ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 2, 1977

EXTREMAL PROPERTIES OF A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. Denote by H the subclass of close-to-convex functions $f(z)$ for which there exists a starlike function $g(z)$ satisfying $\mathrm{Re}\{[zf'(z)]'/g'(z)\} > 0$ $(|z| < 1)$. We find distortion theorems, coefficient bounds, and the closed convex hull of H. We also give a necessary intrinsic condition for a function to be in H.

1. **Introduction.** Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic and univalent in the disk $|z| < 1$. A function $f(z) \in S$ is said to be *starlike* if $\mathrm{Re}\{zf'(z)/f(z)\} > 0(|z| < 1)$, is said to be *convex* if $\mathrm{Re}\{1 + zf''(z)/f'(z)\} > 0(|z| < 1)$, and is said to be *close-to-convex* if there exists a starlike function $g(z)$ such that $\mathrm{Re}\{zf'(z)/g(z)\} > 0(|z| < 1)$. These classes are denoted respectively by S^*, K, and C.

We denote by H the class of functions of the form (1) for which there exists a function $g(z) \in S^*$ such that

$$\mathrm{Re}\left\{ \frac{[zf'(z)]'}{g'(z)} \right\} > 0(|z| < 1). \tag{2}$$

In [5] Sakaguchi shows for $g(z) \in S^*$ that $\mathrm{Re}\{[zf'(z)]'/g'(z)\} > 0$ implies $\mathrm{Re}\{zf'(z)/g(z)\} > 0$. Thus $H \subset C$. Moreover if $f(z) \in K$, then $\mathrm{Re}\{[zf'(z)]'/f'(z)\} = \mathrm{Re}\{1 + zf''(z)/f'(z)\} > 0$. Hence we may take $g(z) = f(z)$ in (2) to show that $f(z)$ is also in H. Thus $K \subset H$.

It is well known that $K \subset S^* \subset C$. Since we also have the inclusion relations $K \subset H \subset C$, it is of interest to inquire as to the relationship between S^* and H. In the next section, we shall show that S^* is neither contained in nor contains H.

Note that the result of Sakaguchi yields a quick proof that $K \subset S^*$, for $\mathrm{Re}\{[zf'(z)]'/f'(z)\} > 0$ implies $\mathrm{Re}\{zf'(z)/f(z)\} > 0$.

Received by the editors on March 29, 1976.
AMS (MOS) 1970 *Subject Classifications.* Primary 30A32; 30A40.
Key words and phrases: univalent, starlike, convex, close-to-convex, extreme points.

2. Distortion and Coefficient Bounds for H.

THEOREM 1. *If* $f(z) \in H$, *then* $(3 + r^2)/3(1 + r)^3 \leqq |f'(z)| \leqq (3 + r^2)/3(1 - r)^3 (|z| = r)$, *with equality only for functions of the form*

$$f(z) = \frac{2}{3}\frac{z}{(1 - xz)^2} - \frac{1}{3}\bar{x}\log(1 - xz)(|x| = 1). \tag{3}$$

PROOF. We may write $[zf'(z)]' = g'(z)p(z)$, where $p(z)$ is a function of positive real part with $p(0) = 1$. It is well known that

$$\frac{1 - r}{(1 + r)^3} \leqq |g'(z)| \leqq \frac{1 + r}{(1 - r)^3}(|z| = r) \tag{4}$$

and

$$\frac{1 - r}{1 + r} \leqq |p(z)| \leqq \frac{1 + r}{1 - r}(|z| = r). \tag{5}$$

Hence

$$\frac{(1 - r)^2}{(1 + r)^4} \leqq |[zf'(z)]'| \leqq \frac{(1 + r)^2}{(1 - r)^4}(|z| = r). \tag{6}$$

Integrating along the straight line segment from the origin to $z = re^{i\theta}$ in the right inequality of (6) we obtain $|zf'(z)| \leqq \int_0^r((1+ t)^2|(1 - t)^4)\,dt = (3r + r^3)/3(1 - r)^3$, which proves the right inequality in the theorem. We now prove the left inequality. For every r choose z_0, $|z_0| = r$, such that $|f'(z_0)| = \min_{|z|=r}|f'(z)|$. If $L(z_0)$ is the pre-image of the segment $\{0, z_0f'(z_0)\}$, then

$$|zf'(z)| \geqq |z_0f'(z_0)| = \int_{L(z_0)} |[zf'(z)]'|\,|dz|$$
$$\geqq \int_0^r \frac{(1 - t)^2}{(1 + t)^4}\,dt = \frac{3r + r^3}{3(1 + r)^3}.$$

The result now follows. Equality in (4) holds for $g(z) = z/(1 - xz)^2$ $(|x| = 1)$ and in (5) for $p(z) = (1 + xz)/1 - xz(|x| = 1)$ from which the functions in (3) may be obtained.

THEOREM 2. *If* $f(z) \in H$, *then*

$$\frac{2}{3}\frac{r}{(1 + r)^2} + \frac{1}{3}\log(1 + r) \leqq |f(z)| \leqq \frac{2}{3}\frac{r}{(1 - r)^2} - \frac{1}{3}\log(1 - r).$$

Equality holds only for functions defined by (3).

PROOF. The result follows from the bounds of Theorem 1 just as Theorem 1 followed from the bounds in (6).

COROLLARY. *If $f(z) \in H$, then $f(z)$ maps the disk $|z| < 1$ onto a domain that contains the disk $|w| < (1 + \log 4)/6$.*

PROOF. Let $r \to 1$ in the left inequality of Theorem 2.

THEOREM 3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H$, then $|a_n| \leqq (2/3)n + 1/3n$. This result is sharp, with equality only for functions defined by* (3).

PROOF. Our proof is similar to Reade's proof of the Bieberbach conjecture for C [4]. Suppose $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and $p(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n$. Then $[zf'(z)]' = \sum_{n=1}^{\infty} n^2 a_n z^{n-1} = [\sum_{n=1}^{\infty} nb_n z^{n-1}][1 + \sum_{n=1}^{\infty} \alpha_n z^{n-1}]$, and $n^2 a_n = nb_n + \sum_{k=1}^{n-1}(n-k)b_{n-k}\alpha_k$. It is well known that $|b_n| \leqq n$ and $|\alpha_n| \leqq 2$ for all n. Hence $n^2|a_n| \leqq n^2 + 2\sum_{k=1}^{n-1}(n-k)^2 = n^2 + n(n-1)(2n-1)/3$, which simplifies to $|a_n| \leqq (2/3)n + 1/3n$. Once again equality holds only for functions of the form (3).

Since the bounds for the starlike Koebe function $z/(1-z)^2$ exceed those of Theorems 1, 2, and 3, we see that $S^* \not\subset H$. Moreover $H \not\subset S^*$, as will be seen by showing that $f(z) = (2/3)z/(1-z)^2 - (1/3)\log(1-z) \notin S^*$. We have

$$\frac{zf'(z)}{f(z)} = \frac{3z + z^3}{(1-z)[2z - (1-z)^2 \log(1-z)]}. \tag{7}$$

Multiplying numerator and denominator in (7) by the conjugate of the denominator, the real part of the numerator at $z = e^{i\theta}$ becomes

$$n(\theta) = 6 - 8\cos\theta + 2\cos 2\theta + (\sin 3\theta - 3\sin\theta)\tan^{-1}\left(\frac{\sin\theta}{1-\cos\theta}\right)$$
$$+ \frac{1}{2}(10 - 15\cos\theta + 6\cos 2\theta - \cos 3\theta)\log[2(1-\cos\theta)].$$

Thus $n(\pi/3) = 1 - 3 \cdot 3^{1/2}/2 \tan^{-1} 3^{1/2} < 0$, which means there is a $\delta > 0$ such that $\mathrm{Re}\{zf'(z)/f(z)\} < 0$ ($z = re^{\pi i/3}$, $1 - \delta < r < 1$).

Since the functions defined by (3) are the only functions extremal for Theorems 1, 2, and 3, they must also be extreme points of the closed convex hull of H. We now determine this closed convex hull.

3. **Convex Hull of H.** In this section we determine the closed convex hull of H, denoted by $\mathrm{cl\,co}\,H$. Letting $\tilde{H} = \{h(z) \mid h(z) = [zf'(z)]', f(z) \in H\}$, we note that $h(z) \in \tilde{H}$ if and only if there is a $g(z) \in S^*$ for which

(8) $$\operatorname{Re}\left\{\frac{h(z)}{g'(z)}\right\} > 0(|z| < 1).$$

In the theorem that follows, we obtain the cl co $\tilde{H}$.

THEOREM 4. *Let $\bar{X}$ be the torus $\{(x, y) \mid |x| = |y| = 1\}$, $\mathcal{P}$ be the set of probability measures on $\bar{X}$, and let $k(z, x, y) = ((1 + xz)/(1 - xz))((1 + yz)/(1 - yz)^3)(|x| = |y| = 1, |z| < 1)$. If $\mathcal{F}$ is the family of functions h_μ on $|z| < 1$ defined by $h_\mu(z) = \int_{\bar{X}} k(z, x, y)\, d\mu(x, y)(\mu \in \mathcal{P})$, then*

(9) $$\mathcal{F} = \text{cl co } \tilde{H}.$$

PROOF. First suppose $h(z) \in \tilde{H}$. By (8) we may write $h(z)/g'(z) = p(z)$, where $p(z)$ is a function having positive real part with $p(0) = 1$. From the Herglotz representation, there is a probability measure $\mu_1(x)$ on $\Gamma = \{x \mid |x| = 1\}$ such that $h(z)/g'(z) = \int_\Gamma (1 + xz)/(1 - xz)\, d\mu_1(x)$. In addition since $g(z) \in S^*$ we have [1] $g'(z) = \int_\Gamma ((1 + yz)/(1 - yz)^3)\, d\mu_2(y)$, where $\mu_2(y)$ is a probability measure on Γ. Thus by Fubini's theorem, $h(z) = \int_{\bar{X}} ((1 + xz)/(1 - xz))((1 + yz)/(1 - yz)^3)\, d\mu(x,y)(\mu = \mu_1 \times \mu_2)$, which shows that $\tilde{H} \subset \mathcal{F}$ Since $\mathcal{F}$ is a closed convex family, we have cl co $\tilde{H} \subset \mathcal{F}$.

Conversely, setting $h(z) = k(z, x, y)$ and $g(z) = z/(1 - yz)^2$ in (8), we see that each kernel function $k(z, x, y)$ is in $\tilde{H}$. Hence $\mathcal{F} \subset$ cl co H, which proves (9).

REMARK. In view of Theorem 1d of [1], the functions $\{k(z, x, y), |x| = |y| = 1\}$ are the only possible extreme points of cl co $\tilde{H}$. Since any real-valued continuous linear functional on $\tilde{H}$ is maximized or minimized at an extreme point of cl co $\tilde{H}$, denoted $\mathcal{E}$(cl co $\tilde{H}$), the bounds in (6) enable us to show that the functions $\{k(z, x, x)\}$ are in $\mathcal{E}$(cl co H). On the other hand $k(z, x, -x) = 1/(1 + xz)^2 = (zf'(z))'$ for some $f(z) \in H$. Since $f(z) = \bar{x} \log(1 + xz)$ is in K but not in $\mathcal{E}$(cl co K), $f(z)$ cannot be an extreme point of the larger family $\tilde{H}$. Hence $k(z, x, -x) \notin \mathcal{E}$(cl co $\tilde{H}$). We are not able to determine if the functions $k(z, x, y), x \neq \pm y$, are in $\mathcal{E}$(cl co $\tilde{H}$).

THEOREM 5. *Let $\bar{X}$ be the torus $\{(x, y) \mid |x| = |y| = 1\}$, $\mathcal{P}$ be the set of probability measures on $\bar{X}$, and let*

$$f(z, x, y) = \int_0^z \left[1/\xi \int_0^\xi ((1 + xw)/(1 - xw))((1 + yw)(1 - yw)^3)\right] d\xi$$
$$(|x| = |y| = 1, |z| < 1).$$

If $\mathcal{F}$ is the family of functions of the form $\int_{\bar{X}} f(z, x, y)\, du(x, y)(u \in \mathcal{P})$, then $\mathcal{F} =$ cl co H.

PROOF. Since the operator L defined by

$$L(h(z)) = \int_0^z \left[1/\xi \int_0^\xi h(w)\, dw \right] d\xi$$

is a linear homeomorphism of $\tilde{H}$ onto H, the result follows from Theorem 4.

REMARK. It is interesting to note that the functions $f(z, x, x)$, extreme points of the closed convex hull of H, are actually a linear combination of extreme points taken from the closed convex hulls of starlike functions and functions convex of order 1/2. See [2].

4. **A Necessary Intrinsic Condition for H.** Kaplan [3] found a necessary and sufficient intrinsic condition for functions to be close-to-convex. Following his lead, we give a necessary intrinsic condition for a function to be in H. We do not, however, have a sufficient condition.

LEMMA. *If $f(z) \in H$, then there exists a function $\phi(z) \in K$ such that $h(z)$ defined by*

$$h'(z) = \frac{(zf'(z))'}{1 + \dfrac{z\phi''(z)}{\phi'(z)}} \tag{10}$$

is in C.

PROOF. For $f(z)$ defined by (2), choose $\phi(z) = \int_0^z g(\xi)/\xi\, d\xi$. Since $g'(z) = \phi'(z)\{1 + (z\phi''(z)/\phi'(z))\}$, (2) is equivalent to $\operatorname{Re} h'(z)/\phi'(z) > 0$.

THEOREM 6. *Let $f(z)$ be in H, and set $F(z) = zf'(z)$. Then*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ 1 + \frac{zF''(z)}{F'(z)} \right\} d\theta > -2\pi$$

$$(0 \leqq \theta_1 < \theta_2 \leqq 2\pi, z = re^{i\theta}).$$

PROOF. By the lemma $h(z)$, given by (10), is in C and hence

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} d\theta > -\pi$$

$$(0 \leqq \theta_1 < \theta_2 \leqq 2\pi, z = \mathrm{r}e^{i\theta}),$$

or equivalently

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{zF''(z)}{F'(z)}\right\} d\theta \tag{11}$$
$$> \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{ z\frac{d}{dz}\log\left[1 + \frac{z\phi''(z)}{\phi'(z)}\right]\right\} d\theta$$

Since $\phi(z) \in K$, $\operatorname{Re}\{1 + z\phi''(z)/\phi'(z)\} > 0(|z| < 1)$, so that

$$\left| \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{ z\frac{d}{dz}\log\left[1 + \frac{z\phi''(z)}{\phi'(z)}\right]\right\} d\theta \right|$$

$$= \left| \arg\left[1 + re^{i\theta_2}\frac{\phi''(re^{i\theta_2})}{\phi'(re^{i\theta_2})}\right] - \arg\left[1 - re^{i\theta_1}\frac{\phi''(re^{i\theta_1})}{\phi'(re^{i\theta_1})}\right]\right| < \pi.$$

The theorem follows upon substituting the last inequality into (11).

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