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ERGODIC THEOREMS FOR MIXING TRANSFORMATION GROUPS

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ABSTRACT. The notions of weak and strong mixing are extended to groups of transformations. Mixing transformations are characterized in terms of ergodic theorems which hold for those transformations.

0. **Introduction.** Let τ be a measure preserving transformation on a probability space (Ω, F, P) and let T denote the induced operator on L^2. We say T(or τ) is ergodic if the only functions left fixed by T are the constants. In this case the mean ergodic theorem says that $(1/N)\ \sum_{n=1}^{N} T^n f$ converges to $\int f\,dP$ in L^2 for f in L^2. Conversely, if $(1/N)\sum T^n f \to \int f\,dP$ for all f in L^2 then T is ergodic. (Convergence is in the L^2 sense throughout this paper.) Thus an ergodic transformation can be characterized as one whose "time averages" converge to the projection onto the constants, i.e., the "space average".

It also follows that T is ergodic if and only if

$$\frac{1}{N}\sum_{n=1}^{N}\left[(T^n f, f) - \left(\int f\,dP\right)^2\right] \to 0$$

for all f in L^2. This less intuitive propery of a transformation led to the definition of a strongly mixing transformation as one for which $(T^n f, f) \to (\int f\,dP)^2$ and of a weakly mixing transformation as one for which

$$\frac{1}{N}\sum_{n=1}^{N}\left|(T^n f, f) - \left(\int f\,dP\right)^2\right|^2 \to 0.$$

At first these concepts were not directly related to the ergodic problem if identifying time averages with space averages. But in 1960 Blum and Hanson [2] showed that a transformation is strongly mixing if and only if

$$\frac{1}{N}\sum_{k=1}^{N} T^{n_k} f \to \int f\,dP$$

for all subsequences n_k. In 1971 L. K. Jones [7] showed that a transformation is weakly mixing if and only if

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$$\frac{1}{N}\sum_{k=1}^{N} T^{n_k} f \rightarrow \int f\,dP$$

for all subsequences n_k of positive lower density. These characterizations showed the fundamental nature of the concepts of weak and strong mixing. Unfortunately the corresponding results do not hold for the individual ergodic theorem (see J. P. Conze [5]). The individual ergodic theorem has been shown to hold, however, for weakly mixing transformations for uniform sequences (see Brunel-Keane [4]) and p-sequences (see Blum-Reich [3]).

In §1 we characterize weak mixing and strong mixing for one parameter groups of transformations in terms of ergodic theorems for subsequences. The results and arguments motivate §2 where we define and characterize weak and strong mixing for groups of transformations in terms of ergodic theorems.

1. **A characterization of mixing for flows.** Let τ_t be a one parameter group of measure preserving transformations on (Ω, F, P) and let T_t denote the corresponding group of unitary operators on L^2. We assume the map t to $T_t f$ is continuous for each f. The group T_t(or τ_t) is ergodic if the only functions left fixed by T_t for all t are the constants. The group is called strongly mixing if $(T_t f, f) \rightarrow (\int P)^2$ for all f in L^2 and it is called weakly mixing if

$$\frac{1}{N}\int_0^N |(T_t f, f) - (\int f\,dP)^2|^2\,dt \rightarrow 0.$$

Strong mixing implies weak mixing which inplies ergodicity and then by the mean ergodic theorem for continuous one parameter groups we have $(1/N)\int_0^N T_t f\,dt \rightarrow \int f\,dP$ for all f in L^2. Theorem 1 and 2 characterize strong and weak mixing in terms of ergodic theorems for subsequences of transformations.

THEOREM 1. *T_t is strongly mixing if and only if*

$$\frac{1}{N}\sum_{k=1}^{N} T_{t_k} f \rightarrow \int f\,dP$$

for all f in L^2 and all subsequences t_k with $t_k - t_{k-1} \geqq \delta > 0$.

THEOREM 2. *T_t is weakly mixing if and only if*

$$\frac{1}{N}\sum_{n=1}^{N} T_{n\alpha} f \rightarrow \int f\,dP$$

for all f in L^2 and all $\alpha \neq 0$.

PROOF OF THEOREM 1. Let T_t be strongly mixing. Let $t_k - t_{k-1} \geqq \delta > 0$.

For any $\epsilon > 0$ choose M so $t \geqq M$ implies

$$|(T_t f, f) - (\int f\, dP)^2| \leqq \epsilon/2.$$

Then

$$\left\| \frac{1}{N} \sum_{k=1}^{N} T_{t_k} f - \int f\, dP \right\|^2$$

$$\leqq \left| \frac{1}{N^2} \sum_{k=1}^{N} \sum_{j=1}^{N} [(T_{t_k - t_j} f, f) - (\int f\, dP)^2] \right|$$

$$\leqq \frac{1}{N^2} \sum_{\substack{k=1 \\ |t_k - t_j| \leqq M}}^{N} \sum_{j=1}^{N} |[T_{t_k - t_j} f, f) - (\int f\, dP)^2]| + \epsilon/2.$$

But

$$|(T_s f, f) - (\int f\, dP)^2| \leqq \|f\|^2 + (\int f\, dP)^2 = C,$$

and the number of terms in the first sum is less than or equal $N(2M/\delta)$ since for any k there are most $2M/\delta$ values t_j within M of t_k. Hence

$$\left\| \frac{1}{N} \Sigma T_{t_k} f - \int f\, dP \right\|^2 \leqq \frac{2M}{\delta N} C + \epsilon/2,$$

which is less than ϵ for N large enough. Thus

$$\frac{1}{N} \sum_{k=1}^{N} T_{t_k} f \to \int f\, dP.$$

Conversely if T_t is not strongly mixing there is a subsequence t_k approaching infinity with $|(T_{t_k} f, f) - (\int f\, dP)^2| \geqq \epsilon$. Without loss of generality choose a subsequence t'_k with $t' - t'_{k-1} \geqq \delta$ and $(T_{t'_k} f, f) - (\int f\, dP)^2 \geqq \epsilon$. Then

$$\left(\frac{1}{N} \sum_{k=1}^{N} T_{t'_k} f - \int f\, dP, f \right) = \frac{1}{N} \Sigma [T_{t'_k} f, f) - (\int f\, dP)^2] \geqq \epsilon.$$

This contradicts the fact that $(1/N) \Sigma T_{t'_k} f \to \int f\, dP$.

PROOF OF THEOREM 2. Assume T_t is weakly mixing. It follows from the spectral theorem and Wiener's theorem that

$$(t_t f, f) - (\int f\, dP)^2 = \int e^{i\lambda t}\, dF(\lambda),$$

where dF is a continuous, finife, positive measure. But then

$$\left\| \frac{1}{N} \Sigma T_{n\alpha} f - \int f\, dP \right\|^2 = \int \left| \frac{1}{N} e^{in\alpha\lambda} \right|^2 dF(\lambda)$$

$$\rightarrow \sum_{k=-\infty}^{\infty} dF\left(\frac{2\pi k}{\alpha}\right) = 0.$$

Conversely, if T_t is not weakly mixing there is a non-constant f with $T_t f = e^{i\lambda t} f$ for some λ. Then $(1/N)\Sigma_{n=1}^{N} T_{2\pi n/\lambda} f = f$ does not converge to $\int f\, dP$.

In fact it is easy to see that the individual ergodic theorem holds for averages $(1/N)\Sigma_1^N T_{n\alpha}$ if T_t is weakly mixing. This follows since T_α is itself weakly mixing for any $\alpha \neq 0$.

1. **Weak and strong mixing for groups of transformations.** The problem in extending these results to more general groups is first the lack of a natural averaging scheme for the mean ergodic theorem itself. For two parameter groups for example one could average $T_{s,t}$ over increasing sequences of circles, squares, or sectors. A unified treatment is possible for continuous unitary representations of locally compact Abelian groups based on the results in Blum-Eisenberg [1]. There it is shown that if U_g is a continuous unitary representation of a locally compact Abelian group G and μ_n is a "generalized summing sequence" then $\int U_g f\, d\mu_n(g)$ converges to the projection of f on the space of elements fixed under U_g for all g. A generalized summing sequence is a sequence of probability measures μ_n on G whose Fourier transforms $\hat{\mu}_n$ converge to zero except on the identity in $\hat{G}$.

If U_g arises from a group of measure preserving transformations we say U_g is ergodic if the only functions left fixed by U_g for all g are the constants. In this case $\int U_g f\, d\mu_n \rightarrow \int f\, dP$ for all f in L^2 and all generalized summing sequences.

We say that U_g is strongly mixing if $(U_g f, f) \rightarrow (\int f\, dP)^2$ as $g \rightarrow \infty$. We say U_g is weakly mixing if $\int |(U_g f, f) - (\int f\, dP)^2|^2\, d\mu_n(\gamma) \rightarrow 0$ for all generalized summing sequences μ_n. We show that if, in fact, the limit is zero for some generalized summing sequence μ_n then it is zero for all generalized summing sequences. This is based on a simple extension of Wiener's theorem (see Katznelson [8, p. 42]).

LEMMA (WIENER'S THEOREM FOR LOCALLY COMPACT ABELIAN GROUPS). *A finite positive measure dF on a locally compact Abelian group $\hat{G}$ is continuous if and only if $\int |\hat{F}(g)|^2\, d\mu_n(g) \rightarrow 0$ where μ_n is a generalized summing sequence on G.*

The proof is exactly like that for measures on the unit circle or real line.

THEOREM 3. *Let U_g be a continuous unitary representation of a locally compact Abelian group G. If* $\int |(U_g f, f) - (\int f\,dP)^2|^2\, d\mu_n(g) \to 0$ *for some generalized summing sequence then it converges to zero for all generalized summing sequences.*

PROOF. By the spectral theorem for unitary groups

$$|(U_g f, f) - (\int f\,dP)^2|^2 = |\hat{F}(g)|^2$$

where F is a finite positive measure on $\hat{G}$. If $\int |\hat{F}(g)|^2\, {}^d\mu_n(g) \to 0$ for some generalized summing sequence μ_n, then by the lemma dF is continuous and again by the lemma $\int |\hat{F}(g)|^2\, d\mu_n(g) \to 0$ for all generalized summing sequences μ_n.

This shows that the property of weak mixing is independent of the choice of generalized summing sequence. Unlike the case of $\mathbf{R}^n$ or $\mathbf{Z}^n$ it is not obvious that strong mixing implies weak mixing.

COROLLARY. *If G is σ compact (and non compact) and U_g is strongly mixing then U_g is weakly mixing.*

PROOF. On a σ compact locally compact Abelian group G there exists a generalized summing sequence μ_n of the form

$$\mu_n(A) = \frac{\mu(A \cap E_n)}{\mu(E_n)}$$

where μ is Haar measure (see Hewitt and Ross [**6**, p. 255] and Blum-Eisenberg [**1**, cor. 2]) on G and $\mu(E_n) \to \infty$. Now if U_g is strongly mixing there is a compact set K with $|(U_g f, f) = (\int f\,dP)^2|^2 \leqq \epsilon/2$ for g in K^c.

But then

$$\int |(U_g f, f) - (\int f\,dP)^2|^2\, d\mu_n(g) \leqq \frac{[\|f\|^2 + (\int f\,dP)^2]\mu(K)}{\mu(E_n)} + \frac{\epsilon}{2} \leqq \epsilon$$

for n large. By Theorem 3 U_g is weakly mixing.

LEMMA. *If U_g is strongly mixing then $\int U_g f\,d\mu_n \to \int f\,dP$ for all sequences of probabilities μ_n such that* sup $\mu_n(K) \to 0$ *for compact sets K.*

PROOF. Choose K compact so $|(U_g f, f) - (\int f\,dP)^2| \leqq \epsilon/2$ for g in K^c. Then $\| \int U_g f\,d\mu_n - \int f\,dP\| \leqq \| \int_{g \epsilon K}(U_g f - \int f\,dP)\,d\mu_n\| + \epsilon/2$. But $\| \int_{g\epsilon K}(U_g f - \int f\,dP)\,d\mu_n\|^2 \leqq M \int_K \int_K d\mu_n(g_1 + g_2)\,d\mu_n(g_2)$. But $\sup/g\ \mu_n(K + g) \to 0$ as $n \to \infty$. The lemma follows.

THEOREM 4. *U_g is strongly mixing if and only if* $(1/\mu(E_n)) \int_{E_n} U_g f \, d\mu \to \int f \, dP$ *for all sequences of sets E_n with $\mu(E_n) \to \infty$, where μ is Haar measure.*

PROOF. Let $\mu_n(A) = \mu(A \cap E_n)/\mu(E_n))$. Then μ_n satisfies the conditions in the lemma so the averages converge.

Conversely, if U_g is not strongly mixing we may assume without loss of generality that there is a sequence $g_n \to \infty$ with $(U_g f, f) - (\int f \, dP)^2 > \epsilon > 0$. Since $(U_g f, f)$ is uniformly continuous there is a neighborhood of the identity V with $(U_g f, f) - (\int f \, dP)^2 > \epsilon/2$ for g in $U_{n=1}^{\infty}(g_n + V)$. Letting $E_n = U_1^n(G_i + V)$ we have that $\mu(E_n) \to \infty$ and $((1/\mu(E_n)) \int_{E_n} U_g f \, d\mu_n - \int f \, dP, f) \nrightarrow 0$.

This implies that for G discrete and U_g strongly mixing, $(1/n) \sum_1^n U_{g_i} f \to \int f \, dP$ for any sequence g_i with $g_i \neq g_j$ for $i \neq j$. The theorem does not imply Theorem 1. Nevertheless, it can easily be shown that analogues of Theorem 1 hold for $G = R^n$.

We now proceed with characterization of weakly mixing transformations. Denote the annihilator of a subgroup H in G by $H^{\perp}$. The annihilator is the closed subgroup of $\hat{G}$ such that $(h, x) = 1$ for h in H (see Katznelson [8, p. 189]).

THEOREM 5. *Let U_g be weakly mixing and let H be a closed subgroup of G with $H^{\perp}$ countable. If μ_n is a generalized summing sequence on H then* $\int_H U_g f \, d\mu_n \to \int f \, dP$.

PROOF. It is seen from the proof of Theorem 3 that.

$$\left\| \int U_g f \, d\mu_n - \int f \, dP \right\|^2 = \int_G \left| \int_H (g, x) \, d\mu_n(g) \right|^2 dF(x)$$

where dF is a continuous measure on $\hat{G}$. Since μ_n is a generalized summing sequence on H, $\int (g, x) \, d\mu_n(g) \to 0$ except for x in $H^{\perp}$. But $H^{\perp}$ is countable and dF is continuous. Hence the right side approaches zero.

If U_g is not weakly mixing there is an x in $\hat{G}$ and f in L^2 with $U_g f = \langle g, x \rangle f$. If $\{x^n\}_{n=-\infty}^{\infty}$ is closed there is a nontrivial closed subgroup H of G (we assume G is not compact) with $H^{\perp} = \{x^n\}_{n=-\infty}^{\infty}$. Then $\int_H U_g f \, d\mu_n = f \nrightarrow \int f \, dP$ for μ_n a generalized summing sequence on H. *Moreover* $H^{\perp}$ is countable. We thus have

THEOREM 6. *Assume $\{x^n\}_{n=-\infty}^{\infty}$ is closed in $\hat{G}$ for all x in $\hat{G}$. Then U_g is weakly mixing if and only if the mean ergodic theorem holds for U_g over all closed subgroups H in G with $H^{\perp}$ countable.*

For example, if $G = \mathbf{Z}$, Theorem 6 does not apply. We cannot characterize weakly mixing transformations by ergodic theorems on subgroups because there are characters $e^{i\theta}$ in T with $\{e^{in\theta}\}$ dense in T.

If $G = \mathbf{R}$ Theorem 6 applies. Moreover every closed subgroup of $\hat{G}$ is countable. Thus every closed subgroup of $\mathbf{R}$(except $\{0\}$) has $H^{\perp}$ countable.

If $G = \mathbf{R}^2$ Theorem 6 applies. However, there are closed subgroups of $\hat{G}$ which are not countable (say (t, t)). Thus the mean ergodic theorem may not hold over all closed subgroups of G for weakly mixing transformations. In fact there is the following counter-example. Assume $(U_{s,t}f, f) - (\int f\,dP)^2 = \int \int e^{i\lambda s + i\mu t}\,dF(\lambda, \mu)$ where $dF(\lambda, \mu)$ has a positive density $g(\lambda)$ with respect to Lebesgue measures on the line $(\lambda, \alpha\lambda)$. Then dF is continuous and $U_{s,t}$ is weakly mixing. Let H be the subgroup $(-\alpha t, t)$. Then

$$\left\| \frac{1}{N} \int_0^N u_{-\alpha t,t} f\,dt - \int f\,dP \right\|^2$$

$$= \frac{1}{N} \int_0^N (U_{-\alpha t,t}f, f) - (\int f\,dP)^2\,dt$$

$$= \frac{1}{N} \int_0^N (\int e^{-\lambda\alpha t + i\lambda\alpha t} g(\lambda)\,d\lambda)\,dt$$

$$= \int g(\lambda)\,d\lambda \neq 0.$$

We do have that

$$\frac{1}{N^2} \sum_1^N \sum_1^N U_{n\alpha,n\beta} f \to \int f\,dP$$

because the annihilator of the subgroup $\{(n\alpha, m\beta)\}$ is $\{(2\pi j/\alpha, 2\pi k/\beta)\}$ which is countable.

Although the results of this paper are phrased in terms of groups of measure preserving transformations all the theorems have analogs for general unitary operators on Hilbert space. This is the direction taken by Jones in defining ergodic weak mixing and strong mixing for operators on Banach spaces.

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