. Elementary.

PROPOSITION 2.18. *Suppose G is a group and* $x, y \in \mathrm{Inn}(G)$ *such that the order of y is* p^α *for some prime p and* $y^x = y^k$ *for some integer k. If* $k \not\equiv 1 \pmod p$ *then there exists* $g \in G$ *such that g induces y by conjugation and the order of g is* p^α.

PROOF. See Theorem VI, [**10**].

PROPOSITION 2.19. *Let A be an abelian p-group with a basis consisting of* n_i *generators of order* p^i, $1 \leqq i \leqq k$. *Then* Aut (A) *has an elementary abelian p-subgroup of rank* $(n_1 + n_2 + \cdots + n_k)(n_2 + n_3 + \cdots + n_k)$.

PROOF. See Satz 113, [**12**].

PROPOSITION 2.20. *If G is a group, x is a nonidentity element of G and S is a generating subset of G such that if* $y \in S$ *then* $x \in \langle y \rangle$, *then there does not exist a group X such that* $X/Z(X) \approx G$.

PROOF. See (3.2.10), [**11**].

THEOREM 2.21. (GASCHUTZ). *Suppose A is an abelian normal subgroup of G with* $\exp(A) = k$ *and U is a subgroup of G such that* $(|G:U|, k) = 1$ *then A has a complement in G if it has a complement in U.*

PROOF. See I, 17.4, [**8**].

PROPOSITION 2.22. *Suppose A is an abelian group. Then* Aut (A) *is abelian when A is cyclic and is nonabelian when A is noncyclic.*

PROOF. See Theorem III, [**10**].

THEOREM 3.1. *If G is a given finite group then there are at most finitely many finite groups X such that* Aut $(X) \simeq G$.

PROOF. Let

$$\pi_0 = \{p \in \pi | p \nmid |G|\},\ \pi_1 = \pi(G) \text{ and}$$
$$\pi_2 = \{p \in \pi | (p-1) \mid |G|\}.$$

For each $p \in \pi_1$, let $|G|_p = p^{h_p}$. By (2.11) there exists a function $\beta : N \rightarrow N$ such that $|G|_p = p^h$ implies that $|X|_p < p^{\beta(h)}$.

Suppose $p \in \pi_0$ and that $p \mid |X|$. By (2.9) it follows that $p^2 \nmid |X|$. Moreover $p \nmid |\mathrm{Inn}\, X|$. Hence $p \mid |Z(X)|$. Therefore there is a subgroup Y of X and a cyclic subgroup C_p of X such that $X = Y \times C_p$. Hence $(p-1) \mid |G|$. So $\pi(X) \cap \pi_0 \subseteq \pi_2$.

Now

$$|X| = \prod_{p \in \pi(X)} |X|_p = \Big(\prod_{p \in \pi(X) \cap \pi_0} |X|_p \Big) \cdot \Big(\prod_{q \in \pi(X) \cap \pi_1} |X|_q \Big)$$

$$= \left(\prod_{p \in \pi(X) \cap \pi_0} p \right) \cdot \left(\prod_{q \in \pi(X) \cap \pi_1} |X|_q \right)$$

$$\leqq \left(\prod_{p \in \pi_2} p \right) \cdot \left(\prod_{q \in \pi_1} |X|_q \right)$$

$$< \left(\prod_{p \in \pi_2} p \right) \cdot \left(\prod_{q \in \pi_1} q^{\beta(h_q)} \right) = \mu(G), \text{ say.}$$

Thus $|X| < \mu(G)$, a positive integer completely determined by the group G. Hence the theorem follows.

NOTE. The author wishes to thank Professor F. I. Gross for the above proof.

DEFINITION 3.2. A finite group G is said to have property (P) if $|G| > 1$ and whenever $1 \neq N \trianglelefteq G$, we have $C_G(N) = 1$.

PROPOSITION 3.3. *A finite group G has property (P) if and only if G has a unique minimal normal subgroup N which is nonabelian.*

PROOF. Elementary.

DEFINITION 3.4. If G is a group with property (P), then its unique minimal normal subgroup is denoted by $P(G)$.

Clearly $P(G)$ is characteristically simple and so $P(G) \approx M_1 \times M_2 \times \cdots \times M_k$ where $M_1, M_2, \cdots, M_k$ are all isomorphic to a nonabelian simple group M.

PROPOSITION 3.5. *Let G have property (P). Suppose $P(G) = N = M_1 \times M_2 \times \cdots \times M_k$ where for each i, $M_i \approx M$, a nonabelian simple group. Let $\theta : G \to \mathrm{Aut}(N)$ be the homomorphism defined by $\theta(g)(x) = gxg^{-1}$ for $g \in G$ and $x \in N$. Then*

(*a*) $\mathrm{Inn}(N) \leqq \theta(G) \leqq \mathrm{Aut}(N)$

(b) *$\theta(G)$ has property (P) and $P(\theta(G)) = \theta(P(G))$.*

(c) *$\theta(G)$ acts transitively on the set $\{\theta(M_1), \theta(M_2), \cdots, \theta(M_k)\}$ by conjugation.*

PROOF. Clearly θ is a monomorphism and $\theta(N) = \mathrm{Inn}(N)$. Now (a) and (b) follow. G acts on $\{M_1, M_2, \cdots, M_k\}$ transitively by conjugation. Hence (c) follows.

PROPOSITION 3.6. *Let $N = M_1 \times M_2 \times \cdots \times M_k$ where $M_1, M_2, \cdots, M_k$ are all isomorphic to a nonabelian simple group M. Let $\theta : N \to \mathrm{Aut}\, N$ be the monomorphism defined by $\theta(x)(y) = xyx^{-1}$ for $x, y \in N$. Suppose G is a finite group such that* $\mathrm{Inn}(N) \leqq G \leqq \mathrm{Aut}(N)$.

Then G has property (P) if and only if G acts transitively on $\{\theta(M_1), \theta(M_2), \cdots, \theta(M_k)\}$ *by conjugation.*

PROOF. Suppose G has property (P). G acts on $\{\theta(M_1), \theta(M_2), \cdots, \theta(M_k)\}$ by conjugation. If the action is not transitive, let $\{\theta(M_{i_1}), \theta(M_{i_2}), \cdots, \theta(M_{i_r})\}$ be an orbit where $1 \leqq i_1 < i_2 < \cdots < i_r \leqq n$ and $r < k$. Let $R = \times_{s=1}^{r} \theta(M_{i_s})$. Clearly then $C_G(R) > 1$ which contradicts the fact that $1 \neq R \trianglelefteq G$. So G acts transitively on $\{\theta(M_1), \theta(M_2), \cdots, \theta(M_k)\}$.

Conversely, suppose G acts transitively on $\{\theta(M_1), \theta(M_2), \cdots, \theta(M_k)\}$. Then $\theta(N)$ must be a minimal normal subgroup of G. Let R be any nontrivial normal subgroup of G. Then $R \cap \theta(N) = 1$ or $\theta(N)$. If $R \cap \theta(N) = 1$ then $R \subseteq C_G(\theta(N))$ and so $A_C(N) \neq 1$. This is a contradiction since $Z(N) = 1$. Thus $R \supseteq \theta(N)$. So $\theta(N) = \mathrm{Inn}(N)$ is the unique minimal normal subgroup of G and is nonabelian. So G has property (P), by (3.3).

THEOREM 3.7. *Suppose G has property (P) and X is a finite group such that* $\mathrm{Aut}(X) \simeq G$. *Then one of the following holds*:

(a) *X is an elementary abelian 2-group of order at least 8 and* $G \simeq GL(n, 2)$ *for some* $n \geqq 3$.

(b) $(X, \theta_x) \in C(N)$ *for some nontrivial normal subgroup N of G.*

(c) $X \simeq R \times C_2$ *where* $(R, \theta_R) \in C(N)$ *for some nontrivial normal subgroup N of G. Moreover* $|Z(R)|$, $|R/R'|$ *are both odd.*

PROOF. If X is abelian, then by (2.5) it is an elementary abelian 2-group. Clearly then $|X| \geqq 8$ and $G \simeq GL(n, 2)$ for some $n \geqq 3$.

Suppose X is nonabelian. Let $X = Y \times A$ where Y is a PN-group and A is abelian. Since $A_c(X)$ is trivial, it follows from Remak-Krull-Schmidt theorem that Y and A are characteristic in X. So $\mathrm{Aut}(X) \simeq \mathrm{Aut}(Y) \times \mathrm{Aut}(A)$. Hence $\mathrm{Aut}(Y) = G$ and $\mathrm{Aut}(A) = 1$, giving $A \simeq 1$ or C_2.
Now $Y/Z(Y)$ is isomorphic to a normal subgroup N of G. If $Y' \not\supseteq Z(Y)$, then $(|Y/Y'|, |Z(Y)|) > 1$. So $A_c(Y) > 1$ by (2.12). This is impossible and so $Y' \supseteq Z(Y)$. Let $\theta : Y \to N$ be an epimorphism with $\mathrm{Ker}\,\theta = Z(Y)$. Hence $(Y, \theta) \in C(N)$. Hence $X = Y$ or $Y \times C_2$, with $\mathrm{Aut}(Y) \simeq G$. However $\mathrm{Aut}(Y \times C_2)$ is isomorphic to G iff $|Z(Y)|$, $|Y/Y'|$ are both odd, by (2.17). Hence the result follows.

REMARK 3.8. Given a group G with property (P), the above result provides a criterion for actually determining all groups X with $\mathrm{Aut}(X) \simeq G$, by examining a finite number of possibilities, viz by examning the set of groups $\{X \mid (X, \theta_X) \in C(N)$, for some $N \trianglelefteq G, N \neq 1\}$.

PROPOSITION 3.9. *Let G be a finite group such that* $|G/G'|$ *and* $|M(G)|$ *are relatively prime. Let* $(\hat{G}, \theta)$ *be the unique covering group of G. Let B* *be a finite group such that* $(\hat{G}, \eta) \in C(B)$. *Suppose* $\operatorname{Ker} \eta$, $\operatorname{Ker} \theta$ *are characteristic in* $\hat{G}$ *and* $\operatorname{Ker} \eta \subseteq \operatorname{Ker} \theta$. *Then* $\operatorname{Aut}(B) \simeq \operatorname{Aut}(G)$.

PROOF. Let $\lambda : B \to G$ be the homomorphism such that $\lambda \circ \eta = \theta$. Thus $\operatorname{Ker} \lambda = \eta(\operatorname{Ker} \theta)$. By (2.16) there exist monomorphisms $\tau_1 : \operatorname{Aut}(\hat{G}) \to \operatorname{Aut}(G)$, $\tau_2 : \operatorname{Aut}(\hat{G}) \to \operatorname{Aut}(B)$ and $\tau_3 : \operatorname{Aut}(B) \to \operatorname{Aut}(G)$, such that for $\alpha \in \operatorname{Aut}(\hat{G})$, $\beta \in \operatorname{Aut}(B)$ we have $\theta \circ \alpha = \tau_1(\alpha) \circ \theta$, $\eta \circ \alpha = \tau_2(\alpha) \circ \eta$ and $\lambda \circ \beta = \tau_3(\beta) \circ \lambda$. But by (2.3) τ_1 is also surjective. Hence $\operatorname{Aut}(G) \simeq \operatorname{Aut}(B)$.

COROLLARY 3.10. *Let* G *be a nonabelian simple group and* $\hat{G}$ *its unique covering group. Then* $\operatorname{Aut}(\hat{G}) \simeq \operatorname{Aut}(G)$.

THEOREM 3.11. *Let* G *be a nonabelian simple group and* $\operatorname{Aut}(X) \simeq G$. *Then one of the following holds.*

(a) X *is an elementary abelian* 2-*group of order greater than* 4 *and* $G \simeq GL(n, 2)$ *for some* $n \geqq 3$.

(b) $X \simeq X_0$ *or* $X_1 \times C_2$ *where* X_0 *is a factor group of* $\hat{G}$ *by a central subgroup and* X_1 *is a factor group of* $\hat{G}$ *by a central subgroup containing the Sylow* 2-*subgroup of* $Z(\hat{G})$.

PROOF. Immediate consequence of Theorem 3.7, and Proposition 2.17. We point out that the converse holds also.

PROPOSITION 3.12. *Let* G *be a nonabelian simple group and* $|M(G)| > 1$. *Let* $\hat{G}$ *be its unique covering group. Suppose* X *is a finite group such that* $\operatorname{Aut}(X) \simeq \hat{G}$. *Then* $X = S \times T$ *where* $|S| = 1$ *or* 2 *and* T *is a* p-*group of class at most* 2 *for some prime* p.

PROOF. If X is abelian, then the result follows from the indecomposability of $\hat{G}$. Suppose X is nonabelian. Then $X/Z(X)$ is isomorphic to $\hat{G}$ or to subgroup of $\hat{G}$ contained in $Z(\hat{G})$. If $X/Z(X) \simeq \hat{G}$, then by (2.4), $Z(X) = 1$ and so $X \simeq \hat{G}$. But then $Z(\hat{G}) = 1$, a contradiction. Thus $X/Z(X)$ is isomorphic to a central subgroup of $\hat{G}$ and so X is nilpotent of class-2. Once again, since $\hat{G}$ is indecomposable, $X = S \times T$ with $|S| \leqq 2$ and T a p-group of class-2 for some prime p.

REMARKS 4.1. The following facts are well known:

(a) $M(S_n) \simeq C_2$ for $n \geqq 4$. S_6 has a unique covering group $\hat{S}_6$ while S_n has two covering groups T_n and T_n^* when $n \geqq 4$, $n \neq 6$.

(b) $M(A_n) \simeq C_2$ for $n \geqq 4$, $n \neq 6, 7$. $M(A_6) \simeq M(A_7) \simeq C_6$.

(c) A_4 has $SL(2, 3)$ as its unique covering group.

(d) $GL(m, 2) \simeq A_n (n \geqq 3)$ iff $m = 4$ and $n = 8$.

(e) $GL(m, 2) \simeq S_n (n \geqq 3)$ iff $m = 2$ and $n = 3$.

THEOREM 4.2. *Suppose X is a finite group such that* Aut $(X) \simeq A_n$ *for some* $n \in N$. *Then one of the following holds*:

(a) $X \simeq C_2$ *and* Aut $(X) \simeq A_1 \simeq A_2 \simeq \{1\}$.

(b) $X \simeq C_2 \times C_2 \times C_2 \times C_2$ *and* Aut $(X) \simeq A_8$.

PROOF. (a) is trivial. So we may assume $n \geqq 3$. Since $A_3 \simeq C_3$, it follows from (2.7) that there is no group X such that Aut $(X) \simeq A_3$.

Suppose $n = 4$ and X is abelian. Then by (2.5) X is an elementary abelian 2-group, which is impossible by (4.1)(d). Thus X is nonabelian. If $X/Z(X) \simeq A_4$ then $A_c(X) = 1$. Let $X = Y \times A$ where Y is a PN-group and A is abelian. Then Inn $(Y) \simeq$ Aut $(Y) \simeq A_4$. Also $A_c(Y) = 1$. So $Y' \supseteq Z(Y)$ and by (4.1) $Y \simeq A_4$ or $SL(2, 3)$. In either case Aut $(Y) \not\simeq A_4$. So $X/Z(X)$ must be isomorphic to $C_2 \times C_2$. Then X is nilpotent of class 2 and the indecomposability of A_4 implies that X is a 2-group. Then by (2.14) it follows that $|X| \leqq 4$ which is impossible.

Suppose $n \geqq 5$. If X is abelian it must be an elementary abelian 2-group. So by (4.1)(d) we obtain that $X \simeq C_2 \times C_2 \times C_2 \times C_2$ and Aut $(X) \simeq GL(4, 2) \simeq A_8$. If X is nonabelian, Theorem 3.11 says that X is isomorphic to A_n, $\hat{A}_n$ or $A_n \times C_2$. This is impossible since everyone of these groups have S_n as their group of automorphisms when $n \neq 6$ and Aut (S_6) as their automorphism group when $n = 6$.

Hence the theorem is proved.

PROPOSITION 4.3. *Let $n \geqq 5$ and R a covering group of S_n. Then* Aut $(R) \simeq S_n \times C_2$ *if* $n \neq 6$. *If* $n = 6$, Aut (R) *has a subgroup X of index 2 where $X \simeq S_6 \times C_2$.*

PROOF. Let $\alpha : R \longrightarrow S_n$ be the epimorphism $x \longrightarrow xZ(R)(x \in R)$. Define θ : Aut $(R) \longrightarrow$ Aut (S_n) *by* $\theta(f)(\alpha(x)) = \alpha(f(x))$ for $x \in R$ and $f \in$ Aut (R). Clearly θ is a homomorphism. When $n \geqq 5$, $n \neq 6$, Aut $(S_n) \simeq$ Inn $(S_n) \simeq S_n$ and since Inn $(R) \simeq S_n$, we conclude that θ is an epimorphism. The same conclusion holds when $n = 6$ in view of (2.3) and (4.1)(a). Let $K = \text{Ker}\,\theta$. It is easily seen that $K = A_c(R)$. Clearly R is a PN-group and so $|A_c(R)| = |\text{Hom}\,(R/R', Z(R))|$. Since $Z(R) \simeq C_2$ and $|R/R'| = 2$, it follows that $K \simeq C_2$. We also observe that $K \cap$ Inn $(R) = 1$, so that Aut $(R) \geqq K \times$ Inn (R). A consideration of orders shows that Aut $(R) = K \times$ Inn (R) when $n \neq 6$ and when $n = 6$, $K \times$ Inn (R) has index 2 in Aut (R). The result now follows.

THEOREM 4.4. *Suppose* Aut $(X) \simeq S_n$ *for some $n \in N$ and some finite group X. Then one of the following holds.*

(a) $n = 1$ *and* $X \simeq 1$ *or* C_2.

(b) $n = 2$ *and* $X \simeq C_3$, C_4 *or* C_6.

(c) $n = 3$ *and* $X \simeq C_2 \times C_2$ *or* S_3.

(d) *$n = 4$ and $X \simeq Q$, A_4, $A_4 \times C_2$, $SL(2, 3)$ or S_4.*
(e) *$n = 7$ and $X \simeq B_0$, $B_1 \times C_2$ or S_7 where $(B_0, \eta_0) \in C(A_7)$ and $(B_1, \eta_1) \in C(A_7)$ with $|\mathrm{Ker}\, \eta_1| = 1$ or 3.*
(f) *$n \geqq 5$, $n \neq 6, 7$ and $X \simeq A_n$, $A_n \times C_2$, $\hat{A}_n$ or S_n.*

Moreover each of the above possibilities for X does give a group where automorphism group is actually S_n for appropriate n.

PROOF. The case $n = 1$ and $n = 2$ are trivial.

The case $n = 3$ and $n = 4$ have been discussed by G. A. Miller. See (2.6).

Suppose $n \geqq 5$. Then S_n has property (*P*). Now the result follows from theorem (3.7), Proposition (4.3), Proposition (2.17) and the fact that $\mathrm{Aut}\,(S_6) \not\simeq S_6$.

5.1. **Dihedral, Dicyclic and Quasidihedral groups.** The dihedral group $D(2n)$ of order $2n$ is the group $\langle x, y \mid x^2 = y^n = 1,\ y^x = y^{-1}\rangle$. Clearly $D(2) \simeq C_2$ and $D(4) \simeq C_2 \times C_2$ and $D(2n)$ is nonabelian when $n \geqq 3$.

The dicyclic group $DC(4n)$ of order $4n$ is the group $\langle x, y \mid x^4 = y^{2n} = 1,\ x^2 = y^n,\ y^x = y^{-1}\rangle$. Clearly $DC(4) \simeq C_4$, $DC(8) \simeq Q$ and $DC(4n)$ is nonabelian when $n \geqq 2$.

The quasidihedral group $QD(8n)$ of order $8n$ is the group $\langle x, y \mid x^2 = y^{4n} = 1,\ y^x = y^{2n-1}\rangle$. $QD(8) \simeq C_4 \times C_2$ while $QD(8n)$ is nonabelian when $n \geqq 2$.

5.2. **Some Properties of the dihedral groups.** The following facts about the dihedral groups $D(2n)$, $n \geqq 3$, are well known.

(a) Suppose n is odd. Then $Z(D(2n)) = 1$. The only noncyclic normal subgroup of $D(2n)$ is itself. $D(2n)$ is indecomposable. The 2-rank of $D(2n)$ is 1. $M(D(2n))$ is trivial.

(b) Suppose $(n, 4) = 2$. Then $Z(D(2n)) \simeq C_2$. The only proper noncyclic normal subgroups of $D(2n)$ are isomorphic to $D(n)$ and their centralizer in $D(2n)$ is $Z(D(2n))$. $D(2n)$ is decomposable and we have $D(2n) \simeq D(n) \times C_2$. The 2-rank of $D(2n)$ is 1. $M(D(2n)) \simeq C_2$ and $D(2n)$ has two covering groups isomorphic to $D(4n)$ and $DC(4n)$ respectively.

(c) Suppose $4 \mid n$. Then $Z(D(2n)) \simeq C_2$ If X is a noncyclic proper normal subgroup of $D(2n)$ then $X \simeq D(n)$ and the centralizer of X in $D(2n)$ is $Z(D(2n))$. $D(2n)$ is indecomposable and has 2-rank 2. $M(D(2n)) \simeq C_2$ and it has three covering groups isomorphic to $D(4n)$, $DC(4n)$ and $QD(4n)$ respectively.

(d) If $n \geqq 3$, $|\mathrm{Aut}\,(D(2n))| = n\phi(n)$ where ϕ is the Euler phi-function.

5.3. **Dicyclic groups.** The following properties of the dicyclic groups $DC(4n)$ $(n \geqq 3)$ are easily verified.

(a) $Z(DC(4n)) \simeq C_2$. It is indecomposable. When n is odd, $DC(4n)$ has no noncyclic proper normal subgroups. If n is even and X is a proper normal subgroup of $DC(4n)$ then $X \simeq DC(2n)$ and the centralizer of X in $DC(4n)$ is $Z(DC(4n))$. The dicyclic groups have 2-rank 1.

(b) $|\mathrm{Aut}\,(DC(4n))| = 2n\phi(2n)$ when $n \geqq 3$, while $\mathrm{Aut}\, Q \simeq S_4$.

5.4. **Quasidihedral groups.** The following properties of the quasidihedral groups $QD(8n)$ are easily verified to be true for $n \geqq 3$.

(a) Suppose n is odd. Then $Z(QD(8n)) \simeq C_4$. We have the decomposition $QD(8n) \simeq D(2n) \times C_4$. If X is a proper normal subgroup of $QD(8n)$ which is noncyclic then $X \simeq D(2n)$, $D(4n)$ or $DC(4n)$ and the centralizer of X in $QD(8n)$ is $Z(QD(8n)$. The 2-rank of $QD(8n)$ is 2.

(b) Suppose n is even. Then $Z(QD(8n)) \simeq C_2$. $QD(8n)$ is indecomposable. If X is a noncyclic proper normal subgroup of $QD(8n)$ then X is isomorphic to $D(4n)$ or $DC(4n)$ and is centralizer in $QD(8n)$ is $Z(QD(8n))$. $QD(8n)$ has 2-rank 2.

(c) $|\mathrm{Aut}\,(QD(8n))| = 2n\ \phi(4n)$ for $n \geqq 2$.

LEMMA 5.5. *Suppose p is a prime and m, $n \in N$. Then*

(a) $GL(m, p) \simeq D(2n) \Longleftrightarrow (m, n, p) = (2, 3, 2)$ *or* $(1, 1, 3)$.

(b) $Gl(m, p) \simeq DC(4n) \Longleftrightarrow (m, n, p) = (1, 1, 5)$.

(c) $GL(m, p) \not\simeq QD(8n)$ *for any m, n, p.*

PROOF. These facts can be easily verified.

LEMMA 5.6. *Suppose X is a PN-group and $X/Z(X) \simeq D(2n)$, $n \geqq 3$. Then $Z(X)$ is a 2-group. If $A_c(X) = 1$, then n is odd and $Z(X)$ is trivial. If $A_c(X) \simeq C_2$, then n is odd and $X \simeq DC(4n)$.*

Moreover $A_c(X)$ cannot be isomorphic to C_4.

PROOF. Let $n = 2^{r_0} \cdot m$, m odd. Let $Z(X) \subseteq K \subseteq H \subseteq X$ be subgroups of X such that $|X : H| = 2$ and $|K : Z(X)| = m$. Let $^- : X \longrightarrow X/Z(X)$ be the canonical epimorphism. Let $x \in X - H$. Let p be a prime dividing m and $y_p \in K$ such that $\langle \bar{y}_p \rangle$ is a Sylow p-subgroup of $\bar{K}$. Since $\bar{X} \simeq D(2n)$ we have $\bar{y}_p^{\bar{x}} = \bar{y}_p^{-1}$. By (2.18) there exists $g_p \in K$ such that $\bar{g}_p = \bar{y}_p$ and $o(g_p) = o(\bar{y}_p)$. Let $g = \prod_{p \in \pi(K)} g_p$. Then $o(g) = m$ and $K = \langle g \rangle \times Z(X)$. If p is an odd prime and P the Sylow p-subgroup of $Z(X)$, then by (2.21) P would be a direct factor of X. Since X is a PN-group $|P| = 1$. Thus $Z(X)$ is a 2-group.

(a) Suppose $A_c(X) = 1$. Since $|X/X'| \geqq 2$, it follows from (2.12) that $Z(X) = 1$. Thus $X \simeq D(2n)$. So n must be odd.

(b) Suppose $A_c(X) \simeq C_2$. If n is even, $|X/X'| \geqq 4$ and X/X' has 2-rank at least 2. Moreover $|Z(X)| > 1$. This implies $|A_c(X)| \geqq 4$, a contradiction. Hence n is odd. If $X' \supseteq Z(X)$ then by (5.2(a)) $Z(X) = 1$

which is not possible. So $X' \not\supseteq Z(X)$. Hence $|X/X'| \geqq 4$ and therefore $|Z(X)| = 2$. This implies that $X' \cap Z(X) = 1$ and $|X/X'| = 4$. If X/X' is elementary abelian, then $|A_c(X)| = 4$ by (2.12). Therefore X/X' is cyclic. Hence there exists $x \in X - H$ such that $x^2 \in Z(X)$ and $o(x) = 4$. In this case we have $H = K$ and so $H = \langle g \rangle \times \langle x^2 \rangle \simeq C_{2n}$. Clearly $g^x = g^{-1}$ and hence $X \simeq DC(4n)$. But then $A_c(X) \simeq C_2 \times C_2$ when n is even and $A_c(X) \simeq C_2$ when n is odd. Hence $X \simeq DC(4n)$ with n odd.

(c) Suppose $A_c(X) \simeq C_4$. Suppose n is odd. Then $X' \not\supseteq Z(X)$. $X'Z(X) = H$. If $|X/X'| = 2^r$ then $|Z(X)| \geqq 2^{r-1}$. Now $4 = |A_c(X)| \geqq \min(|X/X'|, |Z(X)|) \geqq 2^{r-1}$. It follows now that $r = 2$ or 3.

Suppose $r = 3$. Then $|H/X'| = 4$. Hence $|Z(X)| \geqq 4$. But $|Z(X)| > 4 \Rightarrow |A_c(X)| > 4$ by (2.13). Hence $|Z(X)| = 4$. Therefore $X' \cap Z(X) = 1$. If X/X' is cyclic, then $Z(X)$ is cyclic. Moreover there exists $x \in X - H$ such that x^2 is a generator for $Z(X)$. Also $H = \langle g \rangle \times \langle x^2 \rangle$. Clearly $g^x = g^{-1}$ and so $X = \langle g, x \mid g^n = x^8 = 1, g^x = g^{-1} \rangle$. For $i = 1, 2, 3, 4$ define $\alpha_i : X \to X$ by $\alpha_i(x^a g^b) = x^{a(2i-1)} g^b$. It can be verified that $A_c(X) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and that $A_c(X) \simeq C_2 \times C_2$. Hence X/X' has 2-rank 2. However $Z(X)$ is again cyclic. Hence there exists $x \in X - H$ such that $x^2 = 1$. Also $g^x = g^{-1}$ as before and hence $X \simeq D(2n) \times C_4$ which is a contradiction.

So $r = 2$. Therefore $|X/X'| = 4$ and $|Z(x) : Z(X) \cap X'| = 2$. Clearly $H = \langle g \rangle Z(X)$. Let $x \in X - H$. Every element t of X can be expressed as $t = x^i g^j z$ where $0 \leqq i \leqq 1$, $0 \leqq j \leqq n - 1$ and $z \in Z(X)$. It is easily verified that $[x^{i_1} g^{j_1} z_1, x^{i_2} g^{j_2} z_2] \in \langle g \rangle$, so that $X' \subseteq \langle g \rangle$. Hence $X' \cap Z(X) = 1$ and so $|Z(X)| = 2$. Then X/X' must be elementary abelian. So $x^2 = 1$ and thus $X \simeq D(2n) \times C_2$ which is impossible. Hence n must be even. Clearly X/X' has 2-rank at least 2. Suppose $X' \not\supseteq Z(X)$. Then it is easily seen that $|Z(X)| = 2$, $X' \cap Z(X) = 1$ and $X/X' \simeq C_4 \times C_2$. Let $y \in H$ such that $\langle \bar{y} \rangle$ is the Sylow 2-subgroup of $\overline{H}$ so that $\overline{H} = \langle \bar{y} \rangle \times \langle \bar{g} \rangle$. Since $n = 2^{r_0} \cdot m$, $o(\bar{y}) = 2^{r_0}$. So $o(y) = 2^{r_0}$ or 2^{r_0+1}, $r_0 \geqq 1$. Also since $\bar{y}^{\bar{x}} = \bar{y}^{-1}$ for any $x \in X - H$, it follows that $y^x = y^{-1}$ or $y^{-1}z$ where $Z(X) = \langle z \rangle$. Suppose $o(y) = 2^{r_0+1}$. Then $y^{2^{r_0}} = z$. Now $y^x = y^{-1}$ implies that $y^2 \in X'$ and hence $z \in X'$ which is false. If $y^x = y^{-1}z$ then $[x, y] = y^2 z = (y^2)^{(2^{r_0-1}+1)} \in X'$. Hence if $r_0 > 1$, $y^2 \in X'$ implying again that $z \in X'$, so $r_0 = 1$ in which case $[x, y] = 1$. But then $X = \langle g, x \rangle \times \langle y \rangle$ since $0(x) = 2$ in this case. This is impossible as X is a PN-group. So $o(y) = 2^{r_0}$, and $H = \langle g \rangle \times \langle y \rangle \times \langle z \rangle$. Since $X/X' \simeq C_4 \times C_2$, there exists $x \in X - H$ such that $x^2 = z$. However $y^2 \in X'$. So $y^x = y^{-1}z$ implies that $[x, y] = y^2 z \in X'$ and hence $z \in X'$. However $z \notin X'$ and hence $y^x = y^{-1}$. Let $h = gy$ so that $o(h) = 2^{r_0} \cdot m = n$ and $H = \langle h \rangle \times \langle z \rangle$. Moreover $h^x = h^{-1}$. Let $h^i x^j$ be an arbitrary element of X and σ_1, σ_2, σ_3, σ_4 be maps defined by $\sigma_1(h^ix^j) = h^ix^j$, $\sigma_2(h^ix^j) = h^i(xx)^j$, $\sigma_3(h^ix^j) = (hz)^ix^j$ and $\sigma_4(h^ix^j) = (hz)^i(xz)^j$. It is easily verified that $A_c(X) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ and that $A_c(X) \simeq C_2 \times C_2$. This is a contradiction and hence we must conclude that $X' \supseteq Z(X)$. Hence by (5.2) it follows that $X \simeq D(4n)$, $DC(4n)$ or $QD(4n)$ if $4 \mid n$ while if $4 \nmid n$, $X \simeq D(4n)$ or $DC(4n)$. In every case however $A_c(X) \not\simeq C_4$.

LEMMA 5.7. *Let $n \in N$.*

(a) *There is no finite group X such that $X/Z(X) \simeq DC(4n)$.*

(b) *There is no finite group X such that $X/Z(X) \simeq QD(8n)$.*

PROOF.

(a) $DC(4n) = \langle x,y \mid x^4 = y^{2n} = 1,\ x^2 = y^n,\ y^x = y^{-1}\rangle$. Let $n > 1$. Hence if $z = x^2$, we have a generating set $S = \{x, y\}$ for $DC(4n)$ such that $z \in \langle x\rangle$ and $z \in \langle y\rangle$. So by (2.20) the result follows. If $n = 1$, the result is trivial.

(b) $QD(8n) = \langle x, y \mid x^2 = y^{4n} = 1,\ y^x = y^{2n-1}\rangle$. We observe that $(xy)^2 = y^{2n}$ and that $\{xy, y\}$ is a generating set for $QD(8n)$. Hence the result follows by (2.20).

LEMMA 5.8. *Suppose X is an abelian 2-group of order greater than 1 and exponent at most 4. Let r = 2-rank of* Aut(X). *Then*

(a) $r = 1 \Rightarrow\ \simeq C_4$.

(b) $r = 2 \Rightarrow\ \simeq C_2 \times C_4$ or C_4.

PROOF. Let X have a basis consisting of n_1 elements of order 2 and n_2 elements of order 4. Then Aut(X) has an elementary abelian 2-subgroup of order $2^{(n_1+n_2)n_2}$ by (2.19). If $r = 1$, then $(n_1 + n_2)n_2 \leqq 1$ which implies that $n_1 = 0$ and $n_2 = 1$ giving $X \simeq C_4$. If $r = 2$, then $(n_1 + n_2)n_2 \leqq 2$ so that $n_1 = n_2 = 1$ or $n_1 = 0$, $n_2 = 1$. So $X \simeq C_2 \times C_4$ or C_4.

PROPOSITION 6.1. *The following list gives all finite abelian groups X such that* Aut$(X) \simeq D(2n)$ *for some $n \in N$.*

(a) $X \simeq C_3$, C_4 *or* C_6 *and* Aut$(X) \simeq C_2 \simeq D(2)$.

(b) $X \simeq C_8$ *or* $C_4 \times C_3$ *and* Aut$(X) \simeq C_c \times C_2 \simeq D(4)$.

(c) $X \simeq C_2 \times C_2$ *and Aut*$(X) \simeq D(6) \simeq S_3$.

(d) $X \simeq C_2 \times C_4$ *and* Aut$(X) \simeq D(8)$.

(e) $X \simeq C_2 \times C_2 \times C_3$ *and* Aut$(X) \simeq D(12)$.

PROOF. It is easily verified that if Aut$(X) \simeq D(2n)$ and $n = 1$ then $X \simeq C_3$, C_4 or C_6 and that if $n = 2$ then $X \simeq C_8$ or $C_4 \times C_3$. So let us suppose that $n \geqq 3$.

If n is odd then $Z(D(2n)) = 1$ and so X is an elementary abelian 2-

group by (2.5). So $\mathrm{Aut}(X) \simeq GL(m, 2)$ for some $m \in N$. Hence, by (5.6) we must have $n = 3$ so that $X \simeq D(6)$.

Suppose $(n, 4) = 2$. Then $D(2n) \simeq D(n) \times C_2$, and $Z(D(2n)) \simeq C_2$. *By* (2.5) $\pi(X) \subseteq \{2, 3\}$. If X is a 3-group then it must be elementary abelian so that $GL(m, 3) \simeq D(2n)$ for some $m \in N$. This is not possible since $n \geqq 3$. If X is a 2-group then again by (2.5) $\exp(X) \leqq 4$. So by (5.8) $X \simeq C_4$ or $C_2 \times C_4$ and it may be verified that both cases are not possible under our current assumptions.

Thus X cannot be a p-group. Hence $X = S \times T$ where S is a 2-group with $|S| > 2$ and T a 3-group of order at least 3 and exponent 3. So $\mathrm{Aut}(X) \simeq \mathrm{Aut}(S) \times \mathrm{Aut}(T)$. Since $\mathrm{Aut}(T)$ cannot be isomorphic to $D(n)$ by the first part, we must have $\mathrm{Aut}(S) \simeq D(n)$ and $\mathrm{Aut}(T) \simeq C_2$. It follows that $S \simeq C_2 \times C_2$ and $T \simeq C_3$ so that $X \simeq C_2 \times C_2 \times C_3$ and it is easily verified that $\mathrm{Aut}(X) \simeq D(12)$.

Suppose $4 \mid n$. Then $D(2n)$ is indecomposable and $Z(D(2n)) \simeq C_2$. Again by (2.5) $\pi(X) \subseteq \{2, 3\}$. Clearly X cannot be a 3-group. If X is a 2-group then by (5.8) $X \simeq C_4$ or $C_2 \times C_4$. It is easily verified that $\mathrm{Aut}(C_2 \times C_4) \simeq D(8)$. If $X = S \times T$ where S is a 2-group and T a 3-group it follows from the indecomposability of $D(2n)$ that $|S| \leqq 2$ and $\mathrm{Aut}(T) \simeq D(2n)$. However we have already seen that this is not possible.

Hence the proposition is proved.

PROPOSITION 6.2. *The following list gives all nonabelian finite groups* X *such that* $\mathrm{Aut}(X) \simeq D(2n)$ *for some* $n \in N$.

(a) $X \simeq D(6)$ *and* $\mathrm{Aut}(X) \simeq D(6)$.
(b) $X \simeq C_8$ *or* $C_4 \times C_3$ *and* $\mathrm{Aut}(X) \simeq C_c \times C_2 \simeq D(4)$.
(c) $X \simeq D(12)$ *and* $\mathrm{Aut}(X) \simeq D(12)$.
(d) $X \simeq D(6) \times C_3$ *and* $\mathrm{Aut}(X) \simeq D(12)$.
(e) $X \simeq D(8)$ *and* $\mathrm{Aut}(X) \simeq D(8)$.

PROOF. Clearly $n > 1$. If $n = 2$, it is easily seen that X must be a 2-group of class 2 and hence $|X|$ divides 4 (by (2.14)). So $n > 2$.

Suppose n is odd and X is a PN-group. Clearly $X/Z(X) \simeq D(2n)$. So $A_c(X) = 1$. By (5.6) it follows that $Z(X) = 1$. So $X \simeq D(2n)$. Hence $|\mathrm{Aut}\, X| = n\, \phi(n) = 2n$. Thus $\phi(n) = 2$ and so $n = 3$, 4 or 6. Since n is odd, $n = 3$. In fact, $\mathrm{Aut}(D(6)) \simeq D(6)$. If X is not a PN-group then $X = Y \times A$ where Y is a PN-group and A is abelian. It follows that $\mathrm{Inn}(Y) \simeq \mathrm{Aut}(Y) \simeq D(2n)$ and that $\mathrm{Aut}(A) = 1$. Hence $X \simeq D(6)$ or $D(6) \times C_2$. However $\mathrm{Aut}(D(6) \times C_2)$ is isomorphic to $D(12)$ and so $X \simeq D(6)$ is the only possibility satisfying the current hypothesis.

Suppose now that $(n, 4) = 2$ so that $n = 2n_0$, n_0 odd. Clearly $X/Z(X) \simeq D(2n_0)$ or $D(2n)$. Suppose $X/Z(X) \simeq D(2n_0)$ and that X is a PN-group. It follows from (5.6) that $X \simeq DC(4n_0)$, since $|A_c(X)| = 2$. Thus $|\text{Aut}\,(X)| = 2n_0\ \phi(2n_0) = 4n_0$ so that $\phi(2n_0) = 2$. Hence $2n_0\,3$, 4, or 6 yielding $n_0 = 3$ since it is odd. So $X \simeq DC(12)$ and it is easily verified that $\text{Aut}\,(X) \simeq D(12)$ in this case. Suppose $X = Y \times A$ where Y is a PN-group and A is abelian. Then $\text{Inn}\,(Y) \simeq D(2n_0)$ and $\text{Aut}\,(Y) \simeq D(2n_0)$ or $D(2n)$, since $\text{Aut}\,(X)$ must contain a subgroup isomorphic to $\text{Aut}\,(Y)$. If $\text{Inn}\,(Y) \simeq \text{Aut}\,(Y) \simeq D(2n_0)$ then from what has already been established it follows that $n_0 = 3$ and $Y \simeq D(6)$. Moreover $|\text{Aut}\,(A)| \leqq 2$ and so $A \simeq 1$, C_2, C_3, C_4 or C_6. It is easily verified that $X \simeq D(6) \times C_2$ or $D(6) \times C_3$ and in each case $\text{Aut}\,(X) \simeq D(12)$. Note that there is no PN-group Y with $\text{Inn}\,(Y) \simeq D(2n_0)$ and $\text{Aut}\,(Y) \simeq D(2n)$.

Suppose now that $n > 4$ and $4 \mid n$. Again $X/Z(X) \simeq D(2n)$ or $D(n)$ and in either case $|A_c(X)| = 2$. By (5.6) it follows that both possibilities cannot occur if X is a PN-group. So suppose $X = Y \times A$ where Y is a PN-group and A is abelian. If $\text{Inn}\,(X) \simeq D(n)$ then $\text{Inn}\,(Y) \simeq D(n)$ and $\text{Aut}\,(Y) \simeq D(n)$ or $D(2n)$. Neither case can occur, as has already been established. If $\text{Inn}\,(X) \simeq D(2n)$ then $\text{Inn}\,(Y) \simeq \text{Aut}\,(Y) \simeq D(2n)$ which is again impossible. Thus $4 \mid n$ implies $n = 4$.

So suppose that $\text{Aut}\,(X) \simeq D(8)$. By (5.6) this is impossible if $\text{Inn}\,(X) \simeq D(8)$ when X is a PN-group. The same conclusion can be established even if X is not a PN-group, in the usual way. Hence $\text{Inn}\,(X) \simeq C_2 \times C_2$ so that X is nilpotent of class 2. The indecomposability of $D(8)$ implies that X is a 2-group. Hence by (2.14) $|X|$ divides 8. The only possibility is therefore $|X| = 8$ so that $X \simeq D(8)$ or Q. Since $\text{Aut}\,(D(8)) \simeq D(8)$ and $\text{Aut}\,(Q) \simeq S_4$, X must be isomorphic to $D(8)$ in this case.

Hence the proposition has been proved.

THEOREM 6.3. *The following list gives all finite groups X such that* $\text{Aut}\,(X) \simeq D(2n)$ *for some* $n \in N$.

(a) $X \simeq C_3$, C_4 *or* C_6 *and* $\text{Aut}\,(X) \simeq D(2)$.
(b) $X \simeq C_8$ *or* C_{12} *and* $\text{Aut}\,(X) \simeq D(4)$.
(c) $X \simeq C_2 \times C_2$ *or* $D(6)$ *and* $\text{Aut}\,(X) \simeq D(6)$.
(d) $X \simeq C_2 \times C_4$ *or* $D(8)$ *and* $\text{Aut}\,(x) \simeq D(8)$.
(e) $X \simeq C_2 \times C_6$, $D(6) \times C_3$, $DC(12)$ *or* $D(12)$ *and* $\text{Aut}\,(X) \simeq D(12)$.

PROOF. Immediate consequence of (6.1) and (6.2).

PROPOSITION 6.4. *Suppose X is a finite group such that* $\text{Aut}\,(X) \simeq DC(4n)$ *for some* $n \in N$. *Then* $X \simeq C_5$ *or* C_{10} *and* $\text{Aut}\,(X) \simeq C_4 \simeq DC(4)$.

PROOF. Suppose X is nonabelian. Then $n \neq 1$. Also $Z(X) < X$ and $X/Z(X) \simeq DC(4m)$ for some $m \in N$, by (5.3). This is impossible by (5.7). So X must be abelian. Suppose $n > 1$. Then $\pi(X) \subseteq \{2, 3\}$ by (2.5). If X is a 3-group then $\text{Aut}(X) \simeq GL(m, 3)$ for some $m \in N$ since $\exp(X) = 3$ in this case. However by (5.5) this is not possible. If X is a 2-group then $\exp(X)$ is at most 4 and hence by (5.8) $X \simeq C_4$ which is false. Hence $X = S \times T$ where S is a 2-group of order greater than 2 and T is a 3-group of order greater than 1. But then $\text{Aut}(X) \simeq \text{Aut}(S) \times \text{Aut}(T)$ which is impossible since $DC(4m)$ is indecomposable for any $m \in N$. So $n = 1$. Clearly then X must be cyclic and so $X \simeq C_5$ or C_{10}.

PROPOSITION 6.4. *The following list gives all finite abelian groups X such that* $\text{Aut}(X) \simeq QD(8n)$ *for some* $n \in N$.

(a) $X \simeq C_{15}, C_{16}, C_{20}$ *or* C_{30} *and* $\text{Aut}(X) \simeq C_4 \times C_2 \simeq QD(8)$.
(b) $X \simeq C_2 \times C_2 \times C_5$ *and* $\text{Aut}(X) \simeq QD(24)$.

PROOF. If $n = 1$ then $QD(8)$ is abelian and so by (2.22) X must be cyclic. Hence it is easily proved that $X \simeq C_{15}, C_{16}, C_{20}$ or C_{30}. So let us assume that $n > 1$. Then $Z(QD(8n)) \simeq C_2$ if n is even and isomorphic to C_4 if n is odd. Suppose n is even. Then by (2.5) $\pi(X) \subseteq \{2, 3\}$. If X is a 3-group, it is elementary abelian and hence $\text{Aut}(X) \simeq GL(m, 3)$ for some $m \in n$. This is impossible by (5.5). If X is a 2-group then $\exp(X) \leqq 4$ and so by (5.8) $X \simeq C_4$ or $C_2 \times C_4$ both of which are impossible. So $X = S \times T$ where S is a 2-group with $|S| \geqq 4$ and T is a 3-group with $|T| > 1$. So $\text{Aut}(X) \simeq \mathit{Aut}(S) \times \text{Aut}(T)$ and this is impossible since $QD(8n)$ is indecomposable when $n > 2$ and n even. So n is odd and hence $\pi(X) \subseteq \{2, 5\}$. If X is a 5-group then $\text{Aut}(X) \simeq GL(m, 5)$ for some $m \in N$, *which is impossible. As before,* X cannot be a 2-group either. Hence $X = S \times T$ where S is a 2-group with $|S| \geqq 4$ and T is a 5-group with $|T| > 1$. So $\text{Aut}(X) \simeq \text{Aut}(S) \times \text{Aut}(T) \simeq D(2n) \times C_4$. By (6.1) $S \simeq C_2 \times C_2$ and hence $T \simeq C_5$. Thus $X \simeq C_2 \times C_2 \times C_5$ and it is easily verified that $\text{Aut}(X) \simeq QD(24)$.

PROPOSITION 6.5. *If X is a nonabelian finite group such that* $\text{Aut}(X) \simeq QD(8n)$ *for some* $n \in N$ *then* $X \simeq D(6) \times C_5$ *and* $\text{Aut}(X) \simeq QD(24)$.

PROOF. Suppose n is odd. If $n = 1$ then X is nilpotent of class 2. Moreover if an odd prime p divides $|X|$, it is easily seen that X would have to be abelian. Hence X is a 2-group. So by (2.14), $|X| \leqq 8$ which is easily seen to be impossible. So $n > 1$. By (5.4) and (5.7) it follows that $X/Z(X) \simeq D(2n)$ or $D(4n)$. Suppose $X/Z(X) \simeq D(2n)$. Clearly

$A_c(X) \simeq C_4$ and by (5.6) X cannot be a PN-group. Let $X = Y \times A$ where Y is a PN-group and A is abelian. Then $\mathrm{Inn}(Y) \simeq D(2n)$ and $\mathrm{Aut}(Y) \simeq D(2n)$, $D(4n)$ or $QD(8n)$. $\mathrm{Aut}(Y) \simeq D(2n)$ implies that $n = 3$ by (6.2) and $Y \simeq D(6)$. Hence $|\mathrm{Aut}(A)| \leqq 4$ so that $A \simeq 1$, C_2, C_3, C_4, C_5, C_6, C_8, C_{10} or C_{12}. It is easily verified that $\mathrm{Aut}(Y \times A) \simeq QD(24)$ exactly when $A \simeq C_5$. Now suppose that $\mathrm{Aut}(Y) \simeq D(4n)$. Then by (6.2) $Y \simeq D(12)$, $DC(12)$ or $D(6) \times C_3$ and hence $A \simeq 1$, C_2, C_3, C_4 or C_6 and $n = 3$. However, it is easily verified that $\mathrm{Aut}(Y \times A) \not\simeq QD(24)$ in any of these cases. Besides, $\mathrm{Aut}(Y) \simeq QD(8n)$ is impossible since Y is a PN-group and $\mathrm{Inn}(Y) \simeq D(2n)$. Suppose $X/Z(X) \simeq D(4n)$. Again by (5.6) X cannot be a PN-group. Let $X = Y \times A$ as usual. Then $\mathrm{Inn}(Y) \simeq \mathrm{Aut}(Y) \simeq D(4n)$. By (6.2) this is not possible.

Suppose now that n is even. By (5.4) and (5.7) it follows that $X/Z(X) \simeq D(4n)$. Also $A_c(X) \simeq C_2$. By (5.6) X cannot be a PN-group. Let $X = Y \times A$ where Y is a PN-group and A is abelian. We must have $\mathrm{Inn}(Y) \simeq \mathrm{Aut}(Y) \simeq D(4n)$. This is impossible by (6.2).

Thus the result has been established.

THEOREM 6.6. *The following list contains all finite groups X such that* $\mathrm{Aut}(X) \simeq QD(8n)$ *for some* $n \in N$.

(a) $X \simeq C_{15}$, C_{16}, C_{20} *or* C_{30} *and* $\mathrm{Aut}(X) \simeq C_4 \times C_2 \simeq QD(8)$.

(b) $X \simeq C_2 \times C_2 \times C_5$ *or* $D(6) \times C_5$ *and* $\mathrm{Aut}(X) \simeq QD(24)$.

PROOF. Immediate consequence of (6.4) and (6.5).

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