ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 8, Number 3, Summer 1978

A NOTE ON LOCAL FIELD DUALITY

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1. **Introduction.** The additive group of a local field is self-dual. Elementary and explicit constructions of character groups can be obtained for the p-adic and p-series fields as well as for the real numbers. See [4], [8], or [6] chapter 3; there is a brief history in the notes to section 25 of [4]. These constructions together with structure theorems yield proofs of self-duality for all local fields, as noted in [8] and [6]. To the author's knowledge the first explicit mention of self-duality for all local fields is in Tate's thesis [7]. Tate's proof is based on the general Pontryagin duality theorem. This proof is used by Lang in [5] and (with more details) by Goldstein in [3]. Weil [9] gives a different proof, based on a dimension argument but again using Pontryagin duality. The purpose of this note is to add perspective to these proofs by giving a simple proof of the Pontryagin duality theorem for zero-dimensional local fields. The proof is given for any zero-dimensional locally compact Abelian group.

Before beginning we note that use of the term "local field" is not standard and that we follow Weil in meaning a nondiscrete commutative locally compact topological field. A local field is either a finite extension of the real number field (in fact the real numbers or the complex numbers) or it is zero-dimensional (see [9], section 3, theorem 5; the proof does not use duality).

2. **Pontryagin duality for zero-dimensional groups.**

THEOREM 1. *Let G be a locally compact Abelian topological group which is zero-dimensional and has Hausdorff separation. Let $X(G)$ denote the character group of G and $X^2(G)$ the character group of $X(G)$, both $X(G)$ and $X^2(G)$ having the compact-open topologies inherited from continuous functions. The map τ from G to $X^2(G)$ defined by*

$$\tau(x)(\chi) = \chi(x)$$

is a topological isomorphism onto $X^2(G)$.

PROOF. We will write G multiplicatively. It is easy to verify that $\tau(x)$ is in $X^2(G)$ for each $x \in G$ and that τ is a homomorphism. The continuity of τ at the identity e of G can be phrased: "for every neigh-

Received by the editors on October 7, 1975, and in revised form on July 16, 1976.
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borhood U of 1 in the circle group $\mathbf{T}$ and each compact set K in $X(G)$ the set

$$\{x \in G : \chi(x) \in U \text{ for all } \chi \in K\} \left[= \bigcap_{\chi \in K} \chi^{-1}(U) \right]$$

is a neighborhood of e''. So phrased, the continuity of τ is apparent from the general Ascoli-Arzela theorem; it also follows easily from the continuity of the map $(x, \chi) \to \chi(x)$ from $G \times X(G)$ to $\mathbf{T}$.

We have not used the fact that G is zero-dimensional; we use it now to show that τ is one-to-one. Let $x \in G$, $x \neq e$. Since G is zero-dimensional, it contains a compact open subgroup H such that $x \notin H$. The group G/H is discrete. We define a homomorphism σ on the cyclic subgroup of G/H generated by xH to $\mathbf{T}$ satisfying $\sigma(xH) \neq 1$. If xH has finite order, let $\sigma(xH)$ be a nontrivial root of 1 of that order, and if xH does not have finite order let $\sigma(xH)$ be any element of $\mathbf{T}\backslash 1$. Since $\mathbf{T}$ is divisible, σ has an extension to all of G/H ([2] section 21; [4] appendix A). If the extension is also denoted σ and ϕ is the canonical map from G to G/H, then $\chi = \sigma \circ \phi$ is a character of G and $\chi(x) \neq 1$. Thus $\tau(x) \neq \tau(e)$ and so τ is one-to-one.

We prove next that τ is onto. First suppose that G is discrete. We need the fact that $X(G)$ is compact. To prove it note that a subset of G is compact if and only if it is finite. Hence the topology of $X(G)$ is the relative topology of $\mathbf{T}^G$; and, it is immediate that $X(G)$ is closed in $\mathbf{T}^G$. Thus $X(G)$ is compact. Clearly $\tau(G)$ separates points in $X(G)$ and is conjugate closed, so the span of $\tau(G)$ is dense in $C(X(G))$ by the Stone-Weierstrass theorem. By a standard argument (not depending on zero-dimensionality) it follows that $\tau(G) = X(G)$ (e.g., see the last part of the proof of (23.20) in [4]).

The proof so far uses well known methods. The simplification obtained by zero-dimensionality now becomes important, in showing that τ is onto. Suppose then that G is zero-dimensional but not discrete. If H is a subgroup of G, we let

$$A(H) = \{\chi \in X(G) : \chi(h) = 1 \text{ if } h \in H\},$$

the *annihilator* of H. We need the following facts about annihilators and quotients; each makes a routine exercise. The results appear in [4], § 23.

(i) $A(H)$ is a closed subgroup of $X(G)$.

(ii) $A(H)$ is open if H is compact.

(iii) $A(H)$ is compact if H is open.

(iv) G/H is discrete if H is open.

(v) $\rho(\chi)(xH) = \chi(x)$ defines a topological isomorphism ρ of $A(H)$ onto $X(G/H)$ if H is closed.

Let $\psi \in X^2(G)$. We need an x for which $\psi = \tau(x)$. For each compact open subgroup H of G let ψ_H denote ψ restricted to $A(H)$. The subgroup $A(H)$ of $X(G)$ is compact and open. By (v), for the element ψ_H of $X(A(H))$ there is a unique $\psi_H \in X^2(G/H)$ such that $\psi_H(\chi) = \psi_H(\rho(\chi))$ for all $\chi \in A(H)$. Since G/H is discrete there is in turn a unique coset $x_H H$ such that $\psi_H(\sigma) = \sigma(x_H H)$ for all σ in $X(G/H)$. Thus $x_H H$ is the unique coset of H satisfying

$$\psi_H(\chi) = \chi(x_H) \text{ for all } \chi \in A(H).$$

The family $\mathcal{H}$ of compact open subgroups of G is a directed set under reverse inclusion. We show that the net $\{x_H : H \in \mathcal{H}\}$ is Cauchy in the natural uniform structure on G. Fix H. If $L \subset H$, then every χ in $A(H)$ is also in $A(L)$ and so

$$\psi_H(\chi) = \psi_L(\chi) = \chi(x_L)$$

holds for all $\chi \in A(H)$. By the uniqueness of $x_H H$, we have $x_L H = x_H H$. Thus we have $x_L x_H{}^{-1} \in H$ if $L \subset H$, and therefore

$$x_L x_M{}^{-1} \in H \text{ if } L, M \subset H \; (L, M \in \mathcal{H}).$$

Thus $\{x_H : H \in \mathcal{H}\}$ is Cauchy and so converges, say to x (G is complete). For any H there is an $L \subset H$ such that $x x_L{}^{-1} \in H$, and thus

$$x x_H{}^{-1} = (x x_L{}^{-1})(x_L x_H{}^{-1}) \in H.$$

Thus the equality $xH = x_H H$ holds for each H and we have $\psi_H(\chi) = \chi(x)$ for all $\chi \in A(H)$. Since every χ in $X(G)$ is in some $A(H)$, $\psi(\chi) = \chi(x)$ holds for all χ in $X(G)$. Thus the proof that τ is onto is complete.

It remains to prove that τ^{-1} is continuous. For a compact open subgroup H and $x \notin H$, the proof that τ is one-to-one can be adapted to show that there is a $\chi \in A(H)$ such that $\chi(x) \neq 1$. From this and the fact that τ is onto it follows that $\tau(H) = A(A(H))$ for each compact open subgroup H of G. But $A(H)$ is compact, so $A(A(H))$ is open. Thus $\tau(H)$ is open for each compact open subgroup H of G. It follows that τ is an open mapping, and the proof is complete.

Corollary. *With notation as in Theorem* 1, *if Y is a closed subgroup of $X(G)$ that separates points in G, then $Y = X(G)$.*

Proof. Let $\theta \in A(Y)$ (in $X^2(G)$). By Theorem 1 there is an x such that $\theta(\chi) = \chi(x)$ for all $\chi \in X(G)$. If $x \neq e$, then there would be a

$\chi \in Y$ such that $\chi(x) \neq 1$. Thus $x = e$; hence $\theta = 1$, and $A(Y) = \{1\}$. Since Y is closed, $Y = A(A(Y)) = A(\{1\}) = X(G)$.

COROLLARY (TATE). *The additive group of a zero-dimensional local field is self-dual.*

OUTLINE OF PROOF. Let χ be any nontrivial additive character of a local field K. Let $\chi_a(x) = \chi(ax)$, $a \in K$, and let $\rho(a) = \chi_a$. It is elementary that ρ is continuous, one-to-one, and onto. To see that ρ is also open (ρ^{-1} continuous) let H be any compact open subgroup of K. Since a local field is σ-compact ([9], p. 4), K/H is countable. Thus there are $a_n \in K$ such that $\rho(K) = \bigcup_{n=1}^{\infty} \rho(a_n)\rho(H)$. The group $\rho(H)$ is compact, so by an application of the Baire category theorem some $\rho(a_n)\rho(H)$ has nonvoid interior. It follows that $\rho(H)$ is open and hence that ρ is an open mapping. Thus ρ embeds the additive group of K in $X(K)$, $\rho(K)$ is closed, and $\rho(K)$ separates points. Thus the first corollary applies, and $\rho(K) = X(K)$.

The continuity of ρ^{-1} in the above outline also follows from (5.29) of [4], which states that a continuous homomorphism is open if the domain is locally compact and σ-compact and the range is locally compact and Hausdorff. The result is a modification of one appearing in Pontryagin's "Topological Groups"; see [4], p. 51 for a discussion. The openness of continuous homomorphisms is a major minor topic in its own right.

REMARKS. (i) The fact that τ is one-to-one is equivalent to the semisimplicity of the Banach algebra $L_1(G)$. In the general case both results are difficult. As seen above, for discrete groups the result is a consequence of the fact that homomorphisms from subgroups to divisible groups have extensions. The point of this part of the proof is that the zero-dimensional case follows easily from the discrete case.

(ii) There is a proof of Theorem 1 in the case that G is zero-dimensional and compact in [2], section 48. Actually, compact-discrete duality is not terribly difficult in the general case; see (24.3) of [4].

(iii) If V is a finite dimensional vector space over a local field K and B is a continuous regular bilinear form on V over K, then for any nontrivial additive character χ of K the expression

$$\chi_y(x) = \chi(B(x, y))$$

defines a character χ_y of V for each $y \in V$ and the map $y \to \chi_y$ is a topological isomorphism of V onto $X(V)$. A proof such as that outlined following the last corollary above can be given, although more details arise. A similar result appears in [9], p. 40.

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