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QUASI-UNIFORM ABSOLUTE CONTINUITY AND INTEGRAL CONVERGENCE

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ABSTRACT. A generalization of uniform absolute continuity, called quasi-uniform absolute continuity, is introduced. This notion is used to prove Helly-type convergence theorems for refinement-type integrals involving a set function, α, with range a collection of real nonvoid number sets with bounded union, and a generalized sequence with range a uniformly bounded collection of real-valued bounded finitely additive set functions. The assumption of quasi-uniform absolute continuity for the above mentioned generalized sequence in the hypotheses of the aforementioned theorems permits the replacing of "standard" continuity conditions on α with weaker integrability assumptions.

1. **Introduction.** Suppose U is a set, $\mathbf{F}$ is a field of subsets of U and $\mathfrak{p}$ is the set of all functions with domain $\mathbf{F}$ and range a collection of nonvoid number sets. Suppose $\mathfrak{p}_{AB}$ is the set of all real-valued bounded finitely additive functions defined on $\mathbf{F}$ and $\mathfrak{p}_A{}^+$ is the set of all non-negative-valued elements of $\mathfrak{p}_{AB}$. Suppose S is a set and $\leqq^*$ is a partial ordering on S with respect to which S is directed.

One of the principal facts involved in any proof of Helly's (see [8], p. 764) well-known Stieltjes integral convergence theorem is the "uniform approximation of integrals by integral approximation sums". In a previous paper [6] the author showed integral convergence theorems whose hypotheses, while stronger than that of Helly's for boundedness conditions, were weaker for integrability conditions. This paper generalizes those theorems. We state a theorem that is a consequence of the main theorem of this paper.

THEOREM 5.1 (§ 5). *Suppose β is a function from S into $\mathfrak{p}_{AB}$, ξ is a function from $\mathbf{F}$ into $\mathbf{R}$, and α is a function from $\mathbf{F}$ into a collection of nonvoid number sets with bounded union such that:*

(i) *for all V in $\mathbf{F}$, $\beta(y)(V) \to \xi(V)$, for $\leqq^*$,*

(ii) *if $0 < c$, then there are μ in $\mathfrak{p}_A{}^+$ and X in S such that if $X \leqq^* y$, then* (see section 2 for the definition of integral)

$$\int_U \max\{\mu(I), \int_I |\beta(y)(I)|\} - \mu(U) < c, \text{ and}$$

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(iii) *for all* y *in* S, *the integral* $\int_U \alpha(I)\beta(y)(I)$ *exists.*
Then ξ *is in* $\mathfrak{p}_{AB}$, $\int_U \alpha(I)\xi(I)$ *exists, and*

$$\int_U \alpha(I)\beta(y)(I) \to \int_U \alpha(I)\xi(I), \text{ for } \leqq^*.$$

Although the above theorem, among the theorems of this paper, is the more immediate analogue of Helly's theorem, it is, however, a special case of a theorem that we shall state immediately after defining the following generalization of uniform absolute continuity:

(*) Definition. If β is a function from S into $\mathfrak{p}_{AB}$ and $W \subseteq \mathfrak{p}_A{}^+$, then the statement that β is quasi-uniformly absolutely continuous for $\leqq^*$ with respect to W means that if $0 < c$, then there are $d > 0$, μ in W and X in S such that if V is in **F**, $X \leqq^* y$ and $\mu(V) < d$, then $\int_V |\beta(y)(I)| < c$.

Theorem 4.1. *Suppose* β *is a function from* S *into* $\mathfrak{p}_{AB}$, ξ *is a function from* **F** *unto* **R**, *and* α *is a function from* **F** *into a collection of nonvoid number sets with bounded union such that*:
ii (i) *for each* V *in* **F**, $\beta(y)(V) \to \xi(V)$, *for* $\leqq^*$,
i (ii) *there are* X' *in* S *and* M *in* **R** *such that if* $X' \leqq^* y$, *then* $\int_U |\beta(y)(I)| \leqq M$.
(iii) *there is* $W \subseteq \mathfrak{p}_A{}^+$ *such that for all* ζ *in* W, $\int_U \alpha(I)\zeta(I)$ *exists and such that* β *is quasi-uniformly absolutely continuous for* $\leqq^*$ *with respect to* W. *Then* ξ *is in* $\mathfrak{p}_{AB}$, $\int_U \alpha(I)\xi(I)$ *exists, and if* Q *is* L *or* G (see § 2), *then*

$$\int_U Q(\alpha\beta(y))(I) \to \int_U \alpha(I)\xi(I), \text{ for } \leqq^*.$$

The above theorem (and hence Theorem 5.1) also rests on the type of uniform approximation condition mentioned at the beginning of the introduction. The extent to which the theorem requires such a condition is implied by the following characterization theorem (§ 3), a corollary to an E. H. Moore-type limit theorem (Theorem 3.1) which we shall state in § 3 and whose conventional argument we shall leave to the reader:

Corollary 3.1. *Suppose* K *is a function from* S *into a collection of nonvoid number sets,* B *is a function from* S *into* $\mathfrak{p}$, *and* ρ *is in* $\mathfrak{p}$. *With regard to* (a), (b), (1) *and* (2) *below*: *if* (a) *and* (1), *then* (2); *if* (b) *and* (2), *then* (1); *if* (a) *and* (b), *then* (1) *iff* (2).

(a) *For each* V *in* **F**,

$$(\leqq^*)-\lim_y [\sup\{\inf\{|x - z| : x \text{ in } \rho(V)\} : z \text{ in } B(y)(V)\}] = 0.$$

(b) *For each* V *in* $\mathbf{F}$,

$$(\leqq^*) - \lim_{y} [\sup\{\inf\{|x + | : z \text{ in } B(y)(V)\} : x \text{ in } \rho(V)\}] = 0.$$

(1) $\int_U \rho(I)$ *exists and*

$$(\leqq^*) - \lim_{y} \ [\sup\{|w - \int_U \ \rho(I)| : w \text{ in } K(y)\}] = 0.$$

(2) *For each positive real number c, there is* $D \ll \{U\}$ ("$\ll$" means "refinement"; see § 2) *such that if* $\mathfrak{E} \ll \mathfrak{D}$, *then there is* X *in* S *such that if* $X \leqq^* y$ *and for each* I *in* $\mathfrak{E}$, $b(y)(I)$ *is in* $B(y)(I)$, *and* v *is in* $K(y)$, *then* $|v - \sum_{\mathfrak{E}} b(y)(I)| < c$.

The author wishes to thank the referee for directing his attention to Theorem 3.1 and Corollary 3.1 in the form written in § 3 and above, respectively.

2. **Preliminary theorems and definitions.** Throughout this paper we shall refer to definitions and theorems which, in some of the references cited, are for single-valued set functions; these definitions and theorems carry over for appropriate elements of p with only minor and obvious modifications, and we accordingly cite and use them without further elaboration.

We refer the reader to [2] and [4] for the notions of subdivision, refinement and integral that we shall use in this paper, saying here, however, that the integrals that we consider are limits, with respect to refinements of subdivisions, of the appropriate sums. The reader is also referred to [5] for the notions of sum infimum functional G and sum supremum functional L. We shall let "$\mathfrak{E} \ll \mathfrak{D}$" mean "$\mathfrak{E}$ is a refinement of $\mathfrak{D}$".

In [2] there are refinement-sum inequalities that we shall not only refer to as they stand, but which also imply the existence of certain integrals that we shall discuss. We refer the reader again to [2] for Kolmogoroff's [7] notion of differential equivalence and its implications about the existence and equivalence of various integrals.

In this paper, when the existence of an integral or the equivalence of an integral to an integral is an easy consequence of the above mentioned material, the integral need only be written and the proof of existence or equivalence left to the reader.

We state a previous set function theorem of the author.

THEOREM 2.A.1 [5]. *If* α *is in* $\mathfrak{p}_B$ *and* ξ *is in* $\mathfrak{p}_{AB}$, *then* $\int_U \alpha(I)\xi(I)$ *exists iff* $\int_U \alpha(I) \int_I |\xi(J)|$ *exists.*

THEOREM 2.A.2 [3]. *If α is in $\mathfrak{p}_B$, μ is in $\mathfrak{p}_A{}^+$, ξ is in $\mathfrak{p}_{AB}$, $\int_U \alpha(I)\mu(I)$ exists and $|\xi(V)| \leqq \mu(V)$ for all V in* **F**, *then $\int_U \alpha(I)\xi(I)$ exists.*

We close this section by stating a previous theorem of the author [5] that we shall use in proving Theorem 6.1.

THEOREM 2.A.3. *Suppose α is in $\mathfrak{p}_B$, ξ is in $\mathfrak{p}_A{}^+$, and λ is the function from* **F** *into* **R** *given by $\lambda(V) = \sup\{\gamma(V) : \gamma$ in $\mathfrak{p}_A{}^+$, $\xi - \gamma$ in $\mathfrak{p}_A{}^+$, $\int_U \alpha(I)\gamma(I)$ exists$\}$. Then each of λ and $\xi - \lambda$ is in $\mathfrak{p}_A{}^+$ and $\int_U \alpha(I)\lambda(I)$ exists.*

3. Convergence Theorems for Directed Partial Orderings.

THEOREM 3.1. *Suppose (M, d) is a complete metric space and $(T, \leqq^{**})$ is a directed system. Suppose that for all y in S and z in T, each of $A(y, z)$ and $K(y)$ and $H(z)$ is a nonvoid subset of M. With regard to* (a), (b), (1) *and* (2) *below: if* (a) *and* (1), *then* (2); *if* (b) *and* (2), *then* (1); *if* (a) *and* (b), *then* (1) *iff* (2).

(a) *For each z in T,*

$$(\leqq^*) - \lim_y \,[\sup\{\inf\{d(x, w) : w \text{ in } H(z)\} : x \text{ in } A(y, z)\}] = 0.$$

(b) *For each z in T,*

$$(\leqq^*) - \lim_y \,[\sup\{\inf\{d(x, w) : x \text{ in } A(y, z)\} : w \text{ in } H(z)\}] = 0.$$

(1) *For some q in M,*

$$(\leqq^*) - \lim_y \,[\sup\{d(v, q) : v \text{ in } K(y)\}] = 0$$

$$\text{and } (\leqq^{**}) - \lim_z \,[\sup\{d(w, q) : w \text{ in } H(z)\}] = 0.$$

(2) *For each positive real number c there is D in T such that if $D \leqq^{**} E$, then there is X in S such that if $X \leqq^* y$, x is in $A(y, E)$ and v is in $K(y)$, then $d(x, v) < c$.*

We now give an indication of proof for Corollary 3.1, as stated in the introduction.

INDICATION OF PROOF. Let T denote the set of all subdivisions of U, and $\leqq^{**} = \{(\mathfrak{D}, \mathfrak{E}) : \mathfrak{E} \ll \mathfrak{D} \ll \{U\}\}$. Let A denote the function with domain $S \times T$ such that if y is in S and $\mathfrak{E}$ is in T, then $A(y, \mathfrak{E}) = \{\sum_{\mathfrak{E}} b(y)(I) : b(y)(I)$ in $B(y)(I)$ for each I in $\mathfrak{E}\}$. Let H denote the function with domain T such that if $\mathfrak{E}$ is in T, then $H(E) = \{\sum_{\mathfrak{E}} r(I) : r(I)$ in $\rho(I)$ for each I in $\mathfrak{E}\}$.

We leave to the reader the details of showing that for S, T, K, A and

H given above, the conditions of Theorem 3.1 are satisfied, and consequently the conclusions.

4. **Integral Convergence and Quasi-Uniform Absolute Continuity.** In this section we prove Theorem 4.1, as stated in the introduction.

PROOF OF THEOREM 4.1. First, (i) and (ii) immediately imply that ξ is in $\mathfrak{p}_{AB}$. Now let B denote the function with domain S such that if y is in S, then $B(y) = \alpha\beta(y)$. Let $K = \{(y, \{\int_U L(\alpha\beta(y))(I), \int_U G(\alpha\beta(y))(I)\}) : y \text{ in } S\}$. Let ρ denote $\alpha\xi$. It is clear that for B, K and ρ as given, the hypothesis of Corollary 3.1 is satisfied. We shall show that statement (2) of the corollary holds.

Let $M' = \sup\{|x| : x \text{ in range union of } \alpha\}$. Suppose $0 < c$. There are $d > 0$, μ in W and X in S such that $X' \leqq^* X$ and if V is in F, $X \leqq^* y$ and $\mu(V) < d$, then $\int_V |\beta(y)(I)| < c/[16(1 + M' + M)]$. Since $\int_U \alpha(I)\mu(I)$ exists, it follows from the Bochner-Radon-Nikodym Theorem (see [8], p. 315) that there are $\mathfrak{T} \ll \{U]$ and for each V in $\mathfrak{T}$, a number $c(V)$ such that $|c(V)| \leqq M'$ and

$$\sum_{\mathfrak{T}} \int_V \left| c(V)\mu(I) - \int_I \alpha(J)\mu(J) \right| < cd/[32(1 + M' + M)].$$

Now, if V is in $\mathfrak{T}$, then

$$\int_V \left| c(V)\mu(I) - \int_I \alpha(J)\mu(J) \right|$$
$$= \int_V |c(V)\mu(I) - \alpha(I)\mu(I)|,$$

so that there is $\mathfrak{P}(V) \ll \{V\}$ such that if $\mathfrak{Z} \ll \mathfrak{P}(V)$ and for each I in $\mathfrak{Z}$, $a(I)$ is in $\alpha(I)$, then

$$\left| \int_V \left| c(V)\mu(I) - \alpha(I)\mu(I) \right| \right.$$
$$\left. - \sum_{\mathfrak{Z}} |c(V)\mu(I) - a(I)\mu(I)| \right| < cd/[32(1 + M' + M)N],$$

where N is the number of elements in $\mathfrak{T}$. Let $\mathfrak{P} = \bigcup_{\mathfrak{T}}\mathfrak{P}(V)$. Suppose $\mathfrak{E} \ll \mathfrak{P}$ and $X \leqq^* y$. Let $\kappa = \rho(y)$. For each V in $\mathfrak{E}$, let $\mathfrak{E}(V) = \{I : I \text{ in } \mathfrak{T}, I \subseteq V\}$, and for each I in $\mathfrak{E}$, suppose $a(I)$ is in $\alpha(I)$. Then

$$\sum_{\mathfrak{T}} \sum_{\mathfrak{E}(V)} |c(V) - a(I)|\mu(I)$$
$$= \sum_{\mathfrak{T}} \sum_{\mathfrak{E}(V)} |c(V)\mu(I) - a(I)\mu(I)| < cd/[16(1 + M' + M)].$$

Now,

$$\left| \sum_{\mathfrak{T}} c(V)\kappa(V) - \sum_{\mathfrak{E}} a(I)\kappa(I) \right|$$

$$= \left| \sum_{\mathfrak{T}} \sum_{\mathfrak{E}(V)} (c(V)\kappa(I) - a(I)\kappa(I)) \right|$$

$$\leqq \sum_{\mathfrak{T}} \sum_{\mathfrak{E}(V)} |c(V) - a(I)|\, |\kappa(I)|$$

$$\leqq \sum_{\mathfrak{T}} \sum_{\mathfrak{E}(V)} |c(V) - a(I)| \int_I |\kappa(J)|.$$

For each V in $\mathfrak{T}$, let $\mathfrak{E}(V)' = \{I : I \text{ in } \mathfrak{E}(V),\ |c(V) - a(I)| \geqq c/[16(1 + M' + M)]\}$. It follows that

$$cd/[16(1 + M' + M)] > \sum_{\mathfrak{T}} \sum_{\mathfrak{E}(V)} |c(V) - a(I)|\mu(I)$$

$$\geqq \sum_{\mathfrak{T}} \sum_{\mathfrak{E}(V)} (c/[16(1 + M' + M)])\mu(I)$$

$$= (c/[16(1 + M' + M)]) \sum_{\mathfrak{T}} \sum_{\mathfrak{E}(V)'} \mu(I),$$

so that

$$\sum_{\mathfrak{T}} \sum_{\mathfrak{E}(V)'} \mu(I) < d,$$

which implies that

$$\sum_{\mathfrak{T}} \sum_{\mathfrak{E}(V)'} \int_I |\kappa(J)| < c/[16(1 + M' + M)],$$

so that

$$\left| \sum_{\mathfrak{T}} c(V)\kappa(V) - \sum_{\mathfrak{E}} a(I)\kappa(I) \right|$$

$$\leqq \sum_{\mathfrak{T}} \sum_{\mathfrak{E}(V)} |c(V) - a(I)| \int_I |\kappa(J)|$$

$$\leqq \sum_{\mathfrak{T}} \sum_{\mathfrak{E}(V)'} 2M' \int_I |\kappa(J)|$$

$$+ \sum_{\mathfrak{D}} \sum_{\mathfrak{E}(V)-\mathfrak{E}(V)'} [c/[16(1 + M' + M)]] \int_I |\kappa(J)|$$

$$< 2M'c/[16(1 + M' + M)]$$

$$+ (c/[16(1 + M' + M)])M < c/8 + c/8 = c/4.$$

It is easy to see that if $\mathfrak{E} \ll \mathfrak{P}$, $X \leqq^* y$, and for each I in $\mathfrak{E}$, $a(I)$ is in $\alpha(I)$, then

$$\max \left\{ \left| \int_U G(\alpha\beta(y))(I) - \sum_{\mathfrak{E}} a(I)\beta(y)(I) \right|, \right.$$

$$\left. \left| \int_U L(\alpha\beta(y))(I) - \sum_{\mathfrak{E}} a(I)\beta(y)(I) \right| \right\}$$

$$\leqq c/2 < c.$$

We therefore see that for the B, K and ρ defined at the beginning of the proof, statement (2) of Corollary 3.P.1 is satisfied, so that statement (1) of the corollary follows and therefore our theorem follows.

5. **A domination theorem.** In this section we prove Theorem 5.1, as stated in the introduction.

PROOF OF THEOREM 5.1. We show that the hypotheses of Theorem 4.1 are satisfied.

Clearly (i) of the hypothesis of Theorem 4.1 is satisfied.

There are X in S and μ in $\mathfrak{p}_A{}^+$ such that if $X \leqq^* y$, then

$$\int_U \max\{\mu(I), |\beta(y)(I)|\} - \mu(U) < 1.$$

so that

$$\int_U |\beta(y)(I)| \leqq \int_U \max\{\mu(I), |\beta(y)(I)|\} < 1 + \mu(U).$$

Therefore (ii) of the hypothesis of Theorem 4.1 is satisfied.

It remains to be shown that (iii) of the hypothesis of Theorem 4.1 is satisfied. We shall let W denote the set to which κ belongs iff κ is in $\mathfrak{p}_A{}^+$ and $\int_U \alpha(I)\kappa(I)$ exists. Suppose $0 < c$. There are μ in $\mathfrak{p}_A{}^+$ and X in S such that if $X \leqq^* y$, then

$$\int_U \max\{\mu(I), |\beta(y)(I)|\} - \mu(U) < c/2.$$

Let λ be the function from **F** into **R** defined by:

$$\lambda(V) = \sup\{\zeta(V) : \zeta \text{ in } W,\ \mu - \zeta \text{ in } \mathfrak{p}_A^+\}.$$

By Theorem 2.A.3, λ is in $\mathfrak{p}_A^+$ *and* $\int_U \alpha(I)\lambda(I)$ exists. Furthermore, suppose $X \leqq^* y$, V is in **F** and $\lambda(V) < c/2$. Since, by Theorem 2.A.1 and 2.A.2,

$$\int_U \alpha(I) \int_I \min\{\mu(J),\ |\beta(y)(J)|\}$$

exists, it follows that $\lambda - \int \min\{\mu,\ |\beta(y)|\}$ is in $\mathfrak{p}_A^+$, so that, since

$$\int_V |\beta(y)(I)| - \int_V \min\{\mu(I),\ |\beta(y)(I)|\}$$

$$= \int_V \max\{\mu(I),\ |\beta(y)(I)|\} - \mu(V)$$

$$\leqq \int_U \max\{\mu(I),\ |\beta(y)(I)|\} - \mu(U) < c/2,$$

it follows that

$$\int_V |\beta(y)(I)| < c/2 + \int_V \min\{\mu(I),\ |\beta(b)(I)|] \leqq$$

$$\leqq c/2 + \lambda(V) < c/2 + c/2 = c.$$

Therefore (iii) of the hypothesis of Theorem 4.1 is satisfied.

Therefore ξ is in $\mathfrak{p}_{AB}$, $\int_U \alpha(I)\xi(I)$ exists, and, since for each y in S,

$$\int_U G(\alpha\beta(y))(I) = \int_U \alpha(I)\beta(y)(I) = \int_U L(\alpha\beta(y)(I),$$

it follows that

$$\int_U \alpha(I)\beta(y)(I) \to \int_U \alpha(I)\xi(I),$$

for $\leqq^*$.

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