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NON-LINEAR QUANTUM FIELD THEORY*

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0. **Introduction.** This review is a greatly extended version of a set of lectures given at the Soliton conference in Tucson, Arizona.

It is written for mathematicians who have at least a casual acquaintance with systems that support soliton behavior, though one does not have to be a specialist to follow it. The lectures presume no knowledge of quantum field theory, despite the title. The idea was to present methods for investigating non-linear phenomena intrinsic to certain systems of physically interesting equations arising in quantum field theory, and so physics will play a muted role. Some of the material will involve familiar systems, e.g., the sine-Gordon equation, analyzed in an unfamiliar setting; for certain classical perfect systems are so stable they survive intact the process of second quantization.

The methods we discuss are new, this being a review of work done in the area by us and others in the last two years. Our main analytic tool is the functional, or Feynman path integral and approximations to it. Since we assume this is unfamiliar to most mathematicians, we have included a short course on the subject in Section 1, adequate to follow the subsequent development. All the quantum field theory, represented in the language of path integrals, that one needs is introduced where appropriate. There are many excellent texts available for those who wish to know more than we have space to present here. The bibliography contains a short list of relevant papers, which in turn contain references to the already extensive literature.

This is the first time in a century that the frontiers of physics and mathematics share a common ground. The most surprising consequences of nonlinear phenomena for modern physics are in the future, but already there are indications that they play a fundamental role in the field theory description of the sub-nuclear world.

1. **A short course on path integrals.** In the literature, path integrals, functional path integrals, and Feynman path integrals, are names used interchangeably. We will make no attempt at giving a rigorous justifi-

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cation of the method of path integrals. This is neither necessary nor possible. As they are used in quantum field theory (the reader should not become apprehensive if he has never heard of quantum field theory; it will be explained), a functional integral is usually defined as the limit of a finite dimensional approximation. The precise definition of this limit is not needed in anything that follows.

Let $\phi(\cdot)$ denote a field, that is a function of space and time, and let $\mathscr{C}_{\text{b.c.}}$ denote a class of (sufficiently nice) fields subject to fixed boundary conditions (b.c.). For example, $\mathscr{C}_0$ could denote the set of all fields which are continuous and vanish as $|\vec{x}| \to \infty$, t fixed. Furthermore, let K denote an invertible linear operator and adopt the shorthand notation

$$\phi K \phi \equiv \int \int \phi(x) K(x, y) \phi(y) d^n x d^n y$$

Here x and y are points in space-time, and the integrals range over all of space-time.

The Feynman integral is an expression of the form

$$\int_{\phi(\cdot) \in \mathscr{C}_n} \mathscr{D}\phi(\cdot) \left\{ \exp\left[\frac{i}{2} \phi K \phi \right] I(\phi) \right\}, \tag{1.1a}$$

where the integrand $I(\cdot)$ is a functional of the fields, $I : \mathscr{C}_{bc} \to \mathbf{R}$, and one thinks of the integral as a continuous sum over all fields in the class $\mathscr{C}_{bc}$. (The actual definition of the Feynman integral is given in terms of a limit of iterated integrals [8]. One considers a family of partitions of space-time, $\{(x_1, x_2 \cdots, x_n)$ for $N = 1, 2, \cdots\}$ which become finer and finer as $N \to \infty$. For each partition, the field $\phi(x)$ is approximated by the N-vector $\{\phi_j \equiv \phi(x_j), j = 1, 2, \cdots N\}$ and the kernel $K(x, y)$ by the matrix $\{K_{ij} = K(x_i, y_j)\}$. Then the Feynman integral is defined by a formula of the type

$$\begin{aligned} &\int_{\mathscr{C}_{\text{b.c.}}} \mathscr{D}\phi(\cdot) \left\{ \exp\left[\frac{i}{2} \phi K \phi \right] \right\} I(\phi) \\ &\equiv \lim_{N \to \infty} C_N \int \int \cdots \int d\phi_1 \cdots d\phi_N \\ &\qquad \left\{ \exp\left[\frac{i}{2} \phi_i K_{ij} \phi_j \right] \right\} I_N(\phi_1 \cdots \phi_N) \end{aligned} \tag{1.1b}$$

where C_N is a normalization constant and I_N denotes a suitable approximation of the integrand $I(\phi)$.)

In the perturbation calculations of field theory, the integrands I are functionals of the very special form

$$I(\phi) = \prod_{j=1}^{n} \phi(x_j).$$

For this special type of integrand, the definition (1.1b) collapses to a finite-dimensional Gaussian integral. In this case, the natural extension of the formula

$$\begin{aligned}\left(\frac{2\pi}{\alpha}\right)^{1/2} & \int_{-\infty}^{\infty} x^k e^{-\alpha x^2/2}\, dx \\ &= \left(\frac{2\pi}{\alpha}\right)^{1/2}\left[\left(\frac{\partial}{\partial j}\right)^k \int_{-\infty}^{\infty} e^{-\alpha x^2/2)+jx}\, dx\right] \\ &= \left(\frac{\partial}{\partial j}\right)^k \exp(-j^2\alpha^{-1})\Big|_{j=0}\end{aligned}$$

allows us to introduce a working definition of the Feynman integral for this class of integrands [1]:

$$\begin{aligned}&\int_{\mathscr{C}_{b.c.}} \mathscr{D}\phi(\cdot)\left\{\exp\left[\frac{i}{2}\phi K\phi\right]\prod_{n=1}^{n}\phi(x_m)\right\} \\ &= \left((-2)^n \frac{\delta^n}{\delta j(x_1)\cdots\delta j(x_n)} \exp\left[-\frac{i}{2} jk^{-1}j\right]\right)_{j(\cdot)=0}\end{aligned} \tag{1.2}$$

Such a quasi-gaussian functional form has all the properties of an ordinary integral: one can integrate by parts, change measures, define Fourier transforms, iterate, take multiple integrals and define delta functionals. In short, they are like ordinary integrals with somewhat different rules. The most important property that makes all these properties work is that the functional integral of a gaussian is again a gaussian.

Fourier Transform.

$$\begin{aligned}&F(\varphi) = \exp\{i\int \varphi(x)K(x-y)\varphi(y)\,dxdy\}\prod_j \varphi(x_j) \\ &\Rightarrow \exists\, F(j) = \int F(\varphi)\exp\{i\int j(x)\varphi(x)\,dx\}\prod_x d\varphi(x)\end{aligned} \tag{1.3}$$

for any gaussian functional. Hence

$$\begin{aligned}&F(\varphi) = \exp i(\varphi k\varphi)\prod_j \varphi(j) \\ &\Rightarrow \exists\, F(j) = \int (\mathscr{D}\varphi)\, F(\varphi)\exp i(j\varphi).\end{aligned} \tag{1.4}$$

Integration by Parts.

$$\int \left(\frac{\delta}{\delta\varphi(z)} \exp\left\{ \frac{i}{2} \int \varphi(x)K(x-y)\varphi(y)\,dxdy \right\} \right) \exp\left(i \int \varphi(x)j(x)\,dx \right) \prod_x d\varphi(x)$$

(1.5)

$$= - \int \exp\left\{ \frac{i}{2} \int \varphi(x)K(x-y)\varphi(y)\,dxdy \right\} \frac{\delta}{\delta\varphi(z)} \exp\left(i \int \varphi(x)j(x)\,dx \right) \prod_x d\varphi(x),$$

that is,

$$\int (\mathscr{D}\varphi) \left[\frac{\delta}{\delta\varphi} \exp \frac{i}{2} (\varphi k\varphi) \right] \exp i(\varphi j)$$

(1.6)

$$= - \int (\mathscr{D}\varphi) \exp \frac{i}{2} (\varphi k\varphi) \left[\frac{\delta}{\delta\varphi} \exp i(\varphi j) \right].$$

Transformation of variables. The rule for the change of measure under a shift is somewhat different for functional integrals than for ordinary ones and is the origin of the path integrals' power in quantum field theory applications. In the hands of a skilled operator, succinct and transparent proofs can be given, which in another notation would be very complex.

Under the shift

$$\varphi = f(\varphi'); \quad f(\varphi') = c_0(x) + \varphi(x) + \tilde{f}(\varphi') \tag{1.7}$$

the form

$$\int \exp\left\{ \frac{i}{2} \int \varphi(x)K(x-y)\varphi(y)\,dxdy + i \int \varphi(x)j(x)\,dx \right\} \prod_x d\varphi(x) \tag{1.8}$$

$$= \int \exp\left\{ \frac{i}{2} \int f(\varphi'(x))K(x-y)f(\varphi'(y))\,dxdy + i \int f(\varphi'(x))j(x)\,dx \right\} \det\left\{ 1 + \frac{\delta\tilde{f}}{\delta\varphi'} \right\} \prod_x d\varphi',$$

where

$$
\begin{aligned}
&\det\left\{1+\frac{\delta f}{\delta\varphi'}\right\}\\
&\quad\equiv \exp\left\{\operatorname{tr}\ln\left[1+\frac{\delta f(x)}{\delta\varphi'(y)}\right]\right\}\\
&\quad= \exp\left\{\int \frac{\delta f(x)}{\delta\varphi(y)}\,\Big|_{x=y}\,dx\right.\\
&\qquad\left.+\frac{1}{2}\int dxdy\,\frac{\delta f(x)}{\delta\varphi'(y)}\,\frac{\delta f(y)}{\delta\varphi'(x)}+\cdots\right\}.
\end{aligned}
\tag{1.9}
$$

Equation (1.8) becomes

$$
\begin{aligned}
&\int(\mathscr{D}\varphi)\exp\{i/2\,\varphi K\varphi+i\varphi j\}\\
&\quad=\int(\mathscr{D}\varphi')\det\left\{1+\frac{\delta f}{\delta\varphi'}\right\}\exp\left\{\frac{i}{2}fKf+ifj\right\}
\end{aligned}
\tag{1.10}
$$

and (1.9) defines $\det\{1+(\delta f/\delta\varphi')\}$ as a formal power series. This formula is not very useful unless f is independent of ϕ', in which case it can be moved outside the integral. Fortunately the latter is often the case. These are all the properties of path integrals that we will need to analyze the structure of Quantum Field Theory (Q.F.T.).

j as an external source. The reader will have noticed the appearance of the auxiliary variable j in the working definition of the Feynman integral (1.2). With this auxiliary variable it is easy to derive "generating functionals" for various objects. Setting $j=0$ at the end of the calculation returns one to the original system of physical interest. Physicists think of j as an external source. Its use will be demonstrated in the next section.

2. **Feynman's Action Principle, Path Integrals and Q.F.T.** Quantum mechanics, special relativity and the principle of causality imply a description of the world known as Quantum Field Theory (Q.F.T.). Q.F.T. has a long history and this section will be hopelessly inadequate physically. However, since the point is to show how the path integral enters and how one usually works with it, it will be enough to simply state Feynman's action principle, which provides an astonishing and complete connection between the classical description of a system and its relativistic quantum analog.

Feynman originally formulated his path integral representation for a quantum transition probability by a method he called a "sum over histories," If $S=\int d^4x\mathscr{L}(x_\mu)$ is the classical action for a system, where

$\mathscr{L}(x_\mu)$ is the Lagrangian density, then the transition probability is obtained by summing over all possible paths in space time that the system can evolve along, each weighted by the exponential of the action for that path.

Rule 1. *The Action Functional.* The probability for a system to start from the vacuum state and end there under the influence of an external source j is given by the vacuum-vacuum transition amplitude $\langle 0|0\rangle^J$ where

$$\langle 0 \mid 0\rangle^J = N \int (\mathscr{D}\varphi) \exp i\,(S(\varphi) + j\varphi) \tag{2.1}$$

S is the classical action, N is a normalizing factor to be determined.

Rule 2. The complete n leg Green functions (which give the amplitude for n quanta of the theory to interact) is given by the rule

$$G(x_1 \cdots x_n) = \left(\frac{1}{i}\right)^n \frac{\delta^n \langle 0|0\rangle^J}{\delta j(x_1) \cdots \delta j(x_n)} \Bigg|_{j=0} \tag{2.2}$$

These two rules, amended with a prescription for assembling Green functions together to get a physical quantity, called the S matrix, are a complete description of the Q.F.T. corresponding to a given classical S. We will focus our attention on Rule 1, which defines $\langle 0|0\rangle^J$ also known as the vacuum generating functional. There are some assumptions built into Rule 1. The main one is that at large enough distances and times the system under consideration behaves as a collection of free fields, having no interaction with one another. This implies that they are point-like. Unless this boundary condition is altered there is no way one can see systems which are asymptotically of finite extent, coherent extended objects or bound states. This boundary condition selects out only a part of the available Hilbert space of states, missing all coherent nonlinear phenomena. The second point is how one deals with (2.1). As it stands it is intractable, since S is not, in general, quadratic. So one must resort to approximations on it. The most commonly used one is to expand the path integral in a formal series in the coupling constant, about the free field case. This leads to the Feynman rules and Feynman diagrams. To clarify all this, let us choose a very simple model Lagrangian and, using the formulas in Section I, see how linear perturbations and Feynman diagrams come about.

A simple model and some of its Feynman graphs. For simplicity, let us work in Euclidean space-time and consider a model Lagrangian with one real scalar field, interacting with a quartic self interaction. So the action of the system is $S(\varphi)$

$$S(\varphi) = \tfrac{1}{2} \int \varphi(x)\Delta_F^{-1}(x-y)\varphi(y)\,dxdy + \int \mathscr{L}_{\text{int}}(x)\,dx \tag{2.3}$$

where $\mathscr{L}_{\text{int}}(x) \equiv (g/4!)\varphi^4(x)$ and $\Delta_F(x-y)$ satisfies $(\Box - m^2)\Delta_F(x) = \delta^4(x)$ i.e., is the Euclidean Green's function for the free, massive, Klein-Gordon equation. Then by Feynman's action principle, the vacuum functional is $Z[j] \equiv \langle 0|0\rangle^J$,

$$Z[j] = \int (\mathscr{D}\varphi) \exp i\,(S(\varphi) + j\varphi). \tag{2.4}$$

By formula (1.1) we can rewrite this path integral as a j functional operating on the free gaussian, i.e.,

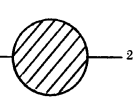
$$Z[j] = \exp\left\{ i \int \mathscr{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta j(x)} \right) dx \right\} \exp\left\{ -\frac{i}{2} \int j(x)\Delta_F(x-y)j(y)\,dxdy \right\}. \tag{2.5}$$

Expanding both exponentials as a formal power series, we see that we get an expansion in powers of the coupling constant g

$$\begin{aligned} Z[j] = \Bigg\{ 1 &+ \frac{ig}{4!} \int \left(\frac{1}{i}\frac{\delta}{\delta j(x)} \right)^4 dx \\ &+ \frac{i^2}{2\cdot 4!}\, g^2 \int \left(\frac{1}{i}\frac{\delta}{\delta j(x)} \right)^4 \left(\frac{1}{i}\frac{\delta}{\delta j(y)} \right)^4 dxdy + \cdots \Bigg\} \\ &\times \Bigg\{ 1 - \frac{i}{2} \int j(x)\Delta_F(x, y)j(y)\,dxdy \\ &+ \frac{i^2}{2\cdot 2} \int j(x)\Delta_F(x, y)j(y)j(x')\Delta_F(x', y')j(y'\,dxdx'dydy' + \cdots \Bigg\} \end{aligned} \tag{2.6}$$

Now, if we agree to represent $\Delta_F(x, y)$ as an undirected line between space-time points x and y, and take integrals over the same point to mean the joining of lines at that point, we can write down Feynman diagrams. As an illustration we write out the first few terms of the connected diagram series (there are many disconnected ones) for the 2- and 4-point functions. Simply apply Rule 2. 2-point function: where the shaded bubble means the complete Green's function

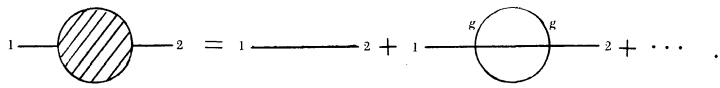
$$\frac{1}{i^2}\frac{\delta}{\delta j(1)}\frac{\delta}{\delta j(2)} Z[j] \Big|_{j=0} \equiv$$



to lowest order in g, because of the $j = 0$ condition we get only the term quadratic in j, i.e.,

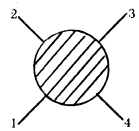
$$\frac{\delta}{\delta j(1)} \frac{\delta}{\delta j(2)} \int j(x)\Delta_F(x, y)j(y)\,dxdy$$
$$= \int \delta(x - 1)\Delta_F(x, y)\delta(y - 2)\,dxdy = \Delta_F(1, 2) = \Delta_F(2, 1). \tag{2.7}$$

So to order g^2



4-point function:

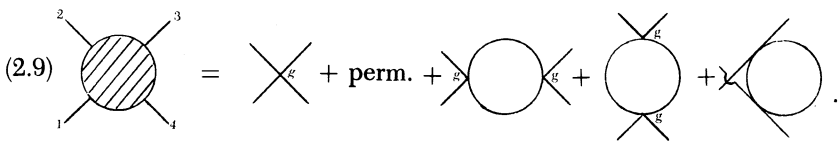
$$\left(\frac{1}{i}\right)^4 \frac{\delta}{\delta j(1)} \frac{\delta}{\delta j(2)} \frac{\delta}{\delta j(3)} \frac{\delta}{\delta j(4)} Z[j] \Big|_{j=0} \equiv \tag{2.8}$$



Consider only connected terms. Examining the series, we see that it begins in order g with the integrand

$$j_1\Delta_{12}j_2j_3\Delta_{34}j_4j_5\Delta_{56}j_6j_7\Delta_{78}j_8$$

where $(\delta/\delta j(x))^4$ contracts 4 points to x and the remaining $(\delta/\delta j(1)) \cdots (\delta/\delta j(4))$ sets the other 4 points to be external lines so



(2.9)

This is the origin of Feynman graph expansions, and is the way he originally invented them. This ends the section on the formal development of linearized Q.F.T.

3. **Quantizing extended solutions (Modified Feynman expansions).** The Feynman diagram technique is obviously a very powerful computational tool. It allows a space-time picture of the structure of various Green's functions, to which one can very often apply physical considerations to reduce their number and complexity. However, it suffers from all the limitations mentioned above. It is, essentially, a weak coupling expansion about a point-like free field theory.

Most of the simple field theories that physicists work with have the property that bounded energy solutions are dissipative. There are, however, very important models, which as classical field theories, admit

non-dissipative solutions of finite energy. The physicist Sidney Coleman has given them the name "lumps" to distinguish them from the true solitons of completely separable systems. This is a good distinction to remember since any theory which tries to use spacially constrained solutions as a model for elementary particles is going to have to deal with "lumps." The reason for the distinction is that solitons have an interaction which is far too simple to be physically useful. For example, particle creation and destruction is missing. Also physical theory must ultimately be formulated in $(3 + 1)$ dimensions (three space and one time dimension) where true solitons probably don't exist.

To begin, however, it seems a good idea to go to $(1 + 1)$ dimensions where sensible field theories having classical soliton modes exist and attempt to find approximation methods which keep intact non-linear effects from the beginning. If we insist on perturbing about a point-like free field solution in the action principle, then in studying non-linear coherent phenomena, no graph can be neglected. One is reduced to considering infinite sums of them, which ruins the utility of the Feynman diagram expansion.

This puts us back to equation (2.1), the action principle. We still have available the entire battery of properties of path integrals described in Section 1. There are two directions we can pursue. Both are approximation methods, both altering the boundary conditions on the class of paths considered. From here on the main development will take place in $(1 + 1)$ dimensions. The reason is simply that we want to use the remarkable properties of soliton systems, both physically and mathematically to guide us in approximating the path integral.

Modified Feynman Expansions. The first approximation scheme is primarily due to Gervais, Jevicki, and Sakita [2] and involves the smallest modification to the path integral. The idea is to develop an expansion analogous to a Feynman diagram expansion, but perturbing about the classical extended lump. We will give only the briefest outline of the idea and some of the troubles it encounters, since, because of its extreme complexity, this approach does not go very far toward a quantum field theory of soliton modes.

Translation Modes. Let us consider only static solutions to a field theory. If we wish to perform a perturbation about a static soliton even classically, it is clear that a naive attempt will fail. Say $\phi(x - vt, t)$ is the functional dependence of the soliton mode and begins at a time $t \approx 0$ with a slight increment in velocity $v \approx \epsilon$, a small expansion parameter. Once set into motion, the soliton will remain at a constant velocity ϵ, so that at small enough times $\sim 1/\epsilon$, fluctuations in the system

become so large that the expansion fails. This is the problem of the translation mode, which in the quantum field theory version causes the inverse kernel in the quadratic piece of the action $(\phi(x)\Delta_F^{-1}(x, y)\phi(y))$ to be undefined. It picks up a translation pole, a singularity which is called an infrared disaster. With a moment's thought, it is apparent that any symmetry of the system will cause like poles. Each such symmetry defines a line along which the potential energy remains constant. Fluctuations, in any approximation around a soliton mode become unbounded along such a line, since there are no energy constraints on them.

Singular Lagrangians. In a more compact language, translation and symmetry poles arise because the original problem contained a singular Lagrangian. Singular Lagrangians, $\mathcal{L}(q, \dot{q})$ are those for which the usual equation $p_i = \partial L(q, \dot{q})/\partial \dot{q}_i$ cannot be inverted to find the $\dot{q}$. This causes the path integral containing them to be ill-defined. The cure for this is to go over to a Hamiltonian formalism for the path integral and find an independent set of functional variables. Effectively, this causes the troublesome terms to factor out of the functional measure. A systematic way to do all of this was given by L. D. Faddeev [3], in a classic paper on path integrals for singular Lagrangian systems.

We will give only a brief outline of the prescription for the case of $(1 + 1)$ dimensions and arbitrary numbers of solitons, treating only the translation mode. The explicit prescription for this case will show the reader why the modified perturbation approach to soliton quantization is bound to become enmeshed in complexity.

Consider a system defined by the Lagrangian $\mathcal{L} = (1/2)(\partial\phi/\partial t)^2 - (1/2)(\partial\phi/\partial x)^2 - V(\phi)$ where $V(\phi)$ is a potential function. Let $\phi_c(x, t)$ be an n-soliton solution to the classical equations of motion. This depends on n constants of integration representing the initial positions of the soliton array, which we call $(\bar{q}_1, \cdots, \bar{q}_n)$. Then $\phi_c(x, t) = \phi_c(x, t, \bar{q}_1, \cdots, \bar{q}_n)$. Translation invariance allows us to define new variables $q_i(t) = v_i t + \bar{q}_i$ $(i = 1, \cdots, n)$, v_i being a set of velocities to be determined. Then $\phi_c(x, t) = \phi_c(x, t, q_1(t), \cdots, q_n(t))$. If we choose to expand $\phi(x, t)$ about some classical n-soliton solution $\phi(x, t) = \phi_c(x, \{q_i(t)\}) + \xi(x, t)$, $i = 1, \cdots, n$, we have effectively chosen the generalized coordinates $\{q_i(t)\}, \xi(x, t)\}$ to describe the system. To complete the description we must find the momenta canonically conjugate to the coordinates and check that they are linearly independent. It turns out that they are not.

The conjugate momenta are given by

$$P_i(t) \equiv \frac{\partial L}{\partial \dot{q}_i(t)} = \dot{q}_j \int \frac{\partial \phi^c}{\partial q_j} \frac{\partial \phi^c}{\partial q_i} + \int \dot{\xi} \frac{\partial \phi_c}{\partial q_i}$$

$$\pi(x, t) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\xi}(z, t)} = \frac{\partial \phi_c}{\partial q_j} \dot{q}_j + \dot{\xi}.$$

These are related by the equations

$$\psi_i \equiv P_i(t) - \int \xi(x, t) \frac{\partial \phi_c}{\partial q_i(t)} = 0,$$

the so-called ψ constraints. This means that $\mathcal{L}$ is singular since one cannot solve for $\dot{q}_1$, ξ as functions of p_i and π, so a straightforward transition to the Hamiltonian formalism is blocked.

The way around it is Faddeev's prescription, and we simply present his form of the Hamiltonian action integral for constrained systems.

Faddeev's formula. Let a mechanical system be defined by the canonical variables $q = (q_1, \cdots, q_n)$ and $p = (p_1, \cdots, p_n)$, the Hamiltonian $H(q, p)$ and a set of functions $\phi^a(q, p)$ called links. Links are the constraint functions $\phi^a(q, p) = 0$, $a = 1, \cdots, n$ which restrict the variables p and q from varying throughout the whole of the phase space Γ. The $\psi_i = 0$ are links. Let the linking functions additionally satisfy (1) $\{\varphi^a, \varphi^b\} = \sum_c C_c{}^{ab}\varphi^c$; (2) $\{H, \varphi^a\} = \sum_b C_b{}^a\varphi^b$, where $\{f, g\}$ is the usual Poisson bracket in Γ and the C's are certain functions. This case, although appearing special, is actually sufficient. Now if one can find m additional conditions called gauges $\chi_a(p, q) = 0$, $a = 1, \cdots, m$ which satisfy (1) $\det |\{\chi_a, \varphi_b\}| \neq 0$; (2) $\{\chi_a, \chi_b\} = 0$, then the action principle, in its Hamiltonian form is:

$$\langle 0|S|0\rangle = \int \exp\{i \int \sum_i P_i\dot{q}_i - H(p, q)\} \, d\mu(q(t), p(t))$$

where the measure $d\mu(p, q) \equiv \prod_a \delta(\chi_a)\delta(\varphi_a) \det|\{\chi_a, \varphi_b\}| \prod_i dp_i dq_i$.

This is the form for finite systems, but goes over to field theory essentially unchanged. From this form for the action one can immediately see difficulties. We have, as promised, resolved the zero eigenvalue translation mode problem and the functional integral is well defined. But the measure is written in a deceptively simple notation. The actual measure is a rather complex object and the Hamiltonian in the new variables is very complex. As the field theories get more sophisticated, this feature becomes alarming. For the case of a Yang-Mills gauge model, with its many symmetry modes, it is almost unmanageable. Now, contemplate developing a Feynman diagram scheme around the result-

ing integrand with such a complex $H(p, q)$, and one sees that it would be wise to look for a more manageable technique. It is fine in principle, but not very efficient.

As a technical aside, the reader should, at this point, be concerned about the validity of a canonical transformation within a Hamiltonian functional integral. It is well known that if you begin with a quantum system described by a Hamiltonian path integral and perform a canonical transformation, one passes to a system different from the original one. Naive canonical mappings under functional integrals are illegal operations. This apparent malfunction of the formalism can be repaired in a consistent way, but doing so further complicates the formalism, adding terms to the potential at the two-loop Feynman diagram level.

Because of all these problems, we will leave the approach of the modified Feynman expansion and instead consider in some detail the last option open to us; a more drastic modification of the action principle itself. This leads us to the main subject of this review, the functional analogy in field theory to the WKB approximation of ordinary quantum mechanics. This method was developed by Dashen, Hasslacher and Neveu [4], [5], [6], [7], and has come to be known in the literature as DHN.

4. **Functional integral analog of the WKB approximation (DHN).** Since the path integral action principle generates Q.F.T. by operating directly on the possible histories of a classical system, one might wonder what would happen if the class of allowed paths were restricted to stable, periodic orbits. In ordinary quantum mechanics, one finds that such motions, when quantized, give a bound state energy spectrum. These classical paths have quantum shadows which are the kinds of objects we want to analyze. But, from a canonical point of view, the transition from classical quantum mechanics to Q.F.T. is a rather drastic process. It only appears simple because we have used Feynman's powerful formulation of it. Do periodic orbits survive the transition to a quantum field theory and if they do, what are the properties of their quantum shadows? The first step in asking such a question is to develop some analytic machinery appropriate to it.

First, we have to find an approximation to the path integral which naturally chooses periodic orbits. Second, for tractability, we must not ask the integral to give us too much information. Third, the method must be non-perturbative. (We have already seen the difficulties involved in any perturbative scheme about an extended classical solution.)

The presentation below will outline such a method, and its application to the periodic motions of an interesting soliton supporting field

theory, the sine-Gordon equation. We outline the development for the case of one degree of freedom, then simply state how it goes over to a field theory.

There are only a small number of operations one can do practically on a path integral, all of them special cases of properties outlined in Section 1. We will use three of them. Given the Hamiltonian of the system we ask only for the bound state energy spectrum. Then we use a saddle point approximation to the integral, a particular shift due to Cameron and Martin and functional integration by parts to analyze the spectrum. The development should be considered heuristic at best. Existence questions will be ignored.

One degree of freedom, a particle in a potential [4]. We begin by outlining a rather complex derivation of the well-known WKB approximation to the bound state energy levels of a particle in a one-dimensional potential well. The derivation uses a path integral representation. The ultimate advantage is that it generalizes immediately to field theory.

Define

$$R(E) = \operatorname{tr} \frac{1}{H - E} = \sum_n \frac{1}{E_n - E} ,$$

E_n being the energy of the nth bound state. Write $R(E)$ as an integral over an exponential.

$$R(E) = i \operatorname{tr} \int_0^\infty \frac{dT}{\hbar} \exp\left\{ \frac{i}{\hbar} (E - H)T \right\} .$$

The advantage of this form is that $\operatorname{tr} \exp[-iHT]$ has a path integral representation, the Feynman propagator:

$$\operatorname{tr} \exp[-iHT] = \int \mathscr{D}\{x(\tau)\} \exp\left[\frac{i}{\hbar} \int_0^T \left(\frac{1}{2} \dot{x}^2 - V(x) \right) d\tau \right]$$

over all paths $x(t + T) = x(t)$. This requirement is due to the trace operation, ($\dot{f}$ stands for time derivative and f' for space derivative). Treating $\hbar$ (a scale factor which we identify with Planck's constant) as a small perturbation parameter, expand the integrand to leading order in $\hbar$ by the method of stationary phase. The dominant contribution to the functional integral comes from paths close to classical orbit $x(\tau)$ with fixed period T. This means that (1) the paths $x(\tau)$ are periodic, and (2)

initial and final momentum are equal for any point on the periodic path. Condition 2 is caused by the stationary phase condition.

$$\frac{\partial S}{\partial q}(q, q') + \frac{\partial S}{\partial q'}(q, q') \Big|_{q=q'}$$

$= P - P' = 0$, q, q' are end points of the path.

Expanding the action about these classical orbits gives

$$\text{tr}\exp(-iHT) \approx \exp(iS_{\text{cl}}(T)/\hbar) \int \mathscr{D}x(\tau)\, dx_{\text{cl}}(0) \times \exp\left\{ \frac{i}{\hbar} \int_0^T \left[\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 V''(x_{\text{cl}}(\tau)) \right] d\tau \right\} \tag{4.1}$$

over paths $x(0) = x(T) = 0$. The measure $dx_{cl}(0)$ reflects the freedom to begin the integration at an arbitrary point along the orbit. The subscript cl means "classical."

Equation (4.1) is evaluated by a shift in the functional integral first used by Cameron and Martin in another context (see [8] for a survey). Introduce the mapping

$$y(\tau) = x(\tau) - \int_0^\tau \frac{\dot{N}(\mu)}{N(\mu)} x(\mu)\, d\mu \tag{4.2}$$

and its inverse

$$x(\tau) = y(\tau) + N(\tau) \int_0^\tau \frac{\dot{N}(\mu)}{N^2(\mu)} y(\mu)\, d\mu \tag{4.3}$$

where N is defined by $\ddot{N} = -V''N$.

Substitution and integration by parts gives:

$$\text{tr}\exp(-iHt) \approx \exp[iS_{\text{cl}}(T)/\hbar] \times \int \mathscr{D}y(\tau) d\alpha\, dx_{\text{cl}}(0) \left| \frac{\mathscr{D}x}{\mathscr{D}y} \right| \times \exp\left\{ \frac{i}{\hbar} \left[\frac{1}{2} \int_0^\tau \dot{y}^2\, d\tau + \alpha \left(y(T) + N(T) \int_0^T \frac{\dot{N}(\tau)}{N^2(\tau)} y(\tau)\, d\tau \right) \right] \right\}.$$

Since (4.2) is Volterra [8], the functional Jacobian $|\mathscr{D}x/\mathscr{D}y| = |N(T)/N(0)|^{1/2}$. Integrating over y and the Lagrange multiplier α gives:

$$\operatorname{tr}\exp(-iHT) \approx \left(\frac{i}{2\pi\hbar}\right)^{1/2} \exp\left[\frac{i}{\hbar} S_{\mathrm{cl}}(T)\right] \times \int dx_{\mathrm{cl}}(0) \left|\frac{N(T)}{N(0)}\right|^{1/2} \left[\int_0^T \frac{N^2(T)}{N^2(\tau)}\, d\tau\right]^{-1/2}. \tag{4.4}$$

To complete the sum over paths of the system, one sums over n traverses of the orbit such that its period is equal to the fundamental orbit. After completing the sum it can be shown that

$$R(E) \approx \left(\frac{i}{2\pi\hbar}\right)^{1/2} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dT}{\hbar}\, T \left| n \frac{dE_{\mathrm{cl}}}{dT} \right|^{1/2} \times \exp\left[in\left(\frac{S_{\mathrm{cl}} + ET}{\hbar} - \pi\right)\right] \tag{4.5}$$

where $E_{\mathrm{cl}} = dS_{\mathrm{cl}}/dT$.

Evaluating this integral by stationary phase, to leading order in $\hbar^{-1}$ gives a stationary point at

$$\frac{dS_{\mathrm{cl}}}{dT} = -E_{\mathrm{cl}} = -E.$$

This reduces $R(E)$ to

$$R(E) \approx \frac{iT(E)}{\hbar} \sum_{n=1}^{\infty} \exp\left[in\left(\frac{W(E)}{\hbar} - \pi\right)\right] = -\frac{iT(E)}{\hbar} \frac{\exp[iW(E)/\hbar]}{1 + \exp[iW(E)/\hbar]} \tag{4.6}$$

where $W(E) \equiv S_{\mathrm{cl}}(T(E)) + E\,T(E)$.

Now if E_m is a root then $W(E_m) = (2m + 1)\pi h$. But from ordinary mechanics, we know that

$$W(E) = 2\int_{x_1}^{x_2} [2(E - V)]^{1/2}\, dx$$

where x_1, x_2 are turning points for the potential V.

So, in this complicated fashion, we derive the ordinary WKB quantization condition from a path integral formulation. Now the transition to field theory is relatively straightforward, involving only one additional concept. The fundamental steps will be the same: An approximation of $\operatorname{tr}\exp[-iHT]$, a stationary phase integral over T which converts $R(E)$ to a factor times $\exp[iW]$, and finally a sum over multiple tra-

verses of the fundamental orbit which gives a geometric series in $\exp[iW]$ and so poles in $R(E)$. To do it properly, one needs a complete classification of the classical periodic orbits.

Extension to Field Theory [4], [5], [6]. Imagine a system with a large number of degrees of freedom. To get to Field Theory let the number become unbounded. When dealing with many degrees of freedom, an additional concept enters, the notion of stability angles. Briefly, $\operatorname{tr}\exp[-iHT]$, after stationary phase approximation and rewriting in a Hamiltonian form becomes:

$$\operatorname{tr}\exp(-iHT) = \int \mathscr{D}P\mathscr{D}X \times \exp\left[i \int_0^T \left(\sum_i P_i X_i - H(P, X) \right) d\tau \right] \tag{4.7}$$

where P and X are generic labels for generalized momenta and coordinates, explicitly depending on t. H is the Hamiltonian corresponding to the quadratic Lagrangian form obtained by evaluating the integral around its stationary point.

Since the integral is gaussian, it is proportional to the inverse square root of the determinant of the operator in the exponential (see Section 1), a differential operator with periodic boundary conditions. To find the determinant, we must find the eigenvalues of this operator. These are given by the solutions to Hamilton's equations $X(t) = \xi_\alpha(t)$, $P(t) = \eta_\alpha(t)$, where η_α and ξ_α satisfy the functional form $\eta_\alpha(t + T) = \exp[-i\nu_\alpha]\eta_\alpha(t)$, ν_α real and $\neq 0$. These are not eigenfunctions, but since the system is first order, the periodic eigenfunctions are:

$$X_{n\alpha}(t) = \xi_\alpha(t) \exp\left[i \frac{t}{T} (\nu_\alpha + 2n\pi) \right], \quad -\infty < n < +\infty \tag{4.8}$$

with eigenvalues $i(\nu_\alpha + 2n\pi)T^{-1}$.

The ν_α's are called stability angles or Floquet indices. For stability, they must be real and the differential system constrains them to occur in pairs $\pm \nu_\alpha$. The condition $\nu_\alpha \neq 0$ is set to remove explicitly any zero stability angles, which reflect the symmetry modes of the problem and cause the determinant to become singular. They factor out of the integral exactly as in previous sections. Using the properties of the ν_α spectrum we write the square root of the eigenvalue product as:

$$\prod_{\nu_\alpha > 0} \prod_{n=1}^{\infty} \nu_\alpha \left[1 - \left(\frac{\nu_\alpha}{2n\pi} \right)^2 \right] = \prod_{\nu_\alpha > 0} \left((e^{-i\pi/2} 2 \sin \frac{\nu_\alpha}{2} \right). \tag{4.9}$$

For each such factor, we can further expand its inverse as

$$\left[e^{-i\pi/2}\, 2 \sin \frac{\nu_\alpha}{2} \right]^{-1} = \frac{e^{i\pi/2} e^{-i\nu_\alpha/2}}{1 - e^{-i\nu_\alpha}} . \tag{4.10}$$

so that

$$\operatorname{tr} \exp[-iHT] = \sum_{\text{periodic orbits}} \exp i[S + \xi]\Delta_1\Delta_2$$

where

$$\xi = -\frac{1}{2} \sum_{\nu_\alpha > 0} \nu_\alpha, \quad \Delta_1 = \text{factor containing only zero modes}$$

and

$$\Delta_2 = \prod_{\nu_\alpha > 0} (1 - e^{-i\nu_\alpha})^{-1} = \prod_{\nu_\alpha > 0} \sum_{q_\alpha = 0}^{\infty} e^{-i\nu_\alpha q_\alpha}.$$

In general the ξ function, the sum over stability angles is divergent and must be regulated and renormalized. This will be explained in the next section on the sine-Gordon system. The factor Δ_2, if there are n discrete stability angles present, can be written as

$$\prod_{i=1}^{n} \frac{1}{1 - e^{-i\nu_i}} = \sum_{\{q_i\}} e^{-i\Sigma_\gamma q_\gamma \nu_\gamma},$$

where $\{q_i\}$ is a set of integers. (4.12)

Under quantization, as before, the quantization condition on $W(m)$ becomes

$$W(m) = 2\pi n - \xi(T(m)) + \sum_i q_i \nu_i(T(m)) \tag{4.13}$$

where $W(m) = S(T(m)) + MT(m)$ and $T(m)$ solves $-dS(T(m))/dT = m$. So each basic particle $n = 1, 2, 3 \cdots$ is accompanied by excited states labeled by the integers $\{q\}$ (see Voros [9]).

5. **Quantization of the sine-Gordon Breather.** We now provide a concrete example of the technique of the previous sections in quantizing, in lowest non trivial approximation, the sine-Gordon soliton and breather. The classical Lagrangian density is:

$$\mathcal{L} = \frac{m^4}{\gamma} \left[-\frac{1}{2} (\partial_\mu \varphi)^2 + \cos \varphi - 1 \right] \tag{5.1}$$

where

$$(\partial_\mu \varphi)^2 = (\partial_x \varphi)^2 - (\partial_t \varphi)^2. \tag{5.2}$$

Let us examine the soliton mode for which the classical field configuration is, in its rest frame

$$\varphi_s(x) = 4 \tan^{-1}\exp(x).$$

The classical mass, M_{cl} of the soliton is given directly by integrating the Hamiltonian density

$$\mathscr{H} = \frac{m^4}{\lambda}\left[\frac{1}{2}(\partial_t \varphi)^2 + \frac{1}{2}(\partial_x \varphi)^2 - \cos\varphi + 1\right] \tag{5.3}$$

hence

$$M_{\text{cl}}\,(\text{soliton}) = \frac{8m^3}{\lambda}\,. \tag{5.4}$$

As explained in previous sections, the first quantum mechanical correction is obtained by adding to M_{cl} the quantum energy of fluctuations around it, computed in the linear approximation. One must solve

$$(\partial_t^2 - \partial_x^2)\delta\varphi - \cos\varphi_s \delta\varphi = 0. \tag{5.5}$$

Separating out time dependence in the form

$$\delta\varphi = e^{-i\omega t}\,\delta\varphi(\omega, x), \quad \omega \geqq 0$$

one finds the Schrödinger equation

$$\partial_x^2 \delta\varphi(\omega, x) - 2\ \text{sech}\ x\ \delta\varphi(\omega, x) = (1 - \omega^2)\delta\varphi(\omega, x).$$

There is one bound state solution at $\omega = 0$

$$\delta\varphi(0, x) = \frac{1}{\cosh x} \tag{5.6}$$

and, for $\omega > 1$

$$\delta\varphi(\omega, x) = e^{ikx}(\tanh x - ik), \quad k^2 = \omega^2 - 1.$$

The first quantum mechanical correction to the energy is $\frac{1}{2}$ of the sum of all values of ω. This is of course divergent, due to the infinite number of modes of a field theory. The divergences are removed in a physically sensible way by the standard procedures of subtracting the vacuum energy and performing a mass renormalization (which turns out, even in higher orders, to be the only manipulations necessary to make the quantum sine-Gordon theory finite). These procedures are now explained on this example:

Since subtraction of infinities is performed, we must be very careful not to miss any finite terms. This is done by first putting cutoffs in the theory, so that everything is finite, and the number of modes is finite, then performing the subtractions, and only then removing the cutoffs. This procedure has been made mathematically rigorous by constructive field theorists, to whom we refer the reader.

The simplest cutoff method is to approximate the theory by a finite lattice with appropriate boundary conditions (periodic, say). This provides an ultraviolet cutoff, of order $\Lambda = \pi/a$ in momentum space (a = lattice spacing), and an infrared cutoff proportional to the length L of the lattice. It would be very convenient to have a lattice version of the sine-Gordon equation that would preserve the infinite set of conservation laws. Fortunately, for our limited purpose in these lectures, it is not needed, and any crude cutoff will do.

One has to count modes of the linearized equation (5.5), and of the same equation with ϕ_s replaced by the vacuum state, that is with ϕ_s replaced by 0 (mod 2π). When the soliton is introduced in the box of length L, each mode shifts slightly in frequency (hence energy) from its frequency in the vacuum. Taking the difference between the two subtracts the infinite vacuum energy. The box has made the modes discrete, so that one can count them and follow what happens when one goes from the vacuum to the one-soliton sector.

Subtracting the vacuum energy cancels the main (quadratic) divergence in the quantum correction. It then turns out that a logarithmic divergence remains. It is cancelled by the mass renormalization counterterm, as explained in [6]. The Lagrangian density (5.1) as such does not lead to finite quantum mechanical results when one computes in perturbation in $\hbar$ (or in λ/m^2). Instead, one must start from the Lagrangian density

$$\mathscr{L} = \frac{m^4}{\lambda}\left[-\frac{1}{2}\,(\partial_\mu\varphi)^2 + \cos\varphi - 1 \right] + \frac{m^2}{\lambda}\,\delta m^2(\cos\varphi - 1) \tag{5.7}$$

where δm^2 is a power series in λ/m^2 whose coefficients are logarithmically divergent. These divergent coefficients are chosen in precisely such a way that they will cancel, order by order in λ/m^2, the divergences that arise in perturbation theory. They are obtained by computing the Feynman diagrams of Figure 1, which turn out to be all the divergent diagrams in this simple two-dimensional theory: the interpretation is that the divergent quantity $(m^2 + \delta m^2)^{1/2}$ is the unobservable bare

mass, which, after quantum corrections becomes the renormalized, finite mass, m. It is then explained in [6] how, in leading order in λ/m^2,the first term of δm^2 cancels the remaining logarithmic divergence in the quantum soliton mass. The result is

$$M(soliton) = \frac{8m^3}{\lambda} - m\pi + O(\lambda/m^2) \tag{5.8}$$

The $O(\lambda/m^2)$ terms require much more work than simple linearized oscillations around the soliton. They have not yet been computed analytically.

We now turn to the quantization of the breather mode, with classical field configuration (in its rest frame):

$$\varphi = 4\tan^{-1}\left(\epsilon \frac{\sin t/(1+\epsilon^2)^{1/2}}{\cosh \epsilon x/(1+\epsilon^2)^{1/2}}\right). \tag{5.9}$$

This mode provides an ideal illustration of the quantization method of Section 4: it is a solution of finite spatial extent, of finite energy, periodic in time, with period $T = 2\pi(1+\epsilon^2)^{1/2}$ ϵ being arbitrary. It is also a classical soliton-antisoliton bound state. Quantum-mechanically, the energy of such a bound state must come out quantized. The quantization method of Section 4 leads to formula (4.13). To apply this quantization formula, we must again solve the linearized equation, analogous to (5.5), which describes the behavior of small perturbations of the breather.

The trick for solving the linearized problem is described in [6]: it was the fact that solutions of the sine-Gordon equation with an arbitrary number of breathers have been constructed analytically [10]. One then takes a two-breather solution which consists of one breather as described by (5.9) coupled to another breather. This second breather is very weak, with amplitude η and velocity v. The first order term in η satisfies the linearized equation. In addition, there are two bound state solutions, given by the time and space derivatives of (5.9), which merely express invariance of the sine-Gordon equations under time and space translations.

The explicit formulas for the solution of the linearized problem are given in Appendix C of [6] and are rather lengthy and not particularly illuminating. Their relevant features are as follows: at $x \to -\infty$, $\cos\phi \to 1$, and $\delta\phi$ is of the type $e^{-i(\omega t - kx)}$ (with $\omega^2 - k^2 = 1$), and at $x \to +\infty$, $\delta\phi \sim e^{-i(\omega t - kx + \delta)}$. Therefore, the stability angle for such a $\delta\phi$ is ωT. The calculation of the sum of stability angles is essentially identical to the sum of frequencies for the quantum correction to the soliton mass. In particular, it is made finite by exactly the same subtraction procedure.

The final result is expressed by the mass formula

$$M_n = \frac{16m}{\gamma'} \sin \frac{n\gamma'}{16},$$

$$n = 1, 2, 3, \cdots < 8\pi/\gamma', \tag{5.10}$$

$$\gamma' = \frac{\lambda}{m^2}\left(1 - \frac{\lambda}{8\pi m^2}\right)^{-1}.$$

There are no other classical particle-like solutions in the sine-Gordon equation. Hence, we can expect that formulas (5.8) and (5.10) give, at least for small coupling ($\gamma' \ll 1$), the complete particle spectrum of the quantum theory.

We now turn to the interpretation of formula (5.10), and its comparison with results obtained by canonical methods. We first discuss what happens for $\gamma' \ll 1$, which is the weak coupling region, where the semi-classical picture is guaranteed to be at least qualitatively correct.

For $\gamma' \ll 1$, we can expand (5.10) at moderate values of n:

$$M_n = nM_1 - \frac{M_1}{6}\left(\frac{\lambda}{16m^2}\right)^2 (n^3 - n) + O(\lambda^3), \tag{5.11}$$

$$M_1 = \frac{m}{16\gamma'} \sin \frac{\gamma'}{16} = m - \frac{m}{6}\left(\frac{\lambda}{16m^2}\right)^2 + O(\lambda^3). \tag{5.12}$$

We see that M_1 is, to order λ^2, the mass m of the "elementary" particle, obtained by the conventional methods: by rescaling ϕ, x and t in (5.1) and expanding the cosine, (5.1) can be written

$$\begin{aligned} \mathscr{L} &= -\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{m^4}{\lambda}\left[\cos\left(\varphi \frac{(\lambda)^{1/2}}{m}\right) - 1\right] \\ &= -\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \cdots . \end{aligned} \tag{5.13}$$

M_n as given by (5.11) corresponds to a nonrelativistic (= small binding energy relative to rest mass) n-body bound state made up of n particles with physical mass M_1. Indeed, the binding energy is exactly the same as one finds upon solving the n-body Schrödinger equation with the δ-function potential obtained from the ϕ^4 term in (5.13).

The emergence of the "elementary" particle as an ultra-relativistic (a very large binding energy) bound-state of a soliton and an anti-soliton suggests that there are actually two equivalent ways of looking at the quantum sine-Gordon equation: one in which ϕ is the fundamental field, with mass $m + 0(\lambda^2)$, and the soliton is a coherent, composite

state with a complicated structure of that field; and the other in which the soliton (and anti-soliton) are fundamental, and "elementary" particles are actually bound states of a soliton and an anti-soliton.

Indeed, Coleman [11] has proved that the quantum sine-Gordon equation is equivalent to the massive Thirring model.

The massive Thirring model is described by the Lagrangian density

$$\mathscr{L} = \bar{\psi} i\gamma_\mu \partial_\mu \psi - M_f \bar{\psi}\psi - \tfrac{1}{2} g(\bar{\psi}\gamma_\mu \psi)^2 \tag{5.14}$$

where ψ is a two-component fermion field, M_f its mass, and g a dimensionless coupling constant. Since fermion fields anticommute with each other, it is impossible to consider (5.14) classically: a fermion field is an intrinsically quantum-mechanical object; the fact that one cannot put more than one excitation in a given mode of a fermion field (the Fermi exclusion principle) makes it meaningless to talk about a large quantum number limit, i.e., a classical limit. Nevertheless, rules for computing in arbitrary order of ordinary perturbation theory with the Lagrangian density (5.14) are perfectly well-defined.

Coleman has proved exact equivalence between (5.14) and (5.1) provided (among other things);

$$\frac{\lambda}{8\pi m^2} = \frac{1}{1 + g/\pi}\,. \tag{5.15}$$

This formula has the very interesting feature that perturbation theory in g corresponds to λ/m^2 around 4π, and perturbation in λ/m^2 corresponds to the extreme strong coupling limit in the Thirring model. One now notices in the mass formula (5.10) that for increasing λ/m^2 fewer and fewer values of n are allowed, until, at $\lambda/m^2 = 4\pi$, $n = 1$ itself becomes forbidden. But this precisely corresponds to $g = 0$. The interpretation is that the quantum sine-Gordon soliton must be identified with the fermion of the massive Thirring model, and, at λ/m^2 just below 4π, the "elementary" particle of the sine-Gordon theory is a weakly bound soliton-antisoliton pair. This coincidence is quite remarkable: recall that the range of validity of the approximation that leads to the mass formula (5.10) is only $\lambda/m^2 \ll 1$. However, it now appears that formula (5.10) is exact as far as mass ratios are concerned. This conjecture is based on the above mentioned coincidence at $\lambda/m^2 = 4\pi$, and on higher order exact corrections in perturbation theory for the ratios M_2/M_1 at $\lambda/m^2 \ll 1$, and M_1/M_s at $g \ll 1$. These calculations are described in [6]. This suggests that the sine-Gordon theory is similar to the hydrogen atom, for which the Bohr-Sommerfeld quantization rules give the energy levels exactly.

Further remarks and problems. The quantum sine-Gordon equation should be a rich mine of problems in mathematical physics. We shall mention only a few:

The mass formula (5.10) is only a part of the quantum theory of the sine-Gordon equation. The rest of the information involves processes somewhat more complicated to analyze, even semi-classically: these are all scattering processes, between all possible kinds of particles (solitons, anti-solitons, and bound states). This will give rise to phase shifts, and lead to a better understanding of the formalism for the scattering of extended objects in quantum field theory. Indeed, an explicit conjecture for the exact soliton-soliton (and soliton-anti-soliton) S-matrix has already been put forward by Faddeev et al. [12] for special values of the coupling constant, and has been verified in first non-trivial order by Gervais and Jevicki [13].

Another very interesting problem would be to use more thoroughly the infinite set of conservation laws [10] in the quantization procedure. Indeed, Pauli used the conservation of the Lenz vector to study the non-relativistic hydrogen atom before the Schrödinger equation was discovered.

6. **Functional WKB and the Method of Trace Identities.** In this section we present another model in $(1 + 1)$ dimensions which we believe is a separable (perfect) system and is fundamentally non-classical. New tools will be used to approximate the functional integral appearing in the WKB method; the trace identities of an associated scattering problem. These are the same trace identities familiar to most readers for their power in constructing the invariant integrals for classical systems. What we are after is a way to improve the efficiency of the functional WKB, and a natural extension to higher dimensions.

Some problems with the functional WKB. From the sine-Gordon system one sees that the functional WKB is a powerful approximate method, but it does require a great deal of classical input. Minimally, a collection of classical static and time-dependent solutions have to be found and some knowledge of how complete a list of solutions it is. In the case of $(1 + 1)$ dimensions, we have available a number of perfect (separable) classical systems which are both well studied and go over to sensible field theories. Since they are perfect, we have complete information of the classical problem. Suppose the system is not perfect or is in some higher number of dimensions. At first it seems one can only guess at particular solutions. As the models become more complex, the zero mode symmetry problem becomes more acute, making the analysis almost intractable. There is a way out of these difficulties which is very

powerful in (1 + 1) dimensions and extends in a natural way to higher dimensions; namely, one may use the trace identities of an appropriately chosen associated scattering problem. This has the following advantages:

(1) It is a method that is deductive, for rather than guessing all the classical solutions it involves only the choice of a clever scattering problem.

(2) It avoids the zero eigenvalue problem.

(3) It gives analytic conditions on the energy spectrum, in some cases merely a simple algebraic equation.

(4) It reconstructs the set of classical solutions corresponding to stable periodic orbits (or all static ones as a special case) by the Gelfand-Levitan method.

An illustration of these points is a field theory model which is used quite frequently as a theoretical laboratory by physicists, but is unfamiliar to most mathematicians. It is a fermion model, and so has no classical analog. Below, we define the model and outline how trace identities come about and are used in the path integral. We believe this technique to be a rich mine for non-linear analysis and can see no other practical way to higher dimensions.

Trace Identities and the Gross-Neveu model (1 + 1) *dimensions.* The Lagrangian for this system is

$$\mathscr{L} = i\bar{\psi}\partial\!\!\!/\psi + \tfrac{1}{2}g^2(\bar{\psi}\psi)^2 \tag{6.1}$$

where $i\bar{\psi}\partial\!\!\!/\psi \equiv \sum_{k=1}^{N} i\bar{\psi}_k i\partial\!\!\!/\psi_k$; $\bar{\psi}\psi \equiv \sum_{k=1}^{N}\bar{\psi}_k\psi_k$, g is a dimensionless parameter, and $\partial\!\!\!/ = \gamma_0\partial_0 - \gamma_1\partial_1$; γ's being the usual Dirac matrices.

The vacuum functional for this system is

$$\langle 0|0\rangle = \int \mathscr{D}\bar{\psi}\mathscr{D}\psi \exp(i \int \mathscr{L}) \tag{6.2}$$

where it is not necessary for the argument to know what are the rules for integrating over anticommuting fields ψ. By the rules of Section 1, this may be rewritten as

$$\langle 0|0\rangle = \int \mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}\sigma \exp[i \int -\tfrac{1}{2}\sigma^2 + \bar{\psi}(i\partial\!\!\!/ - g\sigma)\psi]. \tag{6.3}$$

σ is a scalar field obeying Section I rules. By introducing an additional "auxiliary" field $\sigma(x)$ and another functional integration over it, we have a form which is quadratic in σ and ψ separately. Equation (6.3) is an equivalent field theory to (6.2). One can go a step further and integrate the fermion fields out of the problem completely. This will be done in two ways. The first is purely formal and we just state the result:

$$\begin{aligned}\int \mathscr{D}\bar{\psi}\mathscr{D}\psi \exp[i \int \mathscr{L}] &= \int \mathscr{D}\sigma \det(i\partial\!\!\!/ - g\sigma) \exp(-i/2\,\sigma^2) \\ &= \int \mathscr{D}\sigma \exp i[-\sigma^2/2 \\ &\quad + \operatorname{tr} \ln(i\partial\!\!\!/ - g\sigma)].\end{aligned} \tag{6.4}$$

The system described by (6.4) is a perfect system of a type not yet studied in the literature. It is an extremely complex model and has a rich spectrum. To see this one uses the trace identities of an associated scattering problem, after doing the functional fermion integration in another way.

It can be shown [7] that $\operatorname{tr} \exp[-iHT]$, the fundamental object of the WKB method, can be written as:

$$\begin{aligned}\exp[-iHT] = \sum_{\{n\}} C(N, \{n\}) \int \mathscr{D}\sigma \exp \Bigg[i \int_0^T dt \\ \times \int_{-\infty}^{+\infty} dx \left(\frac{-\sigma^2}{2} + iN\varphi(\sigma) - i \sum_k n_k \alpha_k(\sigma) \right) \Bigg].\end{aligned} \tag{6.5}$$

where $\varphi \equiv \sum_i \alpha_i$, $0 \leqq n_i \leqq 2N$, $N =$ number of fermion species, over all $\sigma(x, t + T) = \sigma(x, t)$ and $|\sigma(x, t)| \to |\sigma_0|$ as $|x| \to \infty$; $C(N, \{m\})$ are some binomial coefficients and the α_i are the Floquet indices for the Dirac problem: $(i\partial\!\!\!/ - g\sigma)\psi_j = 0$ where $\psi_j(x, t + T) = \exp(-i\alpha_j)\psi_j(x, t)$. The problem is to find the stationary phase points of (6.5). To simplify, we will ask only for time independent stationary points, which means that we have to solve the functional equation:

$$\begin{aligned}\frac{\delta}{\delta\sigma(x)} \Bigg\{ - \int_{-\infty}^{+\infty} \frac{Z}{2} (\sigma^2(x') - \sigma_0{}^2)\, dx' \\ + N \sum_{i=0}^{\infty} [\omega_i(\sigma) - \omega_i(0) - n_0\omega_o(\sigma)] \Bigg\} = 0\end{aligned} \tag{6.6}$$

where $\alpha_i \equiv \omega_i T$, Z is some renormalization constant and the argument (0) refers to the vacuum solution. This is a bit schematic but serves to illustrate the method, since only the functional structure is important. The strategy for solving (6.6) and systems like it is to replace the functional variation on σ by two independent variations on the reflection coefficient and bound state energies of an associated scattering problem. The system will be Schrödinger, $-\psi_{xx} + U(x)\psi = k^2\psi$, with the asymptotic data $S\{r(k), k_l,\ l = 1, \cdots, m\}$. In what follows we assume the reader is familiar with the formalism of inverse scattering problems or consults [10] and [14].

Trace identities and dispersion relations. In the standard notation, one can define the function $a(k) \equiv (1/2ik)(f_x g - g_x f)$ where $f; g$ are the solutions to the Schrödinger equation such that $f(x, k) \sim \exp(+ikx)$, $(x \to +\infty)$, $g(x, k) \sim \exp(-ikx)$, $(x \to \infty)$. The function $a(k)$ has a dispersive representation:

$$a(k) = \exp\left\{ \frac{1}{2i\pi} \int_{-\infty}^{+\infty} \frac{\ln(1 - |r(q)|^2}{k - q} \, dq \right\} \times \prod_{c=1}^{m} \frac{k - ik_c}{k + ik_c} \quad (\operatorname{Im} k \neq 0) \tag{6.7}$$

and an asymptotic expansion:

$$\ln a(k) = \sum_{n=1}^{\infty} \frac{c_n}{k^n}$$

where

$$C_{2j+1} = \frac{1}{2i\pi} \int_{-\infty}^{+\infty} k^{2j} \ln(1 - |r(k)|^2) \, dk - \frac{2}{2j+1} \sum_{c=1}^{m} (ik_c)^{2j+1} \tag{6.8}$$

where from other considerations, the first few C_{2j+1}'s also have the representation

$$C_1 = -\frac{1}{2i} \int_{-\infty}^{+\infty} U(x) \, dx; \qquad C_3 = -\frac{1}{8i} \int_{-\infty}^{+\infty} U^2(x) \, dx, \ \cdots . \tag{6.9}$$

The relations involving the C's are the trace identities.

Returning to the original problem, we examine the system $(i\partial\!\!\!/ - g\sigma)\psi(x) = 0$, a Dirac system in the potential $\sigma(x)$, for the moment unknown. Convert this to a Schrödinger problem:

$$\psi_{xx} - [g^2(\sigma^2 - \sigma_0{}^2) \pm g\sigma_x]\psi = -(\omega^2 - g^2\sigma_0{}^2)\psi$$

Now: identify $k^2 \equiv \omega^2 - g^2\sigma_0{}^2$ and the potential $U(x) \equiv g^2(\sigma^2 - \sigma_0{}^2) \pm g\sigma_x$ with boundary condition $\sigma(x) \to \sigma_0$ as $|x| \to \infty$. Use the trace identities and dispersion relation of the preceding paragraph to re-express the variational problem of (6.6) as a function of the scattering data for this Schrödinger system:

$$
\begin{aligned}
&-\frac{1}{2}\int_{-\infty}^{+\infty}(\sigma^2-\sigma_0{}^2)\,dx=\frac{1}{2\pi g^2}\\
&\times\int_{-\infty}^{+\infty}\ln\left[1-|r(k)|^2\right]dk+\frac{2}{g^2}\,k_0.
\end{aligned}
\tag{6.10}
$$

It can be shown that the second piece of (6.6) is

$$
\begin{aligned}
&N\sum_{i=0}^{\infty}\left[\omega_i(\sigma)-\omega_i(0)\right]\\
&\quad=N\left(-\int\delta\,\frac{d\omega}{\pi}+\omega_0(\sigma)+g\sigma_0\right)
\end{aligned}
\tag{6.11}
$$

where

$$
\begin{aligned}
\delta&=\frac{a^*(k)}{a(k)}\\
&=\frac{1}{2\pi}\,P\int_{-\infty}^{+\infty}\frac{\ln[1-|r(q)|^2]}{k-q}\,dq+2\tan^{-1}\frac{k_0}{k}\,.
\end{aligned}
$$

Assembling the pieces of the action in (6.6), written in the new variables $r(k)$, k_c and performing the variations implies (1) $r(k)=0$. The stationary phase points are those belonging to a potential $u(x)$ which is reflectionless. This is a general feature of perfect systems. (2) Algebraic relations from the k_c variation imply the energy spectrum $E_{n_0}=(2/\pi)g\sigma_0 N\sin(\pi n_0/2N)$ where n_0 is the number of occupied fermion states. Knowing that $U(x)$ is reflectionless and the energy spectrum, allows the reconstruction of $\sigma(x)$ by inverse scattering methods. The form is not especially illuminating, but can be found in [7].

Time dependent solutions to the stationary phase approximation of (6.9) are more difficult to find than static ones. The formalism for a Dirac or Schrödinger scattering problem in such a potential has not yet been developed, so there is no general method analogous to that of the last section. By considering the close relationship of static sector results to those of the sine-Gordon system, one looks for σ fields which are related analytically to the sine-Gordon doublet in the same way that the static σ is related to the sine-Gordon soliton. These can be found, essentially by guess. The very simple analytic form of the results [7] leads us to suspect there must exist some tractable formalism for potentials that are reflectionless and periodic in time. The system looks completely separable.

Higher Dimensions. The time dependent sector of the Gross-Neveu model already leads to a higher dimensional problem. It can be thought of as a static, two-space dimensional system, one axis of which is ordi-

nary space, the other the euclidean time axis, along which the system is periodic. The scattering problem looks very much like scattering from a periodic crystal lattice.

Trace identities for higher dimensional systems can be derived in a number of ways [15–20], the most efficient being either from a path integral representation for the partition function of the system or a direct high temperature expansion of the partition function [21]. The strategy for analyzing such systems is the same as that for the Gross-Neveu model. A basic difficulty, which at present we do not know how to solve, is determining the independent scattering data for higher dimensional systems. Since remarkably simple, topologically stable "lump" solutions to very complex Yang-Mills systems are now known [2], we expect there are simple constraints on the scattering data that determine them. This is the outstanding problem in the classical inverse technique for such complex, higher dimensional systems.

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