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SOLITON-LIKE STRUCTURES IN PLASMAS*

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ABSTRACT. Studies of the generation of soliton-like structures in plasmas lead to a class of nonlinear wave equations in which an external source plays a key role. Several related problems are briefly discussed with the hope of stimulating further work on this important topic by the applied mathematics community.

At the present time the generation of spatially localized structures of the soliton-type is of great interest to plasma physicists due to both fundamental and practical reasons. On the fundamental side, the interest arises because plasmas in the laboratory as well as in the astrophysical environment tend to be found in strongly turbulent states containing a rich spectra of nonlinear waves. Since the soliton structures appear to be a general feature of nonlinear systems, it seems quite appropriate that an understanding of their basic properties may provide a much needed insight into the complicated behavior of strong plasma turbulence.

To understand the practical use of soliton-related studies, it should be recognized that there exists an extensive worldwide effort directed toward the achievement of controlled thermonuclear fusion. A central problem in this effort consists of finding a suitable technique for raising the plasma temperature up to the necessary multi-million degree level. To achieve this condition several schemes are being investigated in which one adds external energy to a plasma through a variety of channels. For example, the external energy can be delivered in the form of electromagnetic radiation, powerful electron beams, or fast neutral beams. In the process of delivering the external energy to the plasma it has been found, both theoretically and experimentally, that in some instances the external source may trigger the growth of intense localized electric fields whose features resemble the soliton structures. When such nonlinear entities appear in the plasma, they can interact strongly with the individual electrons and ions, thus leading to a significant modification of the zero-order properties of the medium (e.g., density, temperature, flow velocity) which may or may not be favorable in the achievement of fusion conditions, depending on the particular situation.

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Accordingly, the properties of soliton-like entities are actively being investigated in connection with the lower-hybrid heating of tokamaks, as well as in the design of laser-fusion experiments.

In the present brief communication we would like to bring to the attention of the applied mathematics community some of the recent mathematical problems that have been encountered in the more realistic soliton-related plasma physics studies. For this reason, most of the equations will be presented in their scaled version without reference to physical units. In addition, we shall restrict ourselves to problems in which the fundamental physical source of nonlinearity can be traced to the modification of the local value of the plasma density due to the "ponderomotive force" produced by high-frequency electric fields. Let us proceed to sketch briefly how this force arises.

When an electron, of charge $-e$ and mass m, interacts with a high-frequency field having an amplitude $E(x)$, its motion is governed by

$$\frac{dv}{dt} = -\frac{e}{m}E(x)\sin\omega t. \tag{1}$$

Assuming that the kick imparted to the electron is small, one finds that its rapidly oscillating position is approximately given by

$$x(t) \simeq x_0 + \frac{e}{m\omega^2}E(x_0)\sin\omega t. \tag{2}$$

This expression can then be inserted into $E(x)$ in (1) to demonstrate that the force which is actually felt by the electron contains a mixture of time-scales, i.e., it has both a high-frequency and a low-frequency component. To extract the low frequency part f_L, one averages the term

$$f_L = -eE\left[x_0 + \frac{e}{m\omega^2}E(x_0))\sin\omega t\right]\sin\omega t \tag{3}$$

over one cycle of the fast oscillations. Assuming, consistently, that the kick is small, one has

$$\begin{aligned} &E(x_0 + \frac{e}{m\omega^2}E(x_0)\sin\omega t) \\ &\simeq E(x_0) + \left(\frac{\partial E}{\partial x}\right)_{x_0}\frac{eE(x_0)}{m\omega^2}\sin\omega t \end{aligned} \tag{4}$$

thus,

$$f_L = -\frac{e^2}{m\omega^2}\frac{E(x_0)}{2}\left(\frac{\partial}{\partial x}E\right)_{x_0}, \tag{5}$$

and, since x_0 is the arbitrary location of the electron,

$$f_L(x) = -\frac{e^2}{m\omega^2}\frac{\partial}{\partial x}\left(\frac{|E(x)|^2}{2}\right). \tag{6}$$

To obtain the total force, F_L, exerted on the plasma, one multiplies (6) by the local density of electrons, n_0, to yield

$$F_L = -\left(\frac{4\pi e^2 n_0}{\omega^2}\right)\frac{\partial}{\partial x}\left(\frac{|E|^2}{8\pi}\right) \tag{7}$$

which is the "ponderomotive force" associated with the oscillating electric field. The effect of this force is to push electrons out of those regions where the electric field has a peak, hence it gives rise to the generation of nonlinear density cavities in the plasma. The nonlinearly modified density n is found to be given by

$$n = n_0 \exp\left\{-\frac{|E|^2}{8\pi n_0 T}\right\} \tag{8}$$

where T represents the plasma temperature.

A significant fraction of the nonlinear wave studies in plasmas have been formulated by using the small amplitude expansion of (8), i.e., by setting $n/n_0 \simeq 1 - |E|^2/(8\pi n_0 T)$ in the fundamental wave equation for the particular phenomena of interest. An example of an equation which is obtained by such a procedure is the well-known nonlinear Schrödinger equation

$$i\partial_t E + \partial_x{}^2 E + |E|^2 E = 0 \tag{9}$$

which describes the slow space-time evolution of the envelope of Langmuir waves in unmagnetized plasmas.

Equation (9) has been extensively investigated by Zakharov and Shabat [1] and Zakharov [2] who have, among other things, described the interaction and evolution of the soliton solutions of such an equation. Although the evolution of soliton solutions from (9) constitutes an interesting mathematical problem, its application to a plasma experiment is unrealistic for a variety of reasons. One of these reasons, which has occupied our research, relates to how one can generate the solitons of (9) by means external to the plasma. This is an important issue which needs to be considered if these nonlinear entities are ever to be observable, since in a plasma there are always damping mechanisms which tend to destroy the solitons. To investigate this problem, we have used [3] the related equation

$$i\partial_t E + \partial_x{}^2 E + [i\nu + |E + E_0|^2](E + E_0) = S(t) \tag{10}$$

which contains three additional effects. Firstly, there is a collisional damping factor ν. Then there is the explicit appearance of the spatially uniform external pump field E_0, which simulates the effect produced by a long-wavelength electromagnetic wave. Finally, the inclusion of the function $S(t)$ allows one to consider a variety of coupling possibilities with the external circuitry.

Two interesting possibilities which we have considered in a numerical study of (10) are: (A) set the $k = 0$ Fourier component of E equal to zero, indicating an undepletable external source of energy; and (B) allow the $k = 0$ component of E to attain a finite self-consistent value, thus permitting the possibility of depleting the pump wave. A $k = 0$ component of E can appear in (10) due to the nonlinear beating of two fluctuations k_1, k_2 such that $k_1 + k_2 = 0$. Physically, the generation of such component means that the nonlinearity is capable of modifying the amplitude and phase of the external field inside the plasma. One refers to such a process as "pump depletion", since its consequence is to lower the intensity of the net pumping field as the level of the fluctuations increases. For the undepleted pump case, we have found that in-

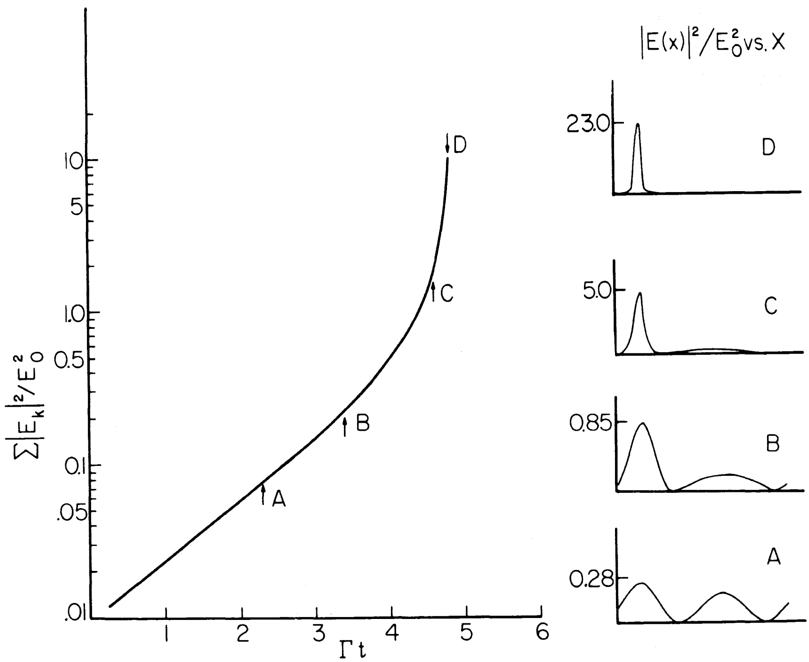

Figure 1. Time evolution of the total energy of the system and corresponding growth of soliton-like structures obtained from equation (10) for the undepleted pump case.

itially, the small random noise grows exponentially in time as predicted by the linear theory of the oscillating two-stream instability [4] (an instability driven by the pump wave and well-known to plasma physicists). However, as the amplitude of the exponentially growing noise increases, we find that spatially localized electric fields are generated, as exhibited in Figure 1. The amplitude of these nonlinear entities is observed to increase rapidly while simultaneously their width decreases steadily. It is of interest to note that structures of this type have been observed in detailed computer simulations [5] in which one follows the exact motion of a large collection of plasma particles. Some investigators in plasma physics like to refer to these localized fields as "spikons" and to the associated density changes as "cavitons".

In contrast with the unbounded growth seen for the case A study of (10), it is found that when one allows the pump wave to be depleted, then the electric field fluctuations attain a finite saturation level. In the saturated level, it is observed that soliton-like structures are present, but instead of growing steadily, they exhibit relaxation oscillations, as shown in Figure 2.

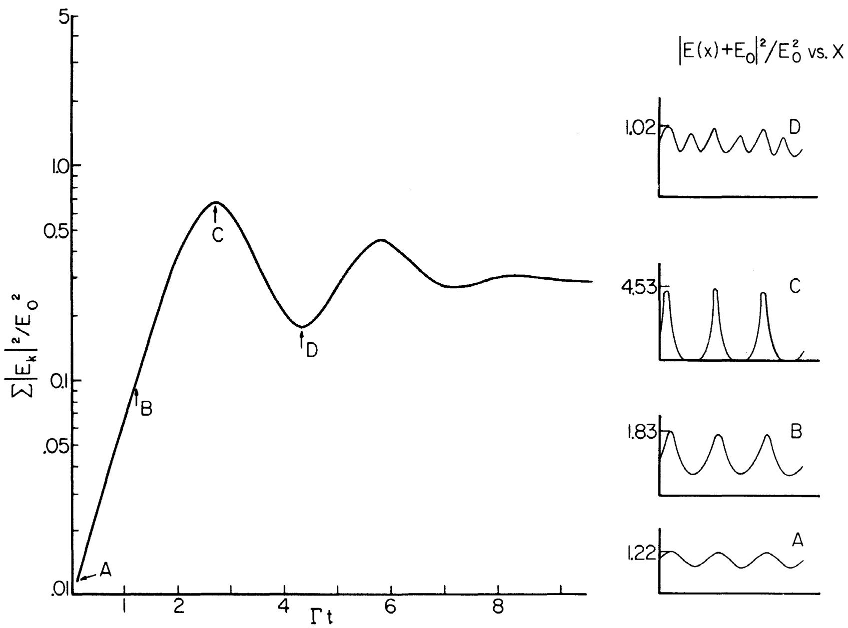

Figure 2. Time evolution of the total energy of the system and corresponding behavior of soliton-like structures obtained from equation (10) for the case in which pump depletion is allowed.

Although (10) is admittedly more realistic than (9), it still falls short of being able to describe an actual laboratory experiment. An important factor absent in (10) relates to the fact that laboratory plasmas tend to be intrinsically non-uniform. We have studied [**6**] this additional complication with the model equation

$$i\partial_t E + \partial_x{}^2 E - (-i\nu + x - p|E|^2)E = 1 \tag{11}$$

which contains the external pumping effect (through the 1 on the right hand side), and in addition, it allows the plasma to have a density profile which depends linearly on x (a situation easily attained in the laboratory). The parameter p in (11) measures the strength of the nonlinearity and satisfies $p \gtrsim 1$ for the cases of interest. A numerical study of (11) reveals that localized fields of the soliton-type can still be generated, as seen in Figure 3. However, these localized structures radiate

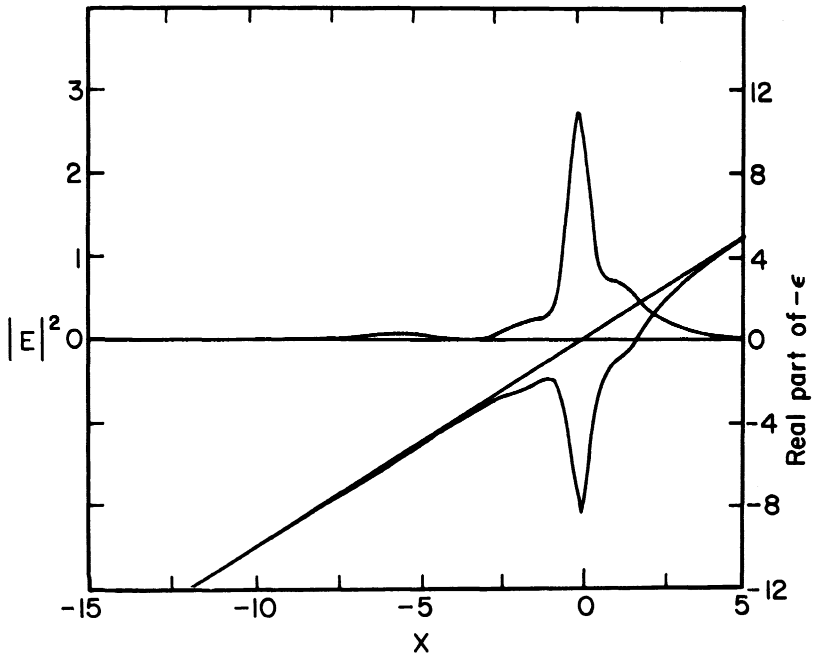

Figure 3. Characteristic localized electric field obtained from equation (11). The physical situation corresponds to the interaction of electromagnetic radiation with a nonuniform plasma. The quantity $\mathrm{Re}(-\epsilon) = x - p|E|^2$ represents the effective dielectric of the plasma, or equivalently, the scaled density profile.

continuum waves which proceed to propagate down the density gradient (i.e., $x < 0$). Accordingly, the amplitude of the localized field exhibits slow relaxation oscillations. The generation of such structures and their oscillations have been observed in the laboratory [7], and the detailed measurements are in remarkable agreement with the predictions of (11).

Buried in the physics which leads to equations (9)–(11) is the assumption that the plasma density can adjust adiabatically to the effects produced by the ponderomotive force, i.e., one assumes that the generation and evolution of the soliton-like structures occurs slowly compared to the transit time of a sound wave across the width of the structure. However, when the parameters of interest are analyzed in detail, it is found that such a condition is rarely attained in plasmas. The consequence of this result is that the realistic description of localized elec-

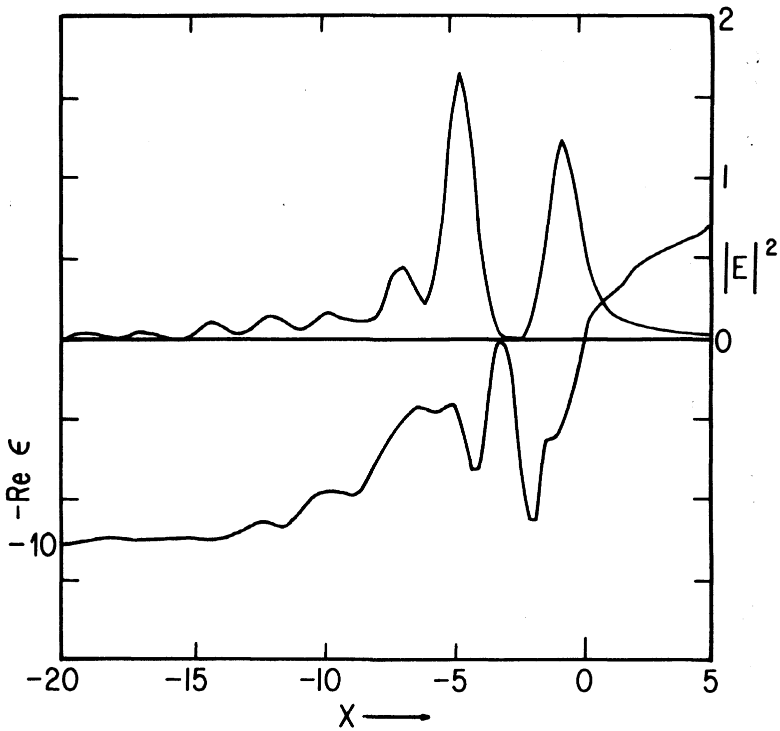

Figure 4. Localized electric fields and modified density profile obtained from equation (12),i.e., with ion inertia included. $\mathrm{Re}(-\epsilon) = x + p\tilde{n}$.

tric fields requires the use of two coupled equations. One describes the evolution of the field, E, while the other describes the evolution of the density fluctuations $\tilde{n}$. For the situation leading to (11) the set takes the form

$$
\begin{aligned}
&i\partial_t E + \partial_x{}^2 E - (-i\nu + x + p\tilde{n})E = 1 \\
&\lambda\partial_t{}^2\tilde{n} - \partial_x{}^2\tilde{n} = \partial_x{}^2|E|^2
\end{aligned}
\tag{12}
$$

in which λ parametrizes the effect of the finite sound speed. It is clear that for $\lambda \ll 1$ (12) reduces to (11).

We have also investigated numerically the properties of (12) and have found, fortunately, that the major qualitative features of (11) are not significantly altered. The localized electric fields are still generated, but as see in Figure 4, the behavior of the density profile is quite different, since now the soliton-like structure not only radiates continuum Langmuir waves, but it can radiate sound waves also.

A specific example of the interesting new effects that appear due to the additional degree of freedom associated with the sound waves can be obtained from the set

$$
\begin{aligned}
&i\partial_t E + \partial_x{}^2 E + (i\nu - \tilde{n})E = 0 \\
&\partial_t{}^2\tilde{n} - \partial_x{}^2\tilde{n} = \partial_x{}^2|E|^2
\end{aligned}
\tag{13}
$$

which we have investigated [8] in connection with the generation of "spiky turbulence" due to the injection of electron beams into a plasma. A typical space-time evolution predicted by (13) is shown in Figure 5, where it is seen that two initially localized fields (previously generated by the presence of the beam) are collisionally damped. As the damping takes place, they no longer remain stationary, as they did when the external energy source was still turned on. While the soliton-like structures move apart, it is observed that they radiate pulses of acoustic waves, which in turn give rise to a train of weakly damped solitons that proceed to propagate through the plasma, trapped inside the density cavities.

Another important effect which has not been considered in enough detail, and which is absent in equations (9)–(13), is associated with the role played by a saturable nonlinearity, an example of which is given in (8). Keeping this effect in full, the analog of (9) takes the form

$$
i\partial_t E + \partial_x{}^2 + (1 - e^{-|E|^2})E = 0 \tag{14}
$$

which constitutes a more appropriate representation for the evolution of perturbations that are large enough to have important physical consequences.

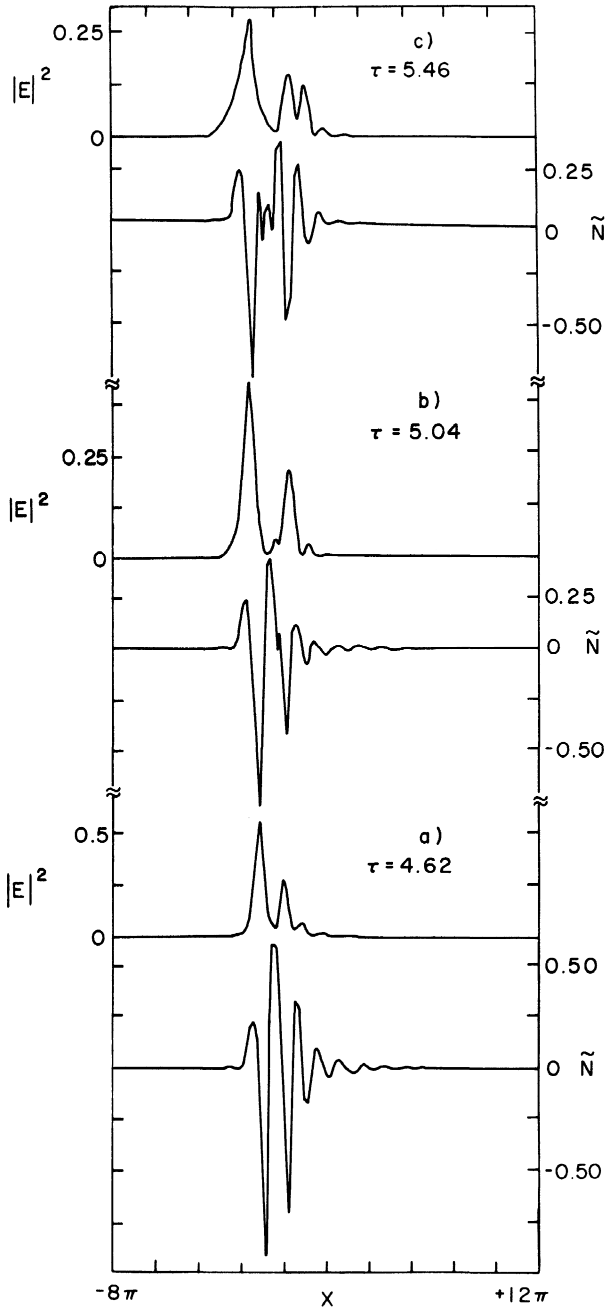

Figure 5. Space-time evolution of two initially localized electric field structures obtained from equation (13).

An equation that encompasses several of the difficulties which can be encountered in the studies of Langmuir wave solitons in plasmas takes the general form

$$i\partial_t E + \partial_x{}^2 E + |E|^2 E = \epsilon S(x, t, E) \tag{15}$$

where S is a function which can include a variety of effects (e.g., collisional damping, external pumping by a capacitor plate field, interaction with an electron beam). It appears that a great deal of progress can be made by developing a perturbation technique which treats the quantity ϵ as a small parameter. We are presently engaged in the investigation of this problem, and to date the salient results are that, to the lowest order in ϵ, the secular time evolution of a slowly varying soliton

$$\psi_0(\xi, t) = A(t) \exp\{i[k(t)\xi - \int_0^t dt'\Omega(t')]\} \operatorname{sech}[\xi/T(t)] \tag{16}$$

can be obtained from the time-averaged equations

$$\frac{d}{dt}(A^2T) = \left\langle 1/2 \operatorname{Im} \left\{ \int_{-\infty}^{\infty} d\xi \psi_0{}^* S \right\} \right\rangle$$

$$\frac{d}{dt}(A^2Tk) = \left\langle 1/2 \operatorname{Re} \left\{ \int_{-\infty}^{\infty} d\xi \frac{\partial}{\partial \xi} \psi_0{}^* S \right\} \right\rangle$$

involving overlap integrals over the source function. This study is still in its preliminary stages and accordingly needs to be considered in more detail. Nevertheless, it should be emphasized that results of this type are of interest to physicists, since they make definite predictions of effects that may be observable in the laboratory or in computer simulations of plasmas.

An area which has just begun to be explored is that of the generation of solitons in magnetized plasmas. The difficulty which is encountered at the outset in this case is that the relevant phenomena intrinsically occurs in more than one spatial dimension. As is well-known, the theory of solitons in more than one dimension is in its infancy, therefore, one suspects that the studies of soliton-like structures with magnetic fields present can be quite difficult. Recently, we have found [9] a case in which the difficulties are fully circumvented because the dependence of the electric field takes the form

$$E(x, z, t) = A(x, z)e^{it} \tag{18}$$

where now A is a soliton in the (x, z) configuration space rather than a soliton propagating in space-time, i.e., in (x, t) space, as is familiar to one dimension nonlinear wave studies. These solutions represent standing wave "cones", whose asymptotic behavior we have found to be described by the modified Korteweg-deVries equation

$$\partial_x A + \partial_z A^3 + \partial_z{}^3 A = 0 \tag{19}$$

for the specific case of lower-hybrid oscillations. Again, as in the case of the nonlinear Schrödinger equation, physicists are interested in finding out what particular set of external boundary conditions in the (x, z) plane can generate the solitons of (19), and how many of them can be generated for a given amount of external energy. In this connection, it is worthwhile to point out that the more realistic study of the effect of boundary conditions should not be aimed at (19), but rather it should consider directly the more general symmetric equation from which (19) arises, namely

$$\partial_x{}^2 A - \partial_z{}^2 A + \partial_x{}^2 A^3 + \partial_x{}^4 A = 0. \tag{20}$$

Such an equation has not yet been studied in much detail, and thus deserves a closer look by the applied mathematics community. It should be mentioned that in a recent laboratory experiment [**10**] filamented lower-hybrid cones and the associated density cavities have been observed.

In summary, our experience with calculations related to the generation of soliton-like structures in plasmas points out the need to investigate further the following mathematical problems: (1) to evaluate the effect of saturable nonlinearities such as (7), and to sort out the scattering properties of (14); (2) to study the properties of soliton-like structures when an additional degree of freedom is present, as in (13); (3) to develop a useful perturbation scheme whereby one can calculate the changes in the soliton parameters as relates to (15); and (4) to consider the explicit effect of external pumps in the nonlinear wave equations which are known to have soliton solutions.

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