INTRODUCTION TO MATHEMATICAL TECHNIQUES
FOR CONTINUOUS PROBLEMS

Certain special, and very important, nonlinear partial differential equations can be understood, and at times solved, through inverse spectral methods. The term “solve” refers to something more explicit than an existence proof, but—in most cases—less explicit than an integral representation such as is used in the classical solutions of the linear heat or wave equations. Many of the equations in question admit the multi-soliton solutions which we mentioned in the preface. These particular solutions can be expressed in terms of elementary functions or, under periodic boundary conditions, in terms of the Riemann theta function. The general initial value problem, however, can only be reduced to a sequence of linear problems for which closed-form solutions are not to be expected.

There are several types of mathematical problems which have been treated in the literature:

1. To derive (preferably, from a unifying point of view) the special equations.
2. To understand the auxiliary linear problems encountered in step 1. (This usually means: study a direct and inverse spectral problem of novel form.)
3. To use the information from step 2 in a qualitative analysis of the solution of the nonlinear p.d.e., and to study the effects small perturbations have on the solution.

The papers in this section of the Proceedings only present a sample of the many novel ideas and techniques that have been developed in response to the problems just listed. In this introduction, we try to identify some of the major trends and open questions.

0. The Korteweg-deVries equation. The reader who is new to the subject will probably find it useful to keep a concrete example in mind while following the necessarily general outline we will present shortly. For this reason, we describe very briefly the basic features of the Korteweg-deVries equation, the first and still the most fundamental example of the inverse spectral methods.

0.1 The KdV equation. In suitable normalized form, the KdV equation is

\[ q_t - 6qq_x + q_{xxx} = 0. \]
It models weakly nonlinear, weakly dispersive waves in one space dimension which propagate predominantly in one direction (see Newell's article). From a theoretical standpoint, the initial-boundary value problem is appropriate: given \( q(x, 0) \), find \( q(x; t) \) for \( t > 0 \), subject to

\[(i) \quad q(x, t) \to c_{\pm} \text{ as } x \to \pm \infty \quad (c_{\pm} = 0 \text{ being the most important; for } c_{\pm} \neq 0, \text{ see E. Ja. Hruslov, Matem. Sb. 99 (1976), 261-281})
\]
or \[(ii) \quad q(x, t) \text{ is periodic in } x.
\]

The asymptotic behavior of \( q(x, t) \) for large \( t \) is of great practical importance.


0.2 Isospectral flows. The KdV equation has a surprising connection with a certain linear eigenvalue problem. We first describe the general setting and then show how the KdV enters.

Consider the eigenvalue problem

\[ Ly = -y'' + qy = \lambda y, \]

where \( q(x) \) satisfies one of the boundary conditions of 0.1. Let \( \lambda \) be an eigenvalue, with eigenfunction \( \phi \) normalized by \( \int \phi^2 \, dx = 1 \) (integration over \(-\infty < x < +\infty\) or over one period of \( q \)). Now we ask: how can \( q \) be changed while keeping \( \lambda \) as an eigenvalue? Put somewhat differently, we want conditions on a one-parameter family \( q(x; t) \) which guarantee that

\[-\phi''(x; t) + q(x; t)\phi(x; t) = \lambda \phi(x; t)\]

for a \( t \)-dependent, normalized \( \phi(x; t) \), but with eigenvalue \( \lambda \) independent of \( t \).

We derive such a condition. Let \( (y, z) \) denote the inner product \( \int y(x)z(x) \, dx \). Differentiate the identity \( \lambda = (L\phi, \phi) \); since \( d\lambda/dt = 0 \), we get 0 = \( (L\phi, \phi) + (L\phi, \phi) + (L\phi, \phi) \). Now \( (L\phi, \phi) = (\phi, L\phi) = \lambda \phi, \phi \phi) = 1/2 \lambda (\phi, \phi) = 0 \) since \( (\phi, \phi) = 1 \). Similarly, one finds \( (L\phi, \phi) = 0 \). Hence \( (L\phi, \phi) = 0 \). Inasmuch as \( L \) is the operator of multiplication by \( q_t = \partial q/\partial t \), we have the following criterion:

The deformation \( q(x; t) \) preserves the eigenvalue \( \lambda \) precisely when

\[ \int q(x; t)\phi^2 \, dx = 0. \]

A family \( q(x; t) \) which preserves all eigenvalues of \( L \) is said to be an isospectral flow. To construct such flows explicitly, one must arrange for
the derivative \( q_t \) to be orthogonal to the squared eigenfunction, \( \phi^2(x; t) \), for all values of \( t \). At first glance, this requirement would seem to impose a hopelessly tangled relationship between \( q(x; t) \) and the eigenfunction \( \phi(x; t) \) (which depends on \( q \)). We now explain how this difficulty can be circumvented.

0.3 Squared eigefunction equations. It will be shown that whenever \( \lambda \) is an eigenvalue, with eigenfunction \( \phi \), of an operator

\[
L = -\frac{d^2}{dx^2} + q,
\]

then

\[
\int (6qq_x - q_{xxx})\phi^2dx = 0.
\]

This is a property of the operator (1), and it is irrelevant whether or not \( q \) is imagined to depend on a parameter \( t \). If we consider a family \( q(x; t) \) which, as function of \( x \) and \( t \), satisfies the KdV equation,

\[
q_t = 6qq_x - q_{xxx},
\]

then

\[
\int q_t\phi^2dx = 0
\]

will hold because of (2): the KdV equation is an isospectral flow.

Relation (2) is only one of an infinite set of similar equations which are derived as follows. Straightforward computation shows:

**Lemma.** When \(-\phi'' + q\phi = \lambda\phi\), then \(-1/4\psi'' + q\psi' + 1/2 q_x\psi = \lambda\psi', \) where \( \psi = \phi^2 \).

In other words, the squares of eigenfunctions satisfy an equation of the form

\[
M\psi = \lambda N\psi
\]

with \( M \) and \( N \) skew-symmetric operators: \( (M\psi_1, \psi_2) = -(\psi_1, M\psi_2) \) etc.

It follows directly from the lemma that

\[
1/2 q_x\psi = [-1/4\psi'' + q\psi - \lambda\psi]',
\]

so that \( q_x\phi^2 \) is a perfect derivative, whence

\[
\int q_x\phi^2dx = 0
\]

under the boundary conditions allowed. This provides a trivial example
of an isospectral flow: if \( q_t = q_x \), or \( q(x; t) = q(x + t) \), then the basic criterion \( \int q \phi^2 = 0 \) is verified.

To obtain the promised result about the KdV equation, integrate by parts in (4) (boundary terms vanish):

\[
0 = \int q_x \phi^2 \, dx = - \int q(\phi^2)_x \, dx.
\]

Now \( (\phi^2)_x = N\phi^2 = 1/\lambda M\phi^2 \) in the notation of (3), and by skew-symmetry of \( M \) we find

\[
0 = - \frac{1}{\lambda} \int q \cdot M \phi^2 \, dx = \frac{1}{\lambda} \int M(q) \cdot \phi^2 \, dx.
\]

As \( M q = -1/4 q_{xxx} + 3/2 q q_x \), we arrive at (2). Repetition of this procedure yields an infinite sequence of isospectral flows governed by the so-called higher KdV equations.

Reference: For detailed information about the KdV equation, see the survey by R. Miura, The Korteweg-deVries Equation; A Survey of Results, SIAM Review 18 (1976), 412–459.

0.4 Scattering theory. We have explained that the KdV equation preserves the eigenvalues of the auxiliary linear operator \(-d^2/dx^2 + q(x)\). These eigenvalues correspond to eigenfunctions satisfying certain boundary conditions; if the interval in question is \(-\infty < x < \infty\), the eigenfunctions will necessarily decay exponentially at \( \pm \infty \). The complete spectral theory of \( L \) will, in this case, involve generalized eigenfunctions as well. These are not square-integrable, being associated with the continuous spectrum of \( L \). It would take us too far afield to discuss this aspect of the KdV theory in any detail, but the fundamental idea (of C. Gardner, J. M. Greene, M. Kruskal, and R. Miura) should at least be mentioned.

Suppose that \( q(x) \to 0 \) sufficiently fast as \( x \to \pm \infty \). Then the equation \(-y'' + qy = \lambda^2 y\) has, for every real \( k \), a solution uniquely specified by the boundary condition

\[
y(x) \sim e^{-ikx} \text{ as } x \to -\infty.
\]

Notice that \( e^{ikx} \) is a solution of the “unperturbed” equation in which \( q(x) = 0 \). At \( x = +\infty \), the solution \( y(x) \) is a superposition of the unperturbed solutions, \( e^{ikx} \) and \( e^{-ikx} \):

\[
y(x) \sim a(k)e^{-ikx} + b(k)e^{ikx} \text{ as } x \to +\infty.
\]

The quantities \( a(k), b(k) \) arise in the study of the quantum-mechanical scattering problem governed by the Hamiltonian \(-d^2/dx^2 + q\). For this reason, they are referred to as “scattering data.”
The inverse spectral theory of the operator $L$ asserts that the coefficient $q(x)$ can be reconstructed from a knowledge of $a(k)$, $b(k)$, of the proper eigenvalues (if any), and of certain normalization constants associated with the (proper) eigenfunctions.

What happens to the generalized eigenfunctions of

\[-y'' + q(x; t)y = k^2y,\]

when $q(x; t)$ satisfied the Korteweg-deVries equation? Elaborating on the arguments of section 0.3, one can show that

(i) $a(k; t)$ is independent of $t$,

(ii) $b(k; t) = b(k; 0) \exp(8ik^2t)$.

The normalization constants referred to above are given by a formula analogous to (ii).

These considerations may be summarized as follows: Certain quantities arising in the spectral theory of $-d^2/dx^2 + q$ completely determine the coefficient $q$. If $q$ changes according to the KdV equation, these spectral quantities change with $t$ in an extremely simple manner.

0.5 Inverse scattering solution. The results listed in 0.4 lead to the following prescription for solving the KdV equation.

**Problem:** Given $q(x; 0)$, solve $q_t - 6qq_x + q_{xxx} = 0$ for $t > 0$.

**Solution:** (1) Find the scattering data, eigenvalues, and normalization constants for the operator $-d^2/dx^2 + q(x; 0)$.

(2) These quantities, for $d^2/dx^2 + q(x; t)$, can now be predicted without knowledge of $q(x; t)$: the eigenvalues and $a(k; t)$ are in fact constant, while $b(k; t)$ and the normalization constants are immediately found from the formula in 0.4.

(3) Knowing these data at time $t$, use the inverse spectral theory to reconstruct the potential, $q(x; t)$.

In order to get explicit solutions by this scheme, one would have to solve, in closed form, the eigenvalue problem of step 1, and then the inverse problem of step 3. (This final step reduces to the solution of a Fredholm integral equation with kernel built from the Fourier transform of $b(k; t)/a(k; t)$.) These explicit computations have been done for the special case $b(k; 0) = 0$. The resulting solutions of the KdV are determined entirely by the eigenvalues and normalization constants of the operator $L$. These closed form solutions are the multi-soliton wave forms, and the preservation of soliton identity upon collisions is the physical counterpart of the constancy of the proper eigenvalues.
Although this solution procedure does not yield a closed form solution of the general initial value problem, it does allow qualitative analysis for large $t$: the difficult asymptotic methods are described by M. Ablowitz and H. Segur (to appear).

References: There are several surveys devoted to various portions of the theory we have sketched. Not many develop all aspects of the inverse-scattering solution from first principles; one which does is M. Toda, Studies of an Exponential Lattice, Physics Reports C, 19 (1975). See also L. D. Faddeev, The Inverse Problem of the Quantum Theory of Scattering II (in Russian), Current Problems of Mathematics, 3 (1974), 93–180. This survey, which should be translated soon in Journal of Soviet Mathematics, treats 1- and 3-dimensional inverse scattering problems as well as isospectral flows.

0.6 Other Boundary Conditions. In § 0.4 and 0.5 it was assumed that $q(x; t) \to 0$ at $x = \pm \infty$. Entirely new techniques are used when $q(x; t)$ is required to be periodic in $x$. In this instance the operator $L = -\frac{d^2}{dx^2} + q$ is known as Hill’s operator. To obtain some feeling about the necessity of new techniques reconsider the case $q(x; t) \to 0$ as $|x| \to \infty$. At any time $t$, the operator $L = -\frac{d^2}{dx^2} + q$ reduces to the $q$-independent operator $L_0 = -\frac{d^2}{dx^2}$ as $|x|$ becomes large. Because of this circumstance, it seems natural to connect the unknown operator $L(t) = -\frac{d^2}{dx^2} + q(x; t)$ with the $q$-independent reference operator $L_0$. This connection ultimately makes the linearization of the KdV equation possible. If, on the other hand, $q(x; t)$ is required to be periodic in $x$, a special location $x_0$ does not exist for which $q(x_0; t)$ is known for all $t$. Thus, there is no natural choice of reference operator, and scattering theory seems inapplicable.

The first steps towards the analysis of the periodic KdV equation were taken by P. D. Lax, in a lecture at the 1972 Potsdam Conference on Nonlinear Waves. Since then, great progress has been made by groups in the USA and the USSR. Certain aspects of this work are described in Novikov’s paper.


A complete study of the periodic KdV was recently published by McKean and Trubowitz, Comm. Pure Appl. Math. 29 (1976), 143–226.

For the remainder of this introduction, we give an overview of some of the work done on the mathematical problems listed at the beginning.
1. Deriving isospectral flows.

1.1 Given an equation—is it isospectral? It would be invaluable, for practical applications, to have a constructive method for deciding whether a given equation is isospectral for some eigenvalue problem, and for finding the eigenvalue problem if it exists. Progress on this question has been delayed by two factors: (1) the computations quickly become incredibly complicated, and (2) there are so many conceivable eigenvalue problems that no one prescription can encompass them all.


1.2 Squared eigenfunction equations. Our derivation of the KdV equation as isospectral flow made use of the equation

\[ M\psi = \lambda N\psi, \]

\[-1/4\psi''' + q\psi' + 1/2 q_x\psi = \lambda\psi,\]

satisfied by the squares of eigenfunctions of \(-d^2/dx^2 + q\). Entirely analogous considerations apply to other linear operators as well. Kaup and Newell have obtained descriptions of isospectral flows associated with the Dirac system, \(Lv = \xi v\), where \(v\) is a 2-vector and

\[ L = \begin{pmatrix} id/dx & -iq \\ ir & -id/dx \end{pmatrix}, \]

basing their arguments on generalized Fourier expansions in quadratic products of eigenfunctions. Their approach is very well suited to the study of perturbations of isospectral equations. These matters are discussed at length by Newell in these Proceedings. A very general variant of this technique has been developed by Calogero and Degasperis (see their articles in Nuovo Cimento).

1.3 Isospectral flows as integrability conditions. We could have deduced the isospectral nature of the KdV equation in a different way. Suppose we want

\[ -\phi'' + q(x; t)\phi = \lambda\phi, \]

with \(\lambda\) independent of \(t\). The eigenfunction \(\phi\) will presumably change with \(t\) in a special way. Make the ansatz that this temporal evolution is of the form

\[ \phi_t = A\phi + B\phi_x \]

where \(A\) and \(B\) are functions of \(q\) and \(\lambda\), and look for conditions on \(A\) and \(B\) under which (5) and (6) are consistent. That is, compute \(\phi_{xxt}\).
from (5) and $\phi_{txx}$ from (6), simplify, and require that the two expressions be equal. The result is:

$$ A = -\frac{1}{2} B_x. $$

Taking for $B$ different polynomials in $\lambda$, one obtains each of the higher KdV equations from (7). One can adapt this method to most eigenvalue problems, even though the computations may get out of hand. Integrability conditions like (7) are being subjected to intensive study, as it now appears that they can be interpreted as the vanishing of the curvature form associated to certain connections in fibre bundles.

References: For a derivation of several important isospectral flows, see Ablowitz, Kaup, Newell, and Segur, Phys. Rev. Lettrs. 31 (1973), 125–127. For the differential geometry: forthcoming papers of R. Hermann, and of Crampin, Pirani and Robinson.

1.4 The Lax Equation. Let $L(t)$ be a one-parameter family of operators whose eigenvalues are constant in $t$. To this end it is sufficient that the operators $L(t)$ be unitarily equivalent to each other, so that

$$ U(t)^{-1}L(t)U(t) = L(0) $$

for some family $U(t)$ of unitary operators. The condition is not necessary, but is in fact satisfied in some form by most of the known operators associated with isospectral flows. Differentiation of (8) yields Lax's equation,

$$ L_t = [B, L], $$

where the skew-adjoint family $B(t)$ is defined by $U_t = BU$. As an example, take $L$ as before, and $B = -4(d^3/dx^3) + 3q(d/dx) + (d/dx)q$; then (9) yields the KdV equation.

Lax's equation leads very efficiently to the time-evolution of the scattering data (the formulas in § 0.4). In fact, one can show quite abstractly that equations of the form (9) become trivial when transformed to scattering data. Equation (9) is also fundamental in the study of space-periodic solutions [Novikov's article in the Proceedings].

1.5 Hamiltonian mechanics. An extremely important feature of the known isospectral equations is this: they are Hamiltonian equations with respect to a certain skew-symmetric bilinear form on the set of coefficients in the eigenvalue problem.

Consider, again, the operator $-d^2/dx^2 + q$. Let $F(q)$, $G(q)$ be two functionals of $q$, and denote by $F'$, $G'$ their (functional, or Gateaux) derivatives. Define

$$\{F, G\} = \int F'(q) \frac{d}{dx} G'(q) \, dx.$$ 

$\{,\}$ is skew-symmetric: $\{G, F\} = -\{F, G\}$, and it can be shown to satisfy the Jacobi identity. One thinks of this bilinear function as generalization of the Poisson bracket familiar from classical mechanics. Hamilton’s equations, derived from a Hamiltonian $H(q)$, then are

$$q_t = \frac{dH'}{dx} (q).$$

The $KdV$ equation is of this form with $H(q) = \int (q^3 + 1/2 q_x^2) \, dx$. As is the case in classical mechanics, if $\{F, H\} = 0$, then $F(q)$ is a constant of motion for (10).

Reconsider equation (2), $\int (6qq_x - q_{xxx}) \phi^2 \, dx = 0$, from this point of view. $6qq_x - q_{xxx} = d/dx H'(q)$, with $H$ given above, and $\phi^2$ is the derivative of $\lambda$ with respect to $q$ (this was essentially proved in 0.1). Hence, equation (2) could have been written in the form $\{H, \lambda\} = 0$, and one sees once again, this time via Hamiltonian mechanics, that the eigenvalues of $L$ are constants of motion for the $KdV$ equation.

Much more is true. The Poisson brackets of distinct eigenvalues $\lambda$, $\mu$ vanish: $\{\lambda, \mu\} = 0$ (one says that $\lambda$ and $\mu$ are in involution). There are, in fact, infinitely many independent functionals $I_j(q)$ which are mutually in involution, and whose values constitute (in a very precise sense) half the information needed to determine $q$. One can generate these $I_j$ quite directly from the operator $L$, and this leads to yet another way of obtaining isospectral flows. We now outline this approach.

The operator $(L - z)^{-1} - (L_0 - z)^{-1}$ is of trace class; here $z$ is complex and $L_0 = -d^2/dx^2$. The analytic function

$$\text{Trace}[(L - z)^{-1} - (L_0 - z)^{-1}]$$

may be expanded asymptotically in powers of $1/\sqrt{z}$, $z \to \infty$, and the coefficients may be computed as functions of $q$. These are the $I_j(q)$ just mentioned. It is remarkable that they are mutually in involution, and that their Poisson brackets with the eigenvalues vanish as well. Thus, any $I_j$ can be used as a Hamiltonian to generate an isospectral flow ac-
cording to (10); the two simplest equations obtained this way are
$q_t = q_x$ and the KdV.

This recipe can be applied to other operators besides $L$: find a Poisson bracket which puts the eigenvalues in involution, compute the coefficients in the expansion of the trace-difference (11), and use them to generate isospectral Hamiltonian equations analogous to (10). According to Faddeev (ref. in § 0.5), the inverse-scattering solution of the nonlinear Schroedinger equation was discovered in precisely this fashion.

One suspects that the existence of a Poisson bracket which puts the eigenvalues in involution is intimately connected with the solvability of the inverse scattering problem, but this speculation, and many related ones, remain to be clarified.


1.6 Volterra Factorization. After one of the methods described above has led from a linear operator to an isospectral equation, there still remains the often difficult task of solving the inverse scattering problem. This would be necessary, in general, for the derivation of multi-soliton formulas, and certainly for any deeper study of the solution. A beautiful idea of Zaharov and Šabat enables one to bypass certain of these obstacles. They take as starting point not a linear operator, but the Marčenko equation of inverse scattering theory, and derive simultaneously: suitable linear operators, associated isospectral flows, and the algebraic systems satisfied by multi-soliton solutions. We must refer the reader to the original paper for details and examples: V. E. Zaharov and A. B. Šabat, Funk. Anal. Prilož. 8, No. 3 (1974), 43–54.

1.7 Riemann Surfaces. The recent realization (see Novikov’s article), that the known space-periodic isospectral flows are most naturally viewed in the language of Riemann surfaces, has evolved into yet another scheme for generating such flows. I. M. Kričever has shown how the construction of a meromorphic function with specified poles and behavior at infinity leads to a class of linear operators, associated isospectral flows, and formulas for periodic multi-soliton solutions. His ideas appear to be the periodic version of the Zaharov-Šabat construction mentioned in § 1.4, and there is undoubtedly a still more general framework which encompasses both the periodic and the rapidly de-
creasing boundary conditions. A few more details of Kričever's work are given in Novikov's article.

2.1 Qualitative behavior of solutions. The previously described work has as its primary goal the derivation of isospectral flows. Explicit formulas for periodic or rapidly decaying multi-soliton solutions are usually byproducts of these methods. As we stated at the outset, however, the inverse-scattering technique has not yielded explicit formulas for any other type of solution. One can at best hope for some sort of asymptotic description of the non-soliton component. It is generally believed that this part of the solution will evolve into a slowly varying, decaying wave train of the kind familiar from the theory of linear dispersive waves (see Whitham's book cited in § 0.1). The decay rates will differ from those of linear problems, however, and there will be intermediate asymptotic regions which have no linear counterparts. The analysis relies heavily on the inverse-scattering formalism and is quite intricate and difficult. Details have been worked out for the \textit{KdV} by Ablowitz and Segur (to appear in Studies in Applied Math.), and the theoretical results agree very well with delicate measurements on water waves (Hammack and Segur, to appear). A very exciting feature of this work is the prospect of a classification of the distinguishable asymptotic regimes in the evolution of a nonlinear wave train. Further discussion of these matters is found in Newell's article.

Rather different qualitative information is obtained through the existence and uniqueness theorems of the "pure" mathematician. This point of view was not represented at the conference, and in any case, not much work on the kind of equations covered by the inverse-scattering method is available. The study of nonlinear dispersive wave equations offers interesting mathematical problems, because the functional analysis techniques which have been so highly developed for, say, parabolic or hyperbolic problems often do not apply. An idea of the wealth of open questions may be gotten from the lectures of B. Benjamin and the abstracts of J. Bona in Lectures in Applied Mathematics, 15, A. C. Newell, ed., AMS, Providence, R.I., 1974.

2.2 Perturbation theory. All the remarkable features of the equations described so far disappear under the slightest perturbation: the explicit multi-soliton formulas, the inverse-scattering linearization, the existence of constants of motion, the possibility of exact asymptotic analysis, and so on. Still, many equations which are probably not integrable by inverse scattering do support soliton-like solutions. These pseudo-solitons, rather than interacting without distortion, exhibit extremely complicated behavior. They may fuse, split, oscillate about each other; they
may be created out of apparently patternless wavelets or decay into random wiggles. Such phenomena are described by several authors in this volume (Bullough and Caudrey, Jackson, Morales and Lee, Maxon, Costabile and Parmentier). The analysis of interacting solitary waves is perhaps the most important open problem in the study of one-dimensional nonlinear dispersive waves. A first step towards this goal is surveyed by Newell. He considers the effect of small perturbations of isospectral equations on the evolution of the scattering data. It is difficult to incorporate the creation or destruction of solitons in this approach; these phenomena have as spectral counterpart the appearance or disappearance of eigenvalues, and this cannot be accomplished by small perturbations. Entirely new ideas are probably needed if one is to understand such drastic changes analytically.