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A COMMON FIXED POINT STRUCTURE

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ABSTRACT. Let X be a set, $\mathscr{P}$ a collection of subsets of X, $\mathscr{F}$ a family of multifunctions o X into itself, and $\mathscr{H}$ a family of single-valued functions of X onto itself. The quadruple (X, $\mathscr{P}$, $\mathscr{F}$, $\mathscr{H}$ is called a common fixed point structure if there are a set of axioms which insure that for each F in $\mathscr{F}$ and h in $\mathscr{H}$ there is an x in X such that $h(x) = x \in F(x)$. A common fixed point structure of semi-trees is developed which overlaps the fixed point structures of Muenzenberger and Smithson and subsumes fixed point theorems of Wallace, Ward, Young, and Mohler.

1. **Introduction.** A *continuum* is a compact connected Hausdorff space. A continuum X is *hereditarily unicoherent* if any two subcontinua of X meet in a continuum. An *arboroid* is an arcwise connected and hereditarily unicoherent continuum. A metric arboroid is called a *dendroid.* If X is locally connected and hereditarily unicoherent then X is called a *tree*. A *multifunction* $F : X \longrightarrow X$ is a point to set correspondence with $F(x) \neq \Phi$ for all x in X. The multifunction $F : X \longrightarrow X$ is said to be *upper semicontinuous* if for each closed set $C \subset X$ the set $F^{-1}(C) = \{x \in X \mid F(x) \cap C \neq \Phi\}$ is closed in X. The single-valued function $f : X \longrightarrow Y$ is *monotone* if $f^{-1}(x)$ is connected for every x in Y.

In [1] Borsuk showed that a dendroid has the fixed point property for continuous single-valued mappings. Then Wallace [6] proved that trees have the fixed point property for upper semicontinuous multifunctions which send points to continua. Also, as a corollary to the above, Wallace showed that if f and g are mappings of a continuum onto a tree with f continuous and g monotone, then f and g have a coincidence point. Ward [7] proved that Wallace's theorem remains true if "trees" are replaced by "dendroids".

Using Muenzenberger and Smithson's development of fixed point structures [4] as motivation, the author develops a common fixed point structure which subsumes the above results and other results of Smithson, Young, and Mohler.

Let X be a set, $\mathscr{P}$ a collection of subsets of X, $\mathscr{F}$ a family of multifunctions of X into itself, and $\mathscr{H}$ a family of single-valued functions of

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X onto itself. The quadruple $(X, \mathscr{P}, \mathscr{F}, \mathscr{H})$ is called a common fixed point structure if there are a set of axioms which insure that each member of $\mathscr{F}$ has a common fixed point with each member of $\mathscr{H}$. In other words, for each F in $\mathscr{F}$ and h in $\mathscr{H}$ there is an x in X such that $h(x) = x \in F(x)$. In § 2 the axioms on $(X, \mathscr{P}, \mathscr{F}, \mathscr{H})$ will be given and in § 3 the main theorem will be proved.

2. **Axioms on $(X, \mathscr{P}, \mathscr{F}, \mathscr{H})$.** Let $(X, \mathscr{P})$ be a pair where X is a set and $\mathscr{P}$ a collection of subsets of X.

Axiom 2. *If $\Phi \neq \mathscr{P}_0 \subset \mathscr{P}$, then $\cap \mathscr{P}_0 = \Phi$ or $\cap \mathscr{P}_0 \in \mathscr{P}$.*

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Define $[x, y] = \cap \{P \in \mathscr{P} : x, y \in P\}$. It follows that $[x, y]$ is the unique minimal element of $\mathscr{P}$ that contains x and y. The set $[x, y]$ is called the *chain in X with endpoints x and y.* The sets $(x, y] = [x, y] \setminus \{x\}$ and $(x, y) = [x, y] \setminus \{x, y\}$.

Axiom 3. *For all $P \in \mathscr{P}$ there exists a unique pair $x, y \in X$ such that $P = [x, y]$.*

Axiom 4. *If $x, y, z \in X$, then $[x, z] \subset [x, y] \cup [y, z]$.*

Axiom 5. *If $\mathscr{P}_0 \subset \mathscr{P}$ is nested, then there exists a $P \in \mathscr{P}$ such that $\cup \mathscr{P}_0 \subset P$.*

Axiom 6. *If $x \neq y$ then $[x, y]$ contains at least three points.*

The pair $(X, \mathscr{P})$ is called a *semitree.* These six axioms of a semitree are essentially the ones given in [4], where Axioms 4 and 6 were replaced by equivalent conditions. Examples have also been found to show that these six axioms are independent [3].

Definition. A subset $A \subset X$ is *chainable* if and only if for all x and y in A the chain $[x, y] \subset A$.

Lemma 2.1. *The union of two chainable sets with nonempty intersection is chainable.*

Proof. Let A and B be two chainable sets. If x and y are both in A or both in B, then $[x, y]$ is in A or in B, so in $A \cup B$. So suppose $x \in A$, $y \in B$ and $x, y \notin A \cap B$. Let $u \in A \cap B$. Then $[x, u] \subset A$ and $[u, y] \subset B$. By Axiom 4, $[x, y] \subset [x, u] \cup [u, y] \subset A \cup B$. Thus $A \cup B$ is chainable.

The following are easy to prove and will be omitted:

(a) If $z \in (x, y)$, then $[x, y] \setminus \{z\}$ is not chainable.
(b) If $x \in X$, then $[x, x] = \{x\}$.
(c) If $z \in [x, y]$, then $[x, z] \subset [x, y]$.
(d) for all $x, y \in X$, $[x, y]$ is chainable.

DEFINITION. Let e be in X. If x and y are in X then $x \leqq y$ means x is in $[e, y]$. The ordering $\leqq$ is called the *chain order* on X with least element e.

Note that $\leqq$ is a partial order.

LEMMA 2.2. *For all $x, y \in X$ with $x < y$, there exists a z in X such that $x < z < y$.*

PROOF. Let $M(x) = \{z \in X \mid x \leqq z\}$ and $x < y$. Now $[e, y] = [e, x] \cup [x, y]$ and $[x, y] \subset M(x)$. So $[x, y] \subset [e, y] \cap M(x)$. We are to show $[x, y] = [e, y] \cap M(x)$. Let $z \in [e, y] \cap M(x)$. Then $x \leqq z \leqq y$. If $z = x$ then $z \in [x, y]$, so suppose that $x < z$. Since $[e, y] = [e, x] \cup [x, y]$ we have $z \in [x, y]$ and so $[e, y] \cap M(x) \subset [x, y]$. Hence $[x, y] = \{z \mid x \leqq z \leqq y\}$ and by Axiom 6, $[x, y]$ contains at least three points. Thus there is a z with $x < z < y$.

The next two lemmas were proved in [4].

LEMMA 2.3. *If $x \leqq y \leqq z$, then $[x, y] \cup [y, z] = [x, z]$.*

LEMMA 2.4. *Each non-empty totally ordered set A has a supremum in X and if $A \subset [x, y]$ then $\sup A \in [x, y]$.*

LEMMA 2.5. *If x and y are in X and $[e, x] \cap [x, y] = \{x\}$, then $[e, y] = [e, x] \cup [x, y]$.*

PROOF. By Axiom 4, $[e, y] \subseteq [e, x] \cup [x, y]$. Let $p = \sup\{[e, x] \cap [e, y]\}$ and $a = \inf\{[e, y] \cap [x, y]\}$. Then $[e, x] \cap [e, y] = [e, p]$ and $[e, y] \cap [x, y] = [a, y]$. Since both a and p are in $[e, y]$ and $[e, y]$ is totally ordered then $a \leqq p$ or $p < a$.

If $a \leqq p$ then $a \in [e, p]$, so $a \in [e, x]$. But since $a \in [x, y]$ we must have $a = x$. Then $x \in [e, y]$, so $[e, x] \cup [x, y] \subseteq [e, y]$.

If $p < a$ then $[p, a]$ is a chain. Suppose $p < c < a$. Since $[e, y] \subseteq [e, x] \cup [x, y]$ either $c \in [e, x] \cap [e, y]$ or $c \in [x, y] \cap [e, y]$, so $c \in [e, p]$ or $c \in [a, y]$, a contradiction. Thus $[p, a] = \{p, a\}$. But this contradicts Axiom 6. Hence $[e, y] = [e, x] \cup [x, y]$.

DEFINITION. A set $A \subset X$ is *closed* if and only if for all $y, z \in X$ with $y \leqq z$, $\inf(A \cap [y, z]) \in A$ and $\sup(A \cap [y, z]) \in A$ whenever $A \cap [y, z] \neq \phi$.

COROLLARY 2.6. *If C is non-empty, chainable, and closed, then C has a minimum element.*

PROOF. If $e \in C$, then e is the minimum element. If $e \notin C$, then let $t \in C$ and $x = \inf(C \cap [e, t])$. Then $x \in C$. Let $y \in C$. Then C chainable implies $[x,] \subseteq C$. Thus $[e, x] \cap [x, y] = \{x\}$, so by Lemma 2.5, $[e, y] = [e, x] \cup [x, y]$. Hence $x \leqq y$ for all $y \in C$.

The following are the four axioms on $\mathscr{F}$ and $\mathscr{H}$.

AXIOM 7. *If $x \leqq y$, then $F([x, y])$ is closed and chainable for all $F \in \mathscr{F}$.*

AXIOM 8. *For all $F \in \mathscr{F}$ and $x \in X$, $F^{-1}(x)$ is closed.*

DEFINITION. A bijection h from X to X is called an *order isomorphism* if $x \leqq y$ implies $h(x) \leqq h(y)$ and $h^{-1}(x) \leqq h^{-1}(y)$.

LEMMA 2.7. *If h is an order isomorphism and $x \leqq y$ then $h[x, y] = [h(x), h(y)]$ and $h^{-1}[x, y] = [h^{-1}(x), h^{-1}(y)]$.*

PROOF. If $z \in [x, y]$ then $x \leqq z \leqq y$. So $h(x) \leqq h(z) \leqq h(y)$. Thus $h(z) \in [h(x), h(y)]$, so $h[x, y] \subset [h(x), h(y)]$. Since $h^{-1}[h(x), h(y)] \subseteq [x, y]$ we have $[h(x), h(y)] \subseteq h[x, y]$. Thus $h[x, y] = [h(x), h(y)]$. The other equality is similar.

AXIOM 9. *Every $h \in \mathscr{H}$ is an order isomorphism.*

AXIOM 10. *For all $F \in \mathscr{F}$ and $h \in \mathscr{H}$, $Fh = hF$, that is $Fh(x) = hF(x)$ for all $x \in X$.*

Thus $(X, \mathscr{P}, \mathscr{F}, \mathscr{H})$ is a quadruple, where X is a set, $\mathscr{P}$ a non-empty collection of subsets of X, $\mathscr{F}$ a non-empty family of multifunctions on X into X, and $\mathscr{H}$ is a non-empty family of single-valued functions of X onto X. Axioms 1–10 are assumed to hold on the quadruple.

3. **The Common Fixed Point Theorem.** Before proving the main theorems we first prove some lemmas. The sets X, $\mathscr{P}$, $\mathscr{F}$, and $\mathscr{H}$ have the meanings assigned in § 2 and all ten axioms are assumed to hold.

LEMMA 3.1. *The fixed point set of h is closed.*

PROOF. Let A be the fixed point set of h. Let $y \leqq z$ and $A \cap [y, z] \neq \Phi$. If $x = \sup(A \cap [y, z])$, then since h is an order isomorphism, $h(x) = h \sup(A \cap [y, z]) = \sup h(A \cap [y, z]) = \sup(A \cap [y, z]) = x$. Similarly, if $p = \inf(A \cap [y, z])$ then $h(p) = p$. Thus A is closed.

LEMMA 3.2. *If $e \leqq p \leqq t$, $e \neq t$, and $p \notin F(p)$, then there is a w in $[e, p]$ and r in $[p, t]$ such that $[w, r] \cap F([w, r]) = \Phi$. If $e \neq p$ then w can be chosen in (e, p) and if $p \neq t$ then r can be chosen in (p, t).*

PROOF. First assume $e < p \leqq t$. Since $p \notin F(p)$ then $p \notin F^{-1}(p)$ and $F^{-1}(p)$ is closed. Thus there are points w' and r' with $w' < p \leqq r'$ such that $[w', r'] \cap F^{-1}(p) = \Phi$. Thus $p \notin F[w', r']$ and $F[w', r']$ is a closed set. So there are points w'' and r'' with $w'' < p \leqq r''$ such that $[w'', r''] \cap F[w', r'] = \Phi$. Let $w = \inf([w'', r''] \cap [w', r'])$ and $r = \sup([w'', r''] \cap [w', r'])$. Hence $[w, r] \cap F([w, r]) = \Phi$.

Finally let $e = p < t$. Since $p \notin F^{-1}(p)$ and $F^{-1}(p)$ is closed there is a point r' with $p < r'$ such that $[p, r'] \cap F^{-1}(p) = \Phi$. Thus $p \notin F[p, r']$, and $F[p, r']$ is a closed set. So there is an r'' with $e = p < r''$ such that $[p, r''] \cap F[p, r'] = \Phi$. Let $r = \sup([p, r''] \cap [p, r'])$. Then $[e, r] \cap F([e, r]) = \Phi$.

The following is a special case of the main theorem of [4].

LEMMA 3.3. *Each $h \in \mathscr{H}$ has a fixed point.*

Recall that $M(x) = \{y \in X \mid x \leqq y\}$ and $\leqq$ is the chain order on X with least element e.

LEMMA 3.4. *Suppose h and F do not have a common fixed point, and let $k \in X$ be a fixed point of h with $F(k) \subset M(k)$. If C is a totally ordered subset of $A = \{x \in M(k) \mid h(x) = x$ and $F(x) \subset M(x)\}$ then* $\sup C$ *is in C.*

PROOF. The set A is not empty since $h(k) = k$ and $F(k) \subset M(k)$. Let C be a totally ordered subset of A and $b = \sup C$. We wish to prove that b is in C. Suppose this is not the case. By Lemma 3.1 the fixed point set of h is closed, so $h(b) = b$. We must then have $F(b) \not\subset M(b)$.

Since $e < b$ then by Lemma 3.2 there is a w in (e, b) for which $[w, b] \cap F([w, b]) = \Phi$. Suppose now that $[e, b] \cap F([w, b]) \neq \Phi$, and let u be the least element of this set. So $u < w$. Since $b = \sup C$ there is an s in $C \cap [w, b]$. Since s is in A the set $F(s) \subset M(s)$ and so there is a $t \in F(s) \subset F([w, b])$. Then s is in $[e, t]$. But e and t are in $[e, u] \cup F([w, b])$, which is a chainable set. Hence $[e, u] \cup F([w, b])$ contains $[e, t]$ but not s, which is a contradiction. Thus

$$[e, b] \cap F([w, b]) = \Phi.$$

Let $q = \min F([w, b])$. We want to show that $[e, b] \cap [b, q] = \{b\}$. Let $r = \min\{[e, b] \cap [b, q]\}$. If $r < b$ then since $b = \sup C$ there is a point s in $(r, b) \cap (w, b) \cap C$. Then there is a t in $F(s) \subset M(s)$. Thus e and t are in $[e, r] \cup [r, q] \cup F([w, b])$, and the last set is a chainable set containing $[e, t]$ but not s. This is a contradiction. Thus $[e, b] \cap [b, q] = \{b\}$, so $[e, q] = [e, b] \cup [b, q]$. Thus $b \leqq q$, and by choice of q we have $F(b) \subset M(b)$. This contradiction shows that b must be in C.

COROLLARY 3.5. *Let $k \in X$ and $F(k) \subset M(k)$. If C is a totally ordered subset of $A = \{x \in M(k) \mid F(x) \subset M(x)\}$ then* $\sup C$ *is in C.*

The main common fixed point theorem generalizes the main theorem of [5].

THEOREM 3.6. *The quadruple $(X, \mathscr{P}, \mathscr{F}, \mathscr{H})$ is a common fixed point structure.*

PROOF. Let $F \in \mathscr{F}$ and $h \in \mathscr{H}$ and suppose F and h have no common fixed point. Choose e in X to be a fixed point of h, and let $\leqq$ be the chain order with least element e. Let

$$A = \{x \in X \mid h(x) = x \text{ and } F(x) \subset M(x)\}.$$

The set A is not empty since $h(e) = e$ and e is the least element of X. Let C be a maximal totally ordered set in A and $b = \sup C$. By Lemma 3.4 we have $b \in C$, and so $h(b) = b$ and $F(b) \subset M(b)$.

Let $t = \min F(b)$. Since h is an order isomorphism then

$$h(t) = h(\min F(b)) = \min hF(b)$$
$$= \min Fh(b) = \min F(b) = t.$$

Now let

$$D = \{x \in [b, t] \mid F(x) \subset M(x)\}.$$

We have that b is in D, and t is not in D since $b = \sup C$ and $h(t) = t$. Let $q = \sup D$. It will be shown that $b < q < t$.

If $e < b$ then by Lemma 3.2 there is a $w \in (e, b)$ and r in (b, t) such that $[w, r] \cap F([w, r]) = \Phi$. In the event that $b = e$, let $w = e$ in what follows. If there were an x in $F([w, r]) \cap [e, r]$, then $x < w$ since $[w, r] \cap F([w, r]) = \Phi$. Since we have x and t in $F([w, r])$, then $[x, t] \subset F[w, r]$, a contradiction of $[w, r] \cap F([w, r]) = \Phi$. Therefore $F[w, r]$, $([w, r])$ cannot meet $[e, r]$, and since t is in $F([w, r])$ we have $\min F([w, r])$ is in $[r, t]$. But this implies $F(r) \subset M(r)$, so r is in D, and $b < r \leqq q$.

By Corollary 3.5 we have q is in D. Since b is maximal in A and $h(t) = t$ we must have $q < t$. Thus $b < q < t$.

Since q is in D and $b < q < t$ then $h(q) \neq q$. Because $h(b) = b$, the point $h(t) = t$, and h is an order isomorphism, we have $h([b, t]) = [b, t]$.

Now $[b, t]$ is totally ordered by $\leqq$, the point q is in $[b, t]$, and $h(q) \neq q$. Thus $h(q)$ and $h^{-1}(q)$ are in $[b, t]$ since $h[b, t] = [b, t]$. Suppose $q < h(q)$. Since $F(q) \subset M(q)$ then $hF(q) \subset hM(q)$ or $F(h(q)) \subset M(h(q))$. That is, $h(q)$ is in D, and the maximality of q is contradicted. If $h(q) < q$, then $q < h^{-1}(q)$ and so $F(h^{-1}(q)) \subset M(h^{-1}(q))$, which again is a contradiction.

So $h(q) = q$. But this contradicts the maximality of b. We conclude that F and h must have a common fixed point.

Let X be an arboroid and $\mathscr{P} = \{A \subset X \mid A \text{ is an arc}\}$. Then $(X, \mathscr{P})$ satisfies Axioms 1–6. If $F: X \rightarrow X$ is an upper semicontinuous multifunction sending points to continua then F sends continua to continua. Also Ward [8] shows that if $h: X \rightarrow X$ is a homeomorphism which leaves e fixed, then h is an order isomorphism under $\leqq$. We then obtain this generalization of a result of Ward [7].

Corollary 3.7. *Let X be an arboroid and $F: X \rightarrow X$ be an upper semicontinuous multifunction which sends points to continua. If h is a self-homeomorphism of X which commutes with F, then F and h have a common fixed point.*

We also obtain the following result.

Corollary 3.8. *If f, g, and h are continuous single-valued functions of an arboroid X into itself, where g is a monotone surjection and h is a homeomorphism which commutes with f and g, then there is a fixed point of h which is also a coincidence point of f and g.*

Proof. Apply Corollary 3.7 to the homeomorphism h and the upper semicontinuous multifunction $F = g^{-1}f$.

The following generalizes a result of Muenzenberger and Smithson [4]:

Corollary 3.9. *Let X be an arboroid and $F: X \rightarrow X$ be a multifunction which sends continua to continua and such that $F^{-1}(x)$ is closed for every x in X. If h is a self-homeomorphism of X which commutes with F, then F and h have a common fixed point.*

The next result generalizes a theorem of Young [9] and Mohler [2]:

Corollary 3.10. *Let X be an arcwise connected space in which every nest of arcs is contained in an arc and $F: X \rightarrow X$ be a multifunction that maps arcs onto arcwise connected closed sets and such that $F^{-1}(x)$ is closed. If h is a self-homeomorphism of X which commutes with F, then F and h have a common fixed point.*

A subset X of a real vector space V is a *closed star* at $e \in X$ in case each line through e intersects X in a closed line segment. Define $[x, y]$ in the following manner: if x and y are on a line through e, then $[x, y]$ is the closed line segment from x to y; otherwise $[x, y] = [e, x] \cup [e, y]$. Let $\mathscr{P} = \{[x, y] \mid x, y \in X\}$. It was shown in [4] that $(X, \mathscr{P})$ satisfies Axioms 1–6. The following generalizes a result of Muenzenberger and Smithson in [4].

COROLLARY 3.11. *Let X be a closed star at e, and $F: X \to X$ be a multifunction such that $F([x, y])$ is closed and chainable and $F^{-1}(x)$ closed for all x in X. If h is an order isomorphism which commutes with F, then F and h have a common fixed point.*

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