ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 9, Number 2, Spring 1979

ON THE FOURIER COEFFICIENTS AND CONTINUITY OF FUNCTIONS OF CLASS $\mathscr{V}_\Phi$*

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ABSTRACT. Let f be a periodic function with a fractional intergral $f_{(r)}$ or fractional derivative $f^{(r)}$ of class $\mathscr{V}_\Phi$*. For a class of Young's functions, this paper presents necessary and sufficient conditions for continuity of $f_{(r)}$ or $f^{(r)}$ in terms of the orders of magnitude of the partial sums of the absolute values of the Fourier coefficients of f. Sufficient conditions are presented for another class of Young's functions. Also, results on the order of magnitude of the Fourier coefficients of f are derived.

1. **Introduction.** For real functions f of period 2π with r-th fractional derivative, $r \geqq 0$, of bounded p-variation, Golubov [4] has obtained conditions for continuity of the r-th derivative in terms of the moduli of the Fourier coefficients. We consider the analogous problem for functions whose fractional derivatives are of Φ-bounded variation, and also obtain estimates on the order of magnitude of the Fourier coefficients. Below, we shall briefly define the terms used and state some elementary properties. For a short summary of the properties of these terms see Cohen [2]. More complete discussions are in Zygmund [11], Krasnosel'skii and Rutickii [5], L. C. Young [10], E. R. Love [6], and Musielak and Orlicz [7].

We define an *N-function* or Young's function to be any convex, strictly increasing function Φ such that $\lim_{u\to\infty} \Phi(u)/u = \infty$ and $\lim_{u\to 0} \Phi(u)/u = 0$. Furthermore, an N-function Φ satisfies the Δ' *condition* (or Φ is Δ') for (small; large) values if there exists $c > 0$ (and $u_0 > c$) such that (for $|x|, |y| \leqq u_0$; $|x|, |y| \geqq u_0$)

$$\Phi(xy) \leqq c\Phi(x)\Phi(y).$$

If Φ is Δ' for small values, say $|u| \leqq u_0$, then we can replace u_0 by any $w > u_0$ but the Δ' constant c increases unboundedly with w unless Φ is Δ' for all values.

For an N-function Φ, define

$$V_\Phi(f; I) = \sup_Q \sum \Phi(f(x_i) - f(x_{i-1}))$$

where the supremum is taken over all partitions Q of the interval I. We call $V_\Phi(f; I)$ the Φ-*varation of f on I*. $V_\Phi(f)$ and "the Φ-variation of f" is used when the interval is of length 2π.

Received by the editors on February 7, 1977.

The class of *functions of Φ-bounded variation* (ΦBV) is

$$\mathscr{V}_{\Phi}(I) = \{f \in L^1(I) \colon V_{\Phi}(f;\, I) < \infty\}.$$

Closely related definitions are

$$V_{\Phi}^{(\delta)}(f;\, I) = \sup_{|Q| \leqq \delta} \sum \Phi(f(x_i) - f(x_{i-1}))$$

and

$$V_{\Phi}{}^{*}(f;\, I) = \lim_{\delta \to 0^+} V_{\Phi}^{(\delta)}(f;\, I),$$

where $|Q|$ denoted the mesh of the partition Q, and

$$\mathscr{V}_{\Phi}{}^{*}(I) = \{f \colon (kf) \in \mathscr{V}_{\Phi}(I) \text{ for some real } k \neq 0\}.$$

Functions f of Φ bounded variation can have simple discontinuities only. We shall assume that they are normalized so that

$$f(x) = \tfrac{1}{2}(f(x+) + f(x-)).$$

It can be shown that a necessary and sufficient condition for $V_{\Phi}(f;\, I)$ to be finite is that $V_{\Phi}{}^{*}(f;\, I)$ is finite. Later proofs will use the two definitions interchangeably.

Suppose f is of period 2π, f has mean value zero ($\int f = 0$) and

$$f(x) \sim {\sum}' c_n e^{inx} = \sum_{n \neq 0} c_n e^{inx}.$$

For $r > 0$ set

$$(in)^{-r} = |n|^{-r} \exp(-\tfrac{1}{2} i\pi r \operatorname{sgn} n),$$

and define

$$D^{(r)}(t) = {\sum}' (in)^{-r} e^{int}.$$

If r is an interger $D^{(r)}$ is a polynomial and

$$\tfrac{1}{2}\pi \int f(t) D^{(r)}(x - t)\,dt$$

is an r-th order primitive of f. For any real $r > 0$ we define the fractional integral of order r of f:

$$f_{(r)}(x) = \int f(t) D^{(r)}(x - t)\,dt.$$

It can be shown that $f_{(r)}(x)$ exists almost everywhere, is integrable, and

$$f_{(r)}(x) \sim {\sum}' \frac{c_n e^{inx}}{(in)^r}.$$

We say that $f^{(r)}$ is the fractional derivative of f of order r if

$$f(x) = \int D^{(r)}(x - t) f^{(r)}(t)dt \qquad \text{a.e.},$$

i.e., f is a fractional integral of $f^{(r)}$ of order r.

We let

$$\mathscr{V}_\Phi^{(r)} = \{f: f^{(r)} \text{ exists and } f^{(r)} \in V_\Phi\}.$$

Define W as the set of 2π periodic functions with no discontinuities of the second kind, and such that

$$\min\{g(t-), g(t+)\} \leqq g(t) \leqq \max\{g(t-), g(t+)\}.$$

Then $V_\Phi \subseteq W$ for all N-functions Φ.

2. Suppose that $g \in \mathscr{V}_\Phi$ for any N-function Φ and that

$$g(t) \sim \sum_{k=1}^{\infty} \alpha_k \cos kt + \beta_k \sin kt \tag{2.1}$$

and

$$\rho_k(g) = (\alpha_k^2 + \beta_k^2)^{1/2}. \tag{2.2}$$

Consider the following conditions on a sequence of non-negative real numbers $\{\rho_k\}$:

$$\sum_{k=1}^{n} k^2 \rho_k^2 = o(n) \tag{I}$$

$$\sum_{k=1}^{n} k\rho_k = o(n) \tag{II}$$

$$\sum_{k=1}^{n} \rho_k = o(\log n) \tag{III}$$

$$\sum_{k=n}^{\infty} \rho_k^2 = o(l/n). \tag{IV}$$

The following result is well known [**3**].

LEMMA 2.1. *Conditions* (I)–(IV) *are related as follows* (IV) $\Rightarrow$ (I) $\Rightarrow$ (II) $\Rightarrow$ (III).

In the same paper Golubov has proven the following result which we shall require.

THEOREM 2.2. *If* $g \in W$, *and* $\{\rho_n\}$ *is as in equation* (2.2), *each of the conditions* (I)–(IV) *is sufficient for* g *to be continuous; however, there is no condition on* $\{\rho_k\}$ *which is necessary and sufficient.*

For a smaller class of functions we make this result somewhat more precise in the following theorem.

THEOREM 2.3. *Let* Φ *be an N-function and let* $g \in \mathscr{V}_\Phi{}^*$, *and equations* (2.1) *and* (2.2) *hold. We have*

(a) *if* $\lim_{u\to 0} u^2/\Phi(u) = 0$, *each of the conditions* (I)–(IV) *is necessary and sufficient for* g *to be continuous,*

(b) *if* $\lim\ \inf_{u\to 0} u^2/\Phi(u) \neq 0$, *each of the conditions* (I)–(IV) *is still sufficient, but there is no condition on* $\{\rho_k\}$ *which is necessary and sufficient.*

PROOF. (a) Since $\mathscr{V}_\Phi{}^* \subset W$ for all I, by Theorem 2.2, we have that each condition is sufficient for the continuity of $g \in \mathscr{V}_\Phi{}^*$. Now, suppose $\lim_{u\to 0} u^2/\Phi(u) = 0$, $g \in \mathscr{V}_\Phi{}^*$ and g is continuous.

$$g(t+h) - g(t-h) \sim 2 \sum_{1}^{\infty} (-\alpha_k \sin kt + \beta_k \cos kt) \sin kh.$$

Cohen [2] has shown that for $f \in \mathscr{V}_\Phi{}^*$ there exists $b \neq 0$ such that

$$\sup_{|h|\leqq\delta} \int_0^{2\pi} \Phi(bf(x\mathrm{th}) - bf(x))dx \leqq 3\delta V_\Phi^{(2\delta)}(bf). \tag{2.3}$$

Now $\lim_{u\to 0} u^2/\Phi(u) = 0$ implies $\mathscr{V}_\Phi \subseteq \mathscr{V}_2$, so using Parseval's formula and equation 2.3, for some $b \neq 0$,

$$\begin{aligned}
&4b^2 \sum_{k=1}^{\infty} \rho_k{}^2 \sin kh \\
&= \frac{1}{\pi} \int_0^{2\pi} b^2 |g(t+h) - g(t-h)|^2\, dt \\
&\leqq \frac{1}{\pi} \sup \left\{ \frac{(bg(x+t) - b(g(x-t))^2}{\Phi(bg(x+t) - bg(x-t))} : \mathrm{x} \in [0, 2\pi];\ |t| \leqq h \right\} \\
&\qquad \cdot \int_0^{2\pi} \Phi(bg(t+h) - bg(t-h))\, dt \\
&= o(1)o(h) = o(h).
\end{aligned}$$

From Wiener [9] and above,

$$\frac{1}{4}\limsup_{n\to\infty}\left(n\sum_{k=n}^{\infty}\rho_k^{\,2}\right)\leqq\limsup_{n\to\infty}\left(n\sum_{k=1}^{\infty}\rho_k^{\,2}\sin\frac{k\pi}{n}\right)=o(1)$$

so condition (IV) is satisfied, and by Lemma 2.1, conditions (I)–(III) are also satisfied.

(b) Consider the functions

$$f_1(t)=\sum_{k=1}^{\infty}k^{-1}\sin kt=\begin{cases}\dfrac{\pi-t}{2} & 0<t<2\pi\\ 0 & t=0\end{cases}$$

with discontinuity at $t=0$, and

$$f_2(t)=\sum_{k=1}^{\infty}k^{-1}\sin k(t+\log k).$$

Both series converge for all t, $f_1(0)=0$, $f_1(0+)=\frac{\pi}{2}$, $f_1(0-)=-\frac{\pi}{2}$, and $f\in\mathscr{V}_1$, while $f_2\in$ Lip 1/2[11, v. I, p. 197] and hence is continuous. We also have $\rho_k(f_1)=\rho_k(f_2)$ for all k.

Now if $\liminf_{u\to 0}u^2/\Phi(u)\neq 0$, then there exists $u_0>0$ and $A>0$ such that for $0\leqq u\leqq u_0$, $\Phi(u)\leqq Au^2$.

Thus, $\mathscr{V}_2^*\subseteq\mathscr{V}_{\Phi}^*$ and since Lip $1/2\subseteq\mathscr{V}_2$,

$$f_2\in\mathscr{V}_{\Phi}^*.$$

Since $\mathscr{V}_1\subseteq\mathscr{V}_{\Phi}$, we have $f_1\in\mathscr{V}_{\Phi}$. Thus, no condition on $\{\rho_k\}$ can be necessary and sufficient for the continuity of a function in this case.

The above theorem admits several generalizations.

COROLLARY 2.4. *Let* $f^{(r)}\in\mathscr{V}_{\Phi}^*$, $r\geqq 0$. *Then*

(a) *if* $\lim_{u\to 0}u^2/\Phi(u)=0$ *each of the following conditions is equivalent to the continuity of* $f^{(r)}$ $(\rho_k=\rho_k(f)$:

(I) $$\sum_{k=1}^{n}k^{2r+2}\rho_k^{\,2}=o(n)$$

(II) $$\sum_{k=1}^{n}k^{r+1}\rho_k=o(n)$$

(III) $$\sum_{k=1}^{n}k^{r}\rho_k=o(\log n)$$

(IV) $$\sum_{k=n}^{\infty}k^{2r}\rho_k^{\,2}=o(1/n).$$

(b) *If* $\lim\inf_{u\to 0} u^2/\Phi(u) \neq 0$ *then each of the conditions* (I)–(IV) *is still sufficient, but there is no condition on* $\{\rho_k\}$ *which is necessary and sufficient.*

PROOF. Since $f(t) = \int D^{(r)}(x - t)f^{(r)}(t)dt$, if $f^{(r)} \sim \Sigma\ \alpha_k \cos kx + \beta_k \sin kx$ then

$$f \sim \sum n^{-r}\cos kx(\alpha_n \cos \tfrac{1}{2}\pi r - \beta_n \sin \tfrac{1}{2}\pi r) + n^{-r}\sin kx(\beta_n \cos \tfrac{1}{2}\pi r + \alpha_n \sin \tfrac{1}{2}\pi r)$$

and

$$\begin{aligned}\rho_k(f) &= n^{-r}[(\alpha_n \cos \tfrac{1}{2}\pi r - \beta_n \sin \tfrac{1}{2}\pi r)^2 \\ &\quad + (\beta_n \cos \tfrac{1}{2}\pi r + \alpha_n \sin \tfrac{1}{2}\pi r)^2]^{1/2} \\ &= n^{-r}[\alpha_n{}^2 + \beta_n{}^2]^{1/2} \\ &= n^{-r}\rho_k(f^{(r)}).\end{aligned}$$

So applying Theorem 2.3 to

$$\rho_k(f^{(r)}) = n^r \rho_k(f),$$

the result follows.

COROLLARY 2.5. *Suppose* $f_{(r)} \in \mathscr{V}_\Phi{}^*$, $0 < r < 1$.

(a) *If* $\lim_{u\to 0} u^2/\Phi(u) = 0$, *then each of the following is equivalent to the continuity of* $f_{(r)}$:

(I) $$\sum_{k=1}^{n} k^{2-2r}\rho_k{}^2 = o(n)$$

(II) $$\sum_{k=1}^{n} k^{1-r}\rho_k = o(n)$$

(III) $$\sum_{k=1}^{n} k^{-r}\rho_k = o(\log n)$$

(IV) $$\sum_{k=n}^{\infty} k^{-2r}\rho_k{}^2 = o(1/n).$$

(b) *if* $\lim\inf_{u\to 0} u^2/\Phi(u) \neq 0$, *then each of the conditions* (I)–(IV) *is still sufficient, but there is no condition on* $\{\rho_k\}$ *which is necessary and sufficient.*

PROOF. $k^{-r}\rho_k(f) = \rho_k(f_{(r)})$.

THEOREM 2.6. *If* $f \in \mathscr{V}_\Phi^*$ *and if* Φ, Ψ *are N-functions such that* $\lim_{x\to 0} \Psi(x)/\Phi(x) = 0$ *then* f *is continuous iff, for some* $b \neq 0$, $\int_0^{2\pi}\Psi(b(f(x+h) - f(x)))dx = o(|h|)$.

PROOF. Let $f \in \mathscr{V}_\Phi^*$ be continuous, Ψ as above. By equation (2.3),

$$\begin{aligned}\int_0^{2\pi} &\Psi(b(f(t+h) - f(t)))\\ &\leqq \sup\left\{\frac{\Psi(bf(t+u) - bf(t))}{\Phi(bf(t+u) - bf(t))} : t \in [0, 2\pi], |u| \leqq h\right\}\\ &\quad\cdot \int_0^{2\pi} \Phi(b(f(t+h) - f(t)))dt\\ &= o(1)\cdot\sigma(|h|)\\ &= o(|h|).\end{aligned}$$

Conversely, suppose f has jump $d > 0$ at x_0. For h sufficiently small, then

$$|f(x+h) - f(x)| > d/2$$

in an interval of length $|h|$. Thus

$$\int_0^{2\pi} \Psi(b(f(x+h) - f(x)))dx > |h|\Psi\left(\frac{db}{2}\right) \neq o(|h|).$$

Therefore, $\int_0^{2\pi} \Psi(b(f(x+h) - f(x)))dx = o(|h|)$ implies f is continuous.

DEFINITION. Let $\omega_\Phi(1; f; \delta) = V_F^{(\delta)}(f)$ and then define

$$\begin{aligned}\omega_\Phi(k; f; \delta) &= \sup_{|h|\leqq\delta} \omega_\Phi(1; \Delta_h^{k-1}f; |h|)\\ &= \sup_{|h|\leqq\delta} V_\Phi^{(|h|)}(\Delta_h^{k-1}f),\end{aligned}$$

where

$$\begin{aligned}\Delta_h{}^k f(t) &= \sum_{v=0}^{k} (-1)^v \binom{k}{v} f(t+vh)\\ &= \Delta_h^{k-1}f(t+h) - \Delta_h^{k-1}f(t)\end{aligned}$$

and $k = 2, 3, \cdots$.

LEMMA 2.7. *Suppose Φ is Δ' and $f \in \mathscr{V}_\Phi^{(r)}$, $0 \leqq r$ and $1 \leqq k$. Then*

$$\max(|a_n|, |b_n|) \leqq \frac{1}{n^r\Phi^{-1}(n)} \cdot \frac{1}{\Phi^{-1}\left(\dfrac{1}{\omega_\Phi(k, f^{(r)}, \pi/n)}\right)} \cdot \frac{1}{2^{k-1}\Phi^{-1}(1/c_\Phi{}^3\pi)\Phi^*(\frac{1}{2}\pi)\pi}.$$

PROOF. Let $g \in \mathscr{V}_\Phi$,

$$g(x) \sim \sum \alpha_n \cos nx + \beta_n \sin nx.$$

Then

$$2^k\alpha_n = \frac{1}{\pi}\int_0^{2\pi} \{\Delta^k_{\pi/n}g(t)\} \cos nt\, dt.$$

Using Hölders inequality,

$$2^k|\alpha_n| \leqq \|\Delta^k_{\pi/n}g(t)\|'_\Phi \left\| \frac{1}{\pi}\cos nt \right\|_{\Phi^*}.$$

Now

$$\|\Delta^k_{\pi/n}g(t)\|_\Phi' = \|\Delta^{k-1}_{\pi/n}\, g(t + \pi/n) - \Delta^{k-1}_{\pi/n}\, g(t)\|'_\Phi$$

$$\leqq \inf\left\{ j : c_\Phi\Phi(1/j) \int_0^{2\pi} \Phi(\Delta^{k-1}_{\pi/n}\, g(t + \pi/n) - \Delta^{k-1}_{\pi/n}\, g(t))dt \leqq 1\right\}$$

$$= \inf\left\{ j : c_\Phi\Phi(1/j)\frac{1}{2n}\int_0^{2\pi}\sum_{k=1}^{2n}\Phi\left(\Delta^{k-1}_{\pi/n}\, g\left(t + \frac{j\pi}{n}\right) - \Delta^{k-1}_{\pi/n}\, g\left(t + \frac{(j-1)\pi}{n}\right)\right) dt \leqq 1\right\}$$

$$\leqq \inf\left\{ j : c_\Phi\Phi(1/j)\frac{\pi}{n}\ \omega_\Phi(k; g; \pi/n) \leqq 1\right\}$$

$$\leqq \inf\left\{ j : \Phi(1/j) \leqq \frac{n}{c_\Phi\pi\omega_\Phi(k; g; \pi/n)}\right\}.$$

Thus

$$\|\Delta^k_{\pi/n} g\|'_\Phi \leqq \frac{1}{\Phi^{-1}\left(\dfrac{n}{c_\Phi \pi \omega_\Phi(k;\, g;\, \pi/n)}\right)}$$

$$\leqq \frac{1}{\Phi^{-1}(n)\Phi^{-1}(1/c_\Phi{}^3\pi)\Phi^{-1}\left(\dfrac{1}{\omega_\Phi(k;\, g;\, \pi/n)}\right)}$$

and

$$2^k|\alpha_n| \leqq \frac{1}{\Phi^{-1}(1/c_\Phi{}^3\pi)\pi\Phi^*\left(\dfrac{1}{2\pi}\right)} \cdot \frac{1}{\Phi^{-1}(n)\Phi^{-1}\left(\dfrac{1}{\omega_\Phi(k;\, g;\, \pi/n)}\right)}.$$

The inequality for $|\beta_n|$ is proved similarly.

Thus

$$\max(|\alpha_n|), |\beta_n|) \leqq \frac{2^{-k}}{\Phi^{-1}(n)\Phi^{-1}\left(\dfrac{1}{\omega_\Phi(k;\, g;\, \pi/n)}\right)} \cdot \frac{1}{\Phi^*(\frac{1}{2}\pi)\Phi^{-1}(1/(c_\Phi{}^3\pi))\pi}.$$

Now, let $g = f^{(r)}$, $r \geqq 0$. Then

$$k^{-r}\rho_k(f^{(r)}) = \rho_k(f),$$

and if

$$f(x) \sim \sum a_n \cos nx + b_n \sin nx,$$

then

$$\max(|a_n|, |b_n|) \leqq \frac{1}{2^{k-1}\Phi^{-1}(n)n^r\Phi^{-1}\left(\dfrac{1}{\omega_\Phi(k;\, f^{(r)};\, \pi/n}\right)} \cdot \frac{1}{\pi\Phi^{-1}(1/c_\Phi{}^3\pi)\Phi^*(\frac{1}{2}\pi)}$$

as desired.

The following class of functions was introduced by E. R. Love [5].

Definition. We say that a function g is *Φ-absolutely continuous* (ΦAC) if given $\epsilon > 0$, there exists a $\delta_0 > 0$ such that

$$\sum \Phi(g(\beta_i) - g(\alpha_i)) < \epsilon$$

holds for any nonoverlapping intervals (α_i, β_i) lying in the period and such that

$$\sum \Phi(\beta_i - \alpha_i) < \delta_0.$$

THEOREM 2.8. *If $f^{(r)} \in \Phi AC$, Φ a Δ' N-function, then*

$$\rho_n(f) = \mathrm{o}\left(\frac{1}{n^r \Phi^{-1}(n)} \right).$$

PROOF. Select $\epsilon > 0$. Then choose an integer p such that $\Sigma\ \Phi(\pi/p) = 2p\Phi(\pi/p) < 2pc\Phi(1/p) < c\delta/\pi$, where $c\delta/\pi$ is the δ_0 in the last definition, and so

$$\sum_{j=1}^{2p} \Phi \left(f^{(r)} \left(\frac{j\pi}{p} \right) - f^{(r)} \left(\frac{(j-1)\pi}{p} \right) \right) < \epsilon.$$

Let $\{x_i\}$ be any partition of a period with mesh less than $1/2p$ and group the elements $\Delta x_i = x_i - x_{i-1}$ so that

$$\frac{1}{p} > \sum_{j_1}^{j_2} \Delta x_i, \geqq \frac{1}{2p}.$$

We have

$$\sum \Phi(\Delta x_i) < \sum_k \Phi \left(\sum_{j=j_k}^{j_{k+1}} \Delta x_i, \right) < 2p\pi\Phi(1/p) < \delta < \delta_0,$$

and therefore

$$\sum \Phi(f^{(r)}(x_i) - f^{(r)}(x_{i-1})) < \epsilon.$$

Thus $V^{(1/(2p))}(f^{(r)}) < \epsilon$, hence

$$V^{(\pi/n)}(f^{(r)}) = \mathrm{o}(1) \text{ as } n \to \infty.$$

Using Lemma 2.7, with $k - 1$,

$$\rho_n(f) = \frac{1}{n^r\Phi^{-1}(n)}\ \mathrm{o}(1) = \mathrm{o}\left(\frac{1}{n^r\Phi^{-1}(n)} \right).$$

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