ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 7, Number 2, 1977

THE DIRICHLET PROBLEM FOR THE MULTIDIMENSIONAL MONGE-AMPÈRE EQUATION

JEFFREY RAUCH* AND B. A. TAYLOR*

1. **Introduction.** In this paper we study the n-dimensional version of the Monge-Ampère operator $Mu = u_{11}u_{22} - u_{12}^2$ where $u_{ij} = \partial^2 u/\partial x_i \partial x_j$. The analogue arises from observing that if $H(u)$ is the Hessian matrix of u then $Mu = \det H(u)$. The main goal is to prove a comparison theorem for nonconvex solutions of the Dirichlet problem for M. In the process we will give a self-contained exposition of the Dirichlet problem for M in the class of convex u.

Our interest in this problem has two sources. First, the existence of convex solutions to the Dirichlet problem for the real Monge-Ampère equation can be used to solve certain Dirichlet problems for plurisubharmonic functions (see [3]). Second, in the study of the buckling of thin elastic shells a Dirichlet problem for the 2 dimensional Monge-Ampère equation $Mu = 0$ arises. What is needed in the latter case is a uniqueness theorem and the difficulty is that u is not necessarily convex. Roughly speaking, for a thin shell under stress there are two obvious solutions, the unbuckled solution and the solution which minimizes potential energy. Using the uniqueness theorem and an "index" argument Rabinowitz [9] is able to show that there is at least one more solution. The uniqueness theorem is a consequence of the following comparison result for possibly nonconvex u. If $\Omega \subset \mathbf{R}^n$ is a bounded open set $v \in C(\bar{\Omega})$ is convex on Ω, $u \in W_{2,n}(\Omega)$ and $Mu \leqq Mv$ on Ω, $u \geqq v$ on $\partial\Omega$ then $u \geqq v$. Note that since the second derivatives of u are in $L_n(\Omega)$, $Mu \in L_1(\Omega)$ and for any convex v, Mv makes sense as a measure on Ω as we will show in § 2.3. This comparison theorem is our main result.

To prove the comparison theorem we need some continuity properties of the operator, M, and an existence theorem for the Dirichlet problem. There is a large and rich Russian literature on this problem (see for example [1, 2, 6, 7, 8] but we were unable to extract precisely the information needed. In this paper we offer a self contained and independent solution to the Dirichlet problem which creates a link between the geometric approach of the Russian school and the more traditional analytic methods of partial differential

Received by the editors on January 9, 1976.

*Research partially supported by the National Science Foundation under Grants NSF GP 34260 and GP 37628.

equations. In particular, we prove that if Ω is a bounded strictly convex subset of $\mathbf{R}^n$ then for any nonnegative $f \in L_1(\Omega)$ and continuous $g : \partial\Omega \rightarrow \mathbf{R}^n$ there is a unique convex function u such that

$$\det H(u) = f \text{ in } \Omega, \quad \text{and} \tag{1.1}$$

$$u = g \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

More general boundary value problems could be treated, for example $\det H(u) = f(x, u, u_x)$ but we have chosen to follow the straightest path to the comparison theorem.

The equation (1.1) is not elliptic unless $f > 0$ and, in general, the solutions will not be differentiable even if f and g are real analytic. Thus the notion of solution must be taken in a generalized sense. To make matters worse the operator $Mu = \det H(u)$ is not in divergence form so the usual method of defining weak solutions using integration by parts is not available. We will discuss two equivalent notions of generalized solution. For the first we follow the approach of Alexandrov (see, e.g., [7]) and rely on a geometrical interpretation.

If $u \in C^\infty(\Omega)$, we can consider the surface $\Sigma \subset \mathbf{R}^n \times \mathbf{R}$ which lies over Ω and is given by the equation $z = u(x)$ in $\mathbf{R}^n \times \mathbf{R}$. The sphere map $\gamma : \Sigma \rightarrow S^n$ from Σ to the lower hemisphere maps $p \in \Sigma$ to the downward point unit normal at p. Analytically the map is given by

$$(x, u(x)) \rightarrow \frac{(\operatorname{grad} u(x), -1)}{(1 + |\operatorname{grad} u(x)|^2)^{1/2}}$$

The scalar curvature, κ, is the Jacobian determinant of this map. For example, when $n = 2$

$$\kappa = \frac{u_{11}u_{22} - u_{12}^2}{(1 + u_1{}^2 + u_2{}^2)^{3/2}}$$

In general, the equation $Mu = 0$ is equivalent to the vanishing of the scalar curvature. Thus the solution of the Dirichlet problem (1.1, 1.2) when $f = 0$ asserts that for any continuous curve, Γ, with simple projection on $\partial\Omega$ there is a unique convex surface with vanishing scalar curvature which passes through Γ. This result is the basic building block in the solution of the more general problem. Fortunately, the scalar curvature can be studied geometrically even when u is not smooth.

Our second point of view following Chern, Levine, and Nirenberg [5] is the conventional analytic technique of approximation by smooth functions. In § 3 we show that this approach agrees with the geometrical method and prove the relevant continuity properties of M. In § 4 these tools are used to show that, somewhat surprisingly, the general Dirichlet problem (1.1, 1.2) can be reduced to the case $f = 0$. The idea is to observe that cones solve certain Dirichlet problems when f is a delta function and then to approximate the measure fdx by finite linear combinations of delta functions. Finally, in §5 the basic comparison theorem for $u \in W_{2,n}$ is proved.

2. **Geometric approach and the Dirichlet problem for the equation of zero curvature.** For smooth surfaces Σ, the total curvature of a subset $\sigma \subset \Sigma$ is defined as the integral of κ over σ; or equivalently, as the area, counting multiplicity, of the spherical image of σ. If Σ is defined by a function u continuous and convex on $\bar{\Omega}$, we can define a generalization of the sphere map. In this case, the set $\hat{\Sigma} = \{(x, z) : x \in \bar{\Omega},\ z \geqq u(x)\}$ is a closed convex set in $\mathbf{R}^{n+1}$, and a vector $(\omega, \tau) \in S^n$ is called a *supporting direction* at $(x_0, u(x_0)) \in \Sigma$ if the plane $(x - x_0) \cdot \omega + \tau(z - u(x_0)) = 0$ is a supporting hyperplane to $\hat{\Sigma}$ at $(x_0, u(x_0))$. This is equivalent to the inequality $u(x) \geqq u(x_0) - (1/\tau)[(x - x_0) \cdot \omega]$ for all $x \in \bar{\Omega}$. If u is also differentiable at $x_0 \in \Omega$ then $(\omega, \tau) = (\operatorname{grad} u(x_0), -1)/(1 + |\operatorname{grad} u(x_0)|^2)^{1/2}$ is the unique supporting direction at $(x_0, u(x_0))$. For nonsmooth convex surfaces, the supporting directions generalize the notion of unit normal. Following Alexandrov [1], given a compact set $E \subset \Omega$, the *spherical image of* E is defined as the set of all supporting directions to points $(x, u(x))$ for $x \in E$. This spherical image is denoted $\gamma(E)$. The next proposition is a simple consequence of the continuity of u and the inequality characterization of supporting directions.

Proposition 2.1. *If $E \subset \Omega$ is compact, then $\gamma(E) \subset S^n$ is a compact subset of S^n.*

Definition 2.2. For a compact subset $E \subset \Omega$, the total curvature, $\kappa(E)$, of E is defined as $m(\gamma(E))$, where m is the surface measure on S_n. The total curvature of the convex surface Σ over Ω is defined as $\sup \kappa(E)$, where the supremum is over all compact subsets of Ω.

As pointed out earlier, if u is differentiable at $x_0 \in \Omega$, the unique supporting direction at $(x_0, u(x_0))$ is the normal vector $(\operatorname{grad} u(x_0), -1)/(1 + |\operatorname{grad} u(x_0)|^2)^{1/2}$. It is clear that we can also find $\operatorname{grad} u$ from the supporting vector. This is important for us and we formalize the map as $\pi : S^n \to \mathbf{R}^n$ by

$$(\omega, \tau) \to -\frac{1}{\tau}\ \omega$$

The map π is, of course, stereographic projection of the lower hemisphere onto $\mathbf{R}^n$. For $E \subset \Omega$ the *gradient image*, $\omega(E)$, is defined as $\pi(\gamma(E))$. A vector $\alpha \in \mathbf{R}^n$ is then in the gradient image of x_0 if and only if $u(x) \geqq u(x_0) + \alpha \cdot (x - x_0)$, $x \in \Omega$. The simplicity of this condition allows us to work effectively with the gradient image. We have the elementary

PROPOSITION 2.3. *If Σ is a convex surface over Ω then the gradient image $\omega(E)$ of a compact subset E of Ω is a compact subset of $\mathbf{R}^n$ and*

$$\operatorname{meas}(\omega(E)) \geqq m(\gamma(E)) = \kappa(E).$$

To illustrate the utility of the gradient image, we prove the following classical fact.

PROPOSITION 2.4. *If u is a convex function on Ω and Γ is a compact subset of Ω, then u is uniformly Lipschitz continuous on Γ.*

PROOF. Let $R = \sup\{|\alpha| : \alpha \in \omega(\Gamma)\}$. We show that $|u(x_1) - u(x_2)| \leqq R|x_1 - x_2|$ for $x_1, x_2 \in \Gamma$. For the proof, notice that if $\alpha \in \omega(x_1)$, then $u(x) \geqq u(x_1) + \alpha \cdot (x - x_1)$, $x \in \Omega$, so that with $x = x_2$ we have

$$u(x_1) - u(x_2) \leqq (x_1 - x_2) \cdot \alpha \leqq R|x_1 - x_2|.$$

A lower bound for the difference is obtained by considering α in the gradient image of $x_2 \in \Gamma$.

In particular, by a classical Theorem of Rademacher [10], u is differentiable at almost all points of Ω.

A basic ingredient in our proof is the following fundamental theorem of Aleksandrov ([1], p. 190).

THEOREM 2.5 [ALEKSANDROV]. *The set of points $\alpha \in \mathbf{R}^n$ which belong to the gradient image of more than one point of Ω has Lebesgue measure zero.*

PROOF. We use the convex function conjugate to u, $u^* : \mathbf{R}^n \to \mathbf{R}$ defined by

$$u^*(\alpha) = \sup\{x \cdot \alpha - u(x) : x \in \Omega\}. \tag{2.1}$$

The function u^* is a locally bounded convex function on $\mathbf{R}^n$. Therefore, u^* is differentiable at almost all points $\alpha \in \mathbf{R}^n$. We will show that if α belongs to the gradient image of two points $x_0, x_1 \in \Omega$, then u^* is not differentiable at α.

First, we note that if $\alpha \in \omega(\{x_0\})$, then $u^*(\alpha) = x_0 \cdot \alpha - u(x_0)$, since $x \cdot \alpha - u(x) = [(x - x_0) \cdot \alpha + u(x_0) - u(x)] +$ constant and for $\alpha \in \omega(\{x_0\})$, the expression in brackets is nonpositive and vanishes at $x = x_0$. Therefore, the supremum of the right hand side of (2.1) is attained at $x = x_0$, as asserted. Therefore, if $\ell(y) = x_0 \cdot y - u(x_0)$, we have $u^*(y) \geqq \ell(y)$ with equality at $y = \alpha$. Thus, if u^* is differentiable at α, we must have $\operatorname{grad} u^*(\alpha) = x_0$. Thus, if α is also in the gradient image of $x_1 \neq x_0$, we would also find $\operatorname{grad} u^*(\alpha) = x_1$, so u^* cannot be differentiable at α.

REMARK. The converse of the assertion is nearly true. Modulo some difficulties involving points $x \in \partial\Omega$, if u^* is not differentiable at α then the graph of u^* must have two supporting planes at $(\alpha, u^*(\alpha))$. If $x_0 \neq x_1$ are the direction vectors of these hyperplanes then α is the direction vector of a hyperplane supporting the graph of u at both x_0 and x_1. In this manner one can prove that the null set of Aleksandrov's theorem is a Borel set.

With the aid of Alexandrov's theorem it is easy to show that $\{E \subset \Omega \mid \omega(E)$ is Lebesgue measurable$\}$ is a σ-algebra. The main problem is complements where we use the identity

$$\omega(\Omega\backslash E) = [\omega(\Omega)\backslash\omega(E)] \cup [\omega(\Omega\backslash E) \cap \omega(E)].$$

By Alexandrov's theorem the intersection in brackets is a null set. Since all compact sets are in this σ-algebra it must contain all Borel sets in Ω. In addition the map which sends a Borel set $E \subset \Omega$ to the Lebesgue measure of $\omega(E)$ defines a measure. To prove the countable additivity observe that if $\{E_i\}$ is a disjoint sequence of Borel sets in Ω then for $i \neq j$ $\omega(E_i) \cap \omega(E_j)$ is a subset of the null set in Aleksandrov's theorem so

$$\begin{aligned}\sum \operatorname{meas}(\omega(E_i)) &\geqq \operatorname{meas} \omega(\textstyle\bigcup E_i)\\ &\geqq \sum \operatorname{meas} \omega(E_i) - \sum_{i,j} \operatorname{meas} \omega(E_i) \cap \omega(E_j)\\ &\geqq \sum \operatorname{meas} \omega(E_i).\end{aligned}$$

DEFINITION 2.6. The Monge-Ampère measure, Mu, of a convex function u on Ω is defined by

$$\mathrm{Mu}\,(E) = \text{Lebesgue measure of } \omega(E)$$

for any Borel set $E \subset \Omega$.

If $u \in C^2(\Omega)$, $\omega(E) = \{\operatorname{grad} u(x) \mid x \in E\}$ and $Mu = \det\{u_{x_i x_j}\}\, dx_1\, dx_2 \cdots dx_n$. From Definition 2.2 we see that $\mathrm{Mu} = 0$ if and only if the total curvature of $\Sigma = \{(x, u(x)) \mid x \in \Omega\}$ is zero. In this case we say that Σ has *zero scalar curvature*. For later use we prove the following comparison theorem for total curvature which is an immediate consequence of the geometric definition.

PROPOSITION 2.7. *Let Ω be a bounded open set in $\mathbf{R}^n$ and u, v functions continuous on $\bar{\Omega}$ and convex on Ω. If $u = v$ on $\partial\Omega$ and $u \geqq v$ in Ω, then the gradient image of Ω under u is a subset of the gradient image of Ω under v.*

PROOF. If $\gamma(\Omega, u)$, $\gamma(\Omega, v)$ are the spherical images of Ω under u and v respectively then it suffices to prove that $\gamma(\Omega, u) \subset \gamma(\Omega, v)$. If $(\omega, \tau) \in \gamma(\Omega, u)$ is normal to a support plane P at $(x_0, u(x_0))$ then it is geometrically obvious that the plane P can be lowered so as to be a support plane to $\Sigma(v)$. To be precise if

$$a = \sup_{x \in \Omega} (\tau(z - u(x_0)) + \omega \cdot (x - x_0) - v(x)) \geqq 0,$$

then the plane $P - (0, 0, \cdots, a)$ is a support plane to $\Sigma(v)$ at some point $(x, v(x))$ with $x \in \Omega$. It is clear that this recipe yields a support plane at some $(x, v(x))$; all that needs to be shown is that $x \in \Omega$. If $a > 0$ then the plane $P - (0, 0, \cdots, a)$ passes below the boundary curve $\{(x, u(x)) \mid x \in \partial\Omega\}$ of both $\Sigma(u)$ and $\Sigma(v)$ so it must be tangent to $\Sigma(v)$ at an interior point. If $a = 0$ then P supports $\Sigma(v)$ at $(x_0, v(x_0))$ so in either case we are done.

With these preliminaries we proceed to solve the Dirichlet problem $Mu = 0$ in Ω and $u = g$ on $\partial\Omega$. Proposition 2.7 suggests that a reasonable candidate for a solution is the largest convex function which is equal to g on $\partial\Omega$. The domain Ω is *strictly convex* iff for any $x_1, x_2 \in \bar{\Omega}$ the open line segment connecting x_1 and x_2 lies in the interior of Ω.

THEOREM 2.8. *If $\Omega \subset \mathbf{R}^n$ is bounded and strictly convex and $g : \partial\Omega \to \mathbf{R}$ is continuous, then there is a unique convex $U \in C(\bar{\Omega})$ such that $MU = 0$ in Ω and $U = g$ on $\partial\Omega$. The solution, U, is the largest convex function in $C(\bar{\Omega})$ which is less than or equal to g on $\partial\Omega$.*

Geometrically, the graph of the solution U is the lower boundary of the convex hull of the set of points $\{(x, g(x)) \mid x \in \partial\Omega\}$.

EXISTENCE PROOF. A function $a : \mathbf{R}^n \to \mathbf{R}$ is called affine if there are real constants $c_0, c_1, \cdots, c_n$ such that

$$a(x_1, \cdots, x_n) = c_0 + c_1 x_1 + \cdots + c_n x_n.$$

Let $\mathcal{G} = \{a \mid a \text{ is affine and } a \leqq g \text{ on } \partial\Omega\}$. Since g is bounded below on $\partial\Omega$ the family $\mathcal{G}$ is nonempty. If

$$U(x) \equiv \sup \{v(x) \mid v \in \mathcal{G}\}, \tag{2.2}$$

then U is automatically convex and $U \leqq g$ on $\partial\Omega$. We show (in this order) that $U = g$ on $\partial\Omega$, $U \in C(\bar{\Omega})$ and $MU = 0$.

To show that $U(\xi) \geqq g(\xi)$ for $\xi \in \partial\Omega$ it is no loss of generality to assume that $\xi = 0$, that $x_1 = 0$ is a supporting hyperplane to Ω at 0, and that $x_1 > 0$ for all $x = (x_1, \cdots, x_n) \in \Omega$. If $\epsilon > 0$ then for $\delta > 0$ sufficiently small, $|g(x) - g(0)| < \epsilon$ for all $|x| < \delta$, $x \in \partial\Omega$. Since $\partial\Omega$ is strictly convex, there is a number $\eta > 0$ so small that $\bar{\Omega} \cap \{x = (x_1, \cdots, x_n) : x_1 < \eta\}$ is contained in $\{x : |x| < \delta\}$. Let $M = \min\{g(x) : x \in \partial\Omega,\ x_1 \geqq \eta\}$. Then the affine function $a(x) = [g(0) - \epsilon] - Ax_1$, where $A \geqq \max\{(g(0) - \epsilon - M)/\eta, 0\}$, satisfies $a(0) \geqq g(0) - \epsilon$, and $a(x) \leqq g(x)$ for $x \in \partial\Omega$. Thus, $a \in \mathcal{G}$ so $U \geqq a$, in particular $U(0) \geqq a(0) \geqq g(0) - \epsilon$. Since this is true for any $\epsilon > 0$ we have $U(0) \geqq g(0)$ so $U = g$ on $\partial\Omega$. Furthermore, if $\{x_n\}$ is a sequence in Ω converging to zero then

$$\underline{\lim}\, U(x_n) \geqq \underline{\lim}\, a(x_n) = a(0) \geqq g(0) - \epsilon$$

so $\underline{\lim}\, U(x_n) \geqq g(0)$.

Since Ω is convex there is a unique harmonic function f on Ω such that $f \in C(\bar{\Omega})$ and $f = g$ on $\partial\Omega$. Since any $a \in \mathcal{G}$ is subharmonic we have $a \leqq f$ so $U \leqq f$, in particular if $x_n \in \Omega$, $x_n \to 0$ then

$$\overline{\lim}\, U(x_n) \leqq \underline{\lim}\, f(x_n) = g(0)$$

which completes the proof that U is continuous at 0.

It only remains to show that $MU = 0$. According to Aleksandrov's theorem, it suffices to show that $\omega(\Omega)$, the gradient image of Ω is a subset of $\{p \in \mathbf{R}^n \mid p$ is in the gradient image of two distinct points of $\bar{\Omega}\}$. Toward this end suppose $p \in \omega(\Omega)$. Then, for some $x_0 \in \Omega$, we have $U(x) - U(x_0) \geqq p \cdot (x - x_0)$. Let $a(x) = U(x_0) + p \cdot (x - x_0)$. We have $a(x) \leqq g(x)$ for all $x \in \partial\Omega$, and equality must hold for at least one $x \in \partial\Omega$. Otherwise, $a(x) + \epsilon \leqq g(x)$ for $x \in \partial\Omega$ and some $\epsilon > 0$, which by (2.2) implies $U(x_0) > a(x_0) = U(x_0)$, a contradiction. Let $\xi \in \partial\Omega$ be such that $a(\xi) = g(\xi)$. The open line segment joining x_0 to ξ lies in Ω. Further, $U(x_0) = a(x_0)$ and $U(\xi) = a(\xi)$. Thus, since a is an affine function, $U(x) \leqq a(x)$ for all x on this segment. But $U(x) \geqq a(x)$ for all $x \in \Omega$. Hence, p is in the gradient image of every point of the line segment from x_0 to ξ and the proof of existence is complete.

Uniqueness proof. Let U be the solution constructed above and suppose $V \in C(\bar{\Omega})$ is convex on Ω and $V = g$ on $\partial\Omega$. If Σ_V is the surface over Ω defined by V then for any $\bar{x} \in \Omega$ there is a support plane to Σ_V at $(\bar{x}, V(\bar{x}))$ so there is an affine function a such that $a(\bar{x}) = V(\bar{x})$ and $a \leqq V$ on $\bar{\Omega}$. In particular $a \leqq g$ on $\partial\Omega$ so $a \in \mathcal{G}$. Thus

$V(\bar{x}) = a(\bar{x}) \leqq U(\bar{x})$ so $V \leqq U$ which proves the last assertion of Theorem 2.8.

We show that if there is an $x_0 \in \Omega$ with $V(x_0) < U(x_0)$ then $MV \neq 0$. Let ν_0 be normal to a supporting plane at $(x_0, U(x_0))$ to the surface Σ_U. For $2\epsilon = U(x_0) - V(x_0)$ the plane through $(x_0, U(x_0) - \epsilon)$ with normal ν lies below Σ_U and above $V(x_0)$ for all ν in the neighborhood of ν_0 on the unit sphere in $\mathbf{R}^{n+1}$. It follows that each such ν is in the spherical image of Σ_V and therefore the total curvature of Σ_V is strictly positive.

REMARK. The same proof can be used to construct a concave solution, $U_{\text{CONCAVE}} \in C(\bar{\Omega})$ with $U_{\text{CONCAVE}} = g$ on $\partial\Omega$ and $Mu_{\text{CONCAVE}} = 0$. In Section 5, we show that for any u with $Mu = 0$ and $u = g$ on $\partial\Omega$ we have

$$U_{\text{CONVEX}} \leqq u \leqq U_{\text{CONCAVE}}.$$

3. **Analytic approach to the curvature measure.** Alexandrov's elegant approach to curvature outlined in § 2 allows the use of the geometry of convex surfaces in the solution of many nonlinear partial differential equations (see e.g., [7, 8])). It is possible, however, to give an analytic definition of the curvature measures, or gradient image measure by the standard method of smoothing and approximation. An analytic approach to the definition of curvature has the advantage that convergence theorems, such as the one we need in the solution of Rabinowitz's problem, follow almost immediately from the definitions, while in the geometric approach, the convergence theorems are more difficult to obtain.

In this section, following the method of [5], we will outline an analytic definition of the gradient image measure, and give some convergence theorems for it. We will also show that the analytic definition is the same as Alexandrov's geometric definition. In particular, this gives an alternative approach to Alexandrov's convergence theorem ([3], p. 197).

We also remark that analytic proofs of all the results of § 2 may be given, although we will not do so here. One reason for wanting such an approach is as a possible outline for the study of analogous results for plurisubharmonic functions. Some progress in that direction appears in [3], where the exact analogue of Proposition 2.7 is used to prove a uniqueness theorem for weak solutions of a complex Monge-Ampère equation.

In order to give the definition precisely, introduce the following notation. For Ω open in $\mathbf{R}^n$, let $K(\Omega)$ denote the cone of convex functions on Ω, and $C^k(\Omega)$, $C^k(\bar{\Omega})$ the usual spaces of k-times differentiable functions on Ω and $\bar{\Omega}$, respectively. Also, let $M_m(\Omega)$, $0 \leqq m \leqq n$, denote the currents of degree m and order 0 on Ω. That is, $M_m(\Omega)$ is the class of differential forms of degree m whose coefficients are Borel measures on Ω. We will suppose that $M_m(\Omega)$ has the topology of weak convergence of measures; i.e., $\mu_j \to \mu$ in $M_m(\Omega)$ if and only if $\int \phi \wedge \mu_j \to \int \phi \wedge \mu$ for all differential forms ϕ of degree $n - m$ whose coefficients are continuous functions with compact support in Ω.

For $I = \{i_1, i_2, \cdots, i_m\}$ a set of integers with $1 \leqq i_1 < i_2 < \cdots < i_m \leqq n$, let

$$\mathscr{M}_I u = d(u_{i_1}) \wedge \cdots \wedge d(u_{i_m})$$

We can think of $\mathscr{M}_I$ as an (unbounded) operator from $C(\Omega)$ to $M_m(\Omega)$ with domain $C^2(\Omega)$. Also, set $\mathscr{M}u = \mathscr{M}_{\{1,2,\cdots,n\}}u = du_1 \wedge du_2 \wedge \cdots \wedge du_n$ so that $\mathscr{M}u = \det [u_{ij}]\, dx_1 \wedge \cdots \wedge dx_n$ is the Monge-Ampère measure. The next Proposition shows that all the $\mathscr{M}_I$ are actually continuous operators on the entire cone $K(\Omega)$ of convex functions.

PROPOSITION 3.1. *For any $I = \{i_1 < \cdots < i_m\}$, $1 \leqq m \leqq n$, the operator $\mathscr{M}_I : C^2(\Omega) \to M_m(\Omega)$ has the property that*

(3.1) *$\mathscr{M}_I$ maps $C(\Omega)$-bounded subsets of $C^2(\Omega) \cap K(\Omega)$ into bounded subsets of $M_m(\Omega)$, and*

(3.2) *if $u^{(j)}, v^{(j)} \in C^2(\Omega)$ are such that*

$$\text{(i)}\ \lim_{j\to\infty} u^{(j)} = \lim_{j\to\infty} v^j = u \text{ in } C(\Omega),$$

$$\text{(ii)}\ \lim_{j\to\infty} \mathscr{M}_I(u^{(j)}) = \mu \text{ in } M_m(\Omega),$$

$$\text{(iii)}\ \lim_{j\to\infty} \mathscr{M}_I(v^{(j)}) = \nu \text{ in } M_m(\Omega),$$

then $\mu = \nu$.

Consequently, $\mathscr{M}_I$ has a unique extension to a continuous operator on all of $K(\Omega)$.

PROOF. For $J = \{j_1 < \cdots < j_m\}$ another multiindex of length m, let $\Delta(I, J)$ be the determinant of the $m \times m$ submatrix $[u_{i_\alpha, j_\beta}]$, of the Hessian of the C^2 function $u \in K(\Omega) \cap C^2(\Omega)$. To prove (3.1), we have to show that the integral of $|\Delta(I, J)|$ over any compact subset of Ω can be estimated by the supremum norm of u over some larger compact subset of Ω. Since u is convex, the Hessian matrix of u is a nonnegative matrix, so $2|\Delta(I, J)| \leqq \Delta(I, I) + \Delta(J, J)$. We can therefore restrict attention to the case when $I = J$. It is no loss of generality to take $I = J = \{1, 2, \cdots, m\}$.

Let K be a compact subset of Ω. Choose a C^∞ function χ with compact support in Ω, $0 \leqq \chi \leqq 1$, and $\chi = 1$ on K. Set $\phi = \chi\, dx_{m+1} \wedge \cdots \wedge dx_n$. Then since $\Delta(I, I) \geqq 0$, we have

$$\int_K \Delta(I, I) \leqq \int du_1 \wedge \cdots \wedge du_m \wedge \phi. \tag{3.3}$$

However, by Stokes Theorem,

$$\int du_1 \wedge \cdots \wedge du_m \wedge \phi$$
$$= - \int u_i\, du_1 \wedge \cdots \wedge du_{i-1} \wedge d\phi \wedge du_{i+1} \wedge \cdots \wedge du_m.$$

Next integrate by parts with respect to x_i in the second integral to take the x_i-derivative from u_i to the other terms and then sum over i. The result is

$$m \int du_1 \wedge \cdots \wedge du_m \wedge \phi \tag{3.4}$$
$$= \sum_{i=1}^{m} \int u\, du_1 \wedge \cdots \wedge du_{i-1} \wedge d\phi_i \wedge du_{i+1} \wedge \cdots \wedge du_m.$$

If $m = 1$, then (3.1) clearly follows from (3.3) and (3.4). If $m > 1$, then we can assume by induction that (3.1) has been proved for multi-indices of length less than m. Thus the L^1 norms of the coefficients of $du_1 \wedge \cdots \wedge du_{i-1} \wedge du_{i+1} \wedge \cdots du_m$ on support ϕ are uniformly bounded when u varies over a $C(\Omega)$-bounded subset of $C^2(\Omega) \cap K(\Omega)$, and (3.1) follows.

To prove (3.2), we use formula (3.4) again. We may again assume $I = \{1, 2, \cdots, m\}$. By induction, we may also assume that

$$\lim_{j \to \infty} \mathcal{M}_J(u^{(j)}) = \lim_{j \to \infty} \mathcal{M}_J(v^{(j)})$$

for all multiindices J of length less than m. Then, since both $u^{(j)} d\phi_i$ and $v^{(j)} d\phi_i$ converge uniformly to $u d\phi_i$, if u is replaced by $u^{(j)}$ and then $v^{(j)}$ in (3.4) and then $j \to \infty$, we find

$$\int \mu \wedge \phi = \int \nu \wedge \phi$$

for all differential forms ϕ of degree m whose coefficients are C^∞ functions with compact support; that is, $\mu = \nu$, as asserted.

The last assertion of the Proposition follows from (3.1), (3.2), and the well-known fact that bounded subsets of $M_m(\Omega)$ are relatively compact. This completes the proof.

As one application, we formalize our earlier definition

Definition 3.2. If u is a convex function on Ω, then $\mathcal{M}u$ is the non-negative Borel measure $\mathcal{M}_{\{1,2\cdots,n\}}u = du_1 \wedge \cdots \wedge du_n$ on Ω.

For another application, we note the following simple inequality.

PROPOSITION 3.3. *If u, v are convex on Ω, then $\mathcal{M}(u + v) \geqq \mathcal{M}u + \mathcal{M}v$.*

PROOF. By Proposition 3.1, it suffices to prove the inequality when u, v are in C^2. Thus, it suffices to show that $\det(A + B) \geqq \det A + \det B$ for A, B two nonnegative, symmetric $n \times n$ matrices. By continuity, we can assume that A is strictly positive. Then $A + B = A^{1/2}(I + A^{-1/2} \; BA^{-1/2})A^{1/2}$ so we may assume without loss of generality that $A = I$. In that case, if $\lambda_1, \cdots, \lambda_n \geqq 0$ are the eigenvalues of B, we have $\det(I + B) = \prod_{i=1}^{n} (1 + \lambda_i) \geqq 1 + \prod_{i=1}^{n} \lambda_i = \det I + \det B$.

We next show that $\mathcal{M}u$ agrees with the geometrically defined Monge-Ampère measure Mu.

PROPOSITION 3.4. *If u is a convex function on Ω, then $\mathcal{M}u = Mu$. That is, for all Borel subsets E of Ω, $(\mathcal{M}u)(E) = (Mu)(E)$.*

PROOF. Since $\mathcal{M}u$ and Mu are Borel measures, it suffices to prove the formula for small balls E centered at $x_0 \in \Omega$. Without loss of generality we may assume that $x_0 = 0$. Let $E(r) = \{|x| \leqq r\}$ denote the closed ball centered at $x_0 = 0$ with radius r.

Next, observe that if $u \in C^2(\Omega)$ we have that

$$Mu = \mathcal{M}u = \det \left\{ \frac{\partial^2 u}{\partial x_i \partial x_j} \right\} dx_1 \wedge \cdots \wedge dx_n.$$

Therefore, if $\{u^{(j)}\}$ is a sequence of C^2 convex functions converging to u in $C(\Omega)$, and if $\mu_j = \mathcal{M}u^{(j)}$, $\nu_j = Mu^{(j)}$, $\mu = \mathcal{M}u$, $\nu = Mu$, then we know

$$\mu_j(E(r)) = \nu_j(E(r)).$$

Now it is not true that we may obtain $\mu(E(r)) = \nu(E(r))$ for every $r > 0$ by letting $j \to \infty$ in this equality, but only for "most" $r > 0$. Precisely, we claim

$$\limsup_{j \to \infty} \nu_j(E(r)) \leqq \nu(E(r)) \tag{3.5}$$

and, for every $0 \leqq \rho < r$,

$$\liminf_{j \to \infty} \nu_j(E(r)) \geqq \nu(E(\rho)). \tag{3.6}$$

These two inequalities suffice to prove the Proposition, because since $r \to \nu(E(r))$ is an increasing function of r, it follows from (3.5) and (3.6) that $\lim_{j \to \infty} \nu_j(E(r)) = \nu(E(r))$ except possibly at points of discontinuity of $r \to \nu(E(r))$. Also, since $\mu_j \to \mu$ weakly, we have $\mu_j(E(r))$

$\to \mu(E(r))$ except possibly at points of discontinuity of $r \to \mu(E((r))$. Thus, $\mu(E(r)) = \nu(E(r))$ for all but countable many r. However, both $r \to \mu(E(r))$ and $r \to \nu(E(r))$ are right continuous functions of r, so they must agree for all $r \geqq 0$.

To prove (3.5), let χ_j, χ denote, respectively, the characteristic function of the gradient image of $E(r)$ under $u^{(j)}$, u. It is routine to verify that $\limsup \chi_j \leqq \chi$ which, together with Fatou's lemma, implies (3.5).

To prove (3.6), fix $\epsilon > 0$ and $0 \leqq \rho < r$. Set $\eta_j = \max\{u(x) - u^{(j)}(x) : |x| = r\}$, and $v^{(j)}(x) = u^{(j)}(x) + \eta_j + \epsilon(|x|^2 - r^2)$. Then $v^{(j)} \geqq u$ on $|x| = r$. Let $\Omega_j = \{x : |x| < r \text{ and } v^{(j)}(x) < u(x)\}$. Because $\eta_j \to 0$ and $u^{(j)} \to u$ uniformly in $|x| \leqq r$, we have $\Omega_j \supset E(\rho)$ for sufficiently large j. Then for such j, Proposition 2.7 yields

$$\begin{aligned} \nu(E(\rho)) &\leqq \nu(\Omega_j) \\ &\leqq (Mv^{(j)})(\Omega_j) \\ &\leqq (Mv^{(j)})(E(r)). \end{aligned}$$

However, $v^{(j)} \in C^2(\Omega)$, so

$$\begin{aligned} Mv^{(j)} &= \det\left[\frac{\partial^2 v^{(j)}}{\partial x_i \partial x_k}\right] dx_1 \wedge \cdots \wedge dx_n \\ &= Mu^{(j)} + A = \nu_j + A, \end{aligned}$$

where A is a sum of terms of the form

$$\pm \epsilon^k \mathcal{M}_I(u^{(j)}) \wedge d(x_{j_1}^2) \wedge \cdots \wedge d(x_{j_k}^2)$$

with $|I| = n - k$, and $k \geqq 1$. From the first assertion of Proposition 3.1, we then have $|A(E(r))| \leqq C\epsilon$, for some constant C independent of j. Thus, $\liminf (Mv^{(j)})(E(r)) \leqq \liminf \nu_j(E(r)) + C\epsilon$. Since $\epsilon > 0$ is arbitrary, (3.6) follows.

The next result is a quantitative version of Proposition 2.7. It asserts that a convex surface over Ω which dips appreciably below its boundary values must have large gradient image.

LEMMA 3.5. *Let Ω be a bounded, open, convex set in $\mathbf{R}^n$ with diameter, Δ, and for $x \in \Omega$, let $d(x)$ be the distance from x to $\partial\Omega$. If $u \in C(\bar{\Omega})$ is a convex function on Ω with $u \geqq 0$ on $\partial\Omega$ and $u(x) = -h < 0$, then*

$$\int_\Omega Mu \geqq \frac{c_n h^n}{d(x)\Delta^{n-1}}$$

where c_n is a constant depending only on the dimension n.

PROOF. It is no loss of generality to assume $h = 1$. Set $G = \{\xi \in \Omega : u(\xi) < 0\}$. Then G is a bounded, open convex set in $\mathbf{R}^n$, $u \in C(\bar{G})$ and $u = 0$ on ∂G. Construct the inverted cone in $\mathbf{R}^{n+1}$ with depth 1, top equal to $G \times \{0\}$ and vertex at $(x, -1)$. That is, the surface $z = c(\xi)$ in $\mathbf{R}^{n+1}$ such that $c(x) = -1$, $c(\xi) = 0$ if $\xi \in \partial G$, and $\xi \to c(\xi)$ is linear on each line segment $[x, y]$ from $x \in G$ to $y \in \partial G$. Since G is convex, the surface $\{(\xi, c(\xi) : \xi \in \bar{G}\}$ is convex. Also $u \leqq c$ on G, since u is convex. Therefore, by Proposition 2.7, we have $\int_\Omega Mu \geqq \int_G Mu \geqq \int_G Mc =$ measure of gradient image of cone. The explicit form of the estimate of the lemma is the "worst case" estimate of this volume. It is easy to compute by noting the following three facts. First, the gradient image of a cone is convex. Second, it contains at least one vector of length $1/d(x)$ (take a hyperplane of support along the line segment joining x to the closest point of ∂G). Third, it contains the ball centered at the origin of radius $1/\Delta$. It is easy to verify that any set with these three properties has volume at least $2^{n-1}/(n!\, d(x)\Delta^{n-1})$.

REMARK. In case $\partial\Omega$ is smooth with strictly positive curvature, the estimate of the lemma can be replaced by a constant times $[h/d(x)]^n$, where the constant depends on Ω. In case Ω is a ball of radius R centered at the origin, the best result is $\tau_n(Rh)^n/(R^2 - |x|^2)^n$, where τ_n is the volume of the unit ball in $\mathbf{R}^n$.

Next, we prove a minimum principle for the Monge-Ampère equation. This result clearly implies the uniqueness of convex solutions to the Dirichlet problem for M.

LEMMA 3.6. *Let Ω be a bounded open set in $\mathbf{R}^n$, and $u, v \in C(\bar{\Omega})$ be convex functions in Ω. If $Mu \leqq Mv$ in Ω, then*

$$\min\{u(x) - v(x) \,|\, x \in \bar{\Omega}\} = \min\{u(x) - v(x) \,|\, x \in \partial\Omega\}.$$

PROOF. Assume by way of contradiction that $a = \min\{u(x) - v(x) \,|\, x \in \bar{\Omega}\} < \min\{u(x) - v(x) \,|\, x \in \partial\Omega\} = b$. Let $x_0 \in \Omega$ be such that $a = u(x_0) - v(x_0)$. It is no loss of generality to assume $x_0 = 0$. For $\delta > 0$ a small number, namely $\delta[\operatorname{diam}\Omega]^2 < (b - a)/2$, consider the function

$$w(x) = v(x) + \delta|x|^2 + \frac{b-a}{2}.$$

Let $G = \{x \in \bar{\Omega} : u(x) < w(x)\}$. If $x \in \partial\Omega$, then $w(x) \leqq u(x) - b + \delta|x|^2 + (b - a)/2 = u(x) + \delta|x|^2 + (a - b)/2 < u(x)$, so G does not meet $\partial\Omega$. Also, $x_0 = 0 \in G$, since $u(0) = v(0) + a = w(0) + (b - a)/2 < w(0)$. Thus, by Proposition 2.7, $(Mw)(G) \leqq (Mu)(G)$. But, by

Propositions (3.3) and (3.4), $(Mw)(G) = [M(v + \delta|x|^2)](G) \geqq (Mv)(G) + [M(\delta|x|^2)](G) = (Mv)(G) + (2\delta)^n$ volume $(G) > (Mv)(G)$. This contradicts the hypothesis $Mu \leqq Mv$, so the Lemma is proved.

We conclude this section with a convergence theorem which is needed in our solution to Rabinowitz' problem and to the Dirichlet problem.

THEOREM 3.7. *Let Ω be a bounded, open, convex set in* $\mathbf{R}^n$. *Let $u, u_j \in C(\bar{\Omega})$, $j = 1, 2, \cdots$ be convex functions in Ω such that*

(i) *$u_j \to u$ uniformly in $\partial\Omega$;*

(ii) *$Mu_j \to Mu$ weakly in Ω; and*

(iii) *$\int_\Omega Mu_j \leqq A < +\infty$.*

Then $u_j \to u$ in $C(\bar{\Omega})$.

PROOF. First of all, the functions u_j are uniformly bounded above in Ω. In fact, if w is the unique function in $C(\bar{\Omega})$ harmonic in Ω and equal to u in $\partial\Omega$, then since convex functions are subharmonic we have from the maximum principle,

$$u_j(x) \leqq w(x) + \epsilon_j, \quad x \in \Omega \tag{3.7}$$

where $\epsilon_j = \max\{|u_j(x) - u(x)| : x \in \partial\Omega\}$.

We now want to obtain a good lower bound for the $u_j(x)$. Let ζ be a point of $\partial\Omega$. Let $\mathfrak{L}(x)$ be a linear function which supports the surface $x \to u(x)$ at ζ; i.e., $u(x) - u(\zeta) \geqq \mathfrak{L}(x - \zeta)$, $x \in \bar{\Omega}$. Let $v_j(x) = u_j(x) - u(\zeta) - \mathfrak{L}(x - \zeta) + \epsilon_j$ with ϵ_j as in (3.7). We then have $v_j(x) \geqq 0$ for all $x \in \partial\Omega$. Therefore, by Lemma 3.5, for any $x \in \Omega$ either $v_j(x) \geqq 0$, or

$$[-v_j(x)]^n \leqq \text{const. } d(x) A\, \Delta^{n-1}. \tag{3.8}$$

In either case, we have $\liminf_{x \to \zeta \in \partial\Omega} v_j(x) \geqq 0$ uniformly in j; that is for any $\epsilon > 0$ there is a $\delta > 0$ and j_0 such that

$$u_j(x) > u(\zeta) - \epsilon \tag{3.9}$$

when $j > j_0$, $x \in \bar{\Omega}$, and $|x - \zeta| < \delta$.

Now, from (3.8), we see that the functions u_j are uniformly bounded below. Moreover, uniformly bounded families of convex functions are compact in $C(\Omega)$, so the u_j have a subsequence which converges to a convex function v in $C(\Omega)$. In particular, by Proposition 3.1, $Mv = \lim Mu_j = Mu$. Also, from (3.7) and (3.9) it follows that $v \in C(\bar{\Omega})$ and $v = u$ in $\partial\Omega$. Thus, from Lemma 3.6 applied to $u - v$ and $v - u$, we have $v = u$ in Ω. Finally, we see that $u_j \to u$ in $C(\bar{\Omega})$, since $u_j - u$ is small near $\partial\Omega$ by (3.7) and (3.9), while $u_j \to v = u$ in $C(\Omega)$ already. This completes the proof.

4. **The Dirichlet Problem.** We now give the construction of continuous, convex solutions of the Dirichlet problem $Mu = \mu$ in Ω, $u = g$ on $\partial\Omega$. The solution will be given, as suggested by Lemma 3.6, by the familiar Perron method. The original method (which goes back to Minkowski) of finding such solutions was to first construct polygonal solutions, and then find the solution in the general case by a limiting argument. Our method is only slightly different. However, what it shows is that the existence of solutions of $Mu = \mu$ follows essentially from knowing just the special case $Mu = 0$, which was treated in § 2.

To construct solutions via the Perron method, let Ω be a bounded, open, convex set in $\mathbf{R}^n$. Then define, for μ a Borel measure on Ω and g a continuous function on $\partial\Omega$,

$$
\begin{aligned}
\mathcal{V}(u, g) = \{v \in C(\bar{\Omega}) &: v \text{ is convex} \\
&\text{and } Mv \geqq \mu \text{ in } \Omega, \\
&\text{and } v = g \text{ in } \partial\Omega\}.
\end{aligned} \tag{4.1}
$$

According to Lemma 3.6, the solution to the Dirichlet problem $Mu = \mu$, $u = g$ in $\partial\Omega$ should be given by

$$U(x) = \sup \{v(x) : v \in \mathcal{V}(\mu, g)\}. \tag{4.2}$$

Of course, the class $\mathcal{V}(\mu, g)$ may be empty if no restrictions are placed on the measure μ. However, an existence theorem which suffices for our application is the following.

THEOREM 4.1. *If Ω is strictly convex, and if μ is a nonnegative Borel measure on Ω with $\mu(\Omega) < +\infty$, then there is a unique convex $U \in C(\bar{\Omega})$ such that*

$$
\begin{aligned}
MU &= \mu && \text{in } \Omega \\
U &= g && \text{in } \partial\Omega.
\end{aligned}
$$

PROOF. The uniqueness is a consequence of Lemma 3.6. To prove existence, we see from Theorem 3.7 that it suffices to prove the Theorem when $\mu \in P$, the convex cone generated by the positive point masses. For, if it is known in this case, and μ is a nonnegative Borel measure on Ω with $\mu(\Omega) < +\infty$, then there exist $\mu_n \in P$ such that $\mu_n \to \mu$ weakly in Ω and $\mu_n(\Omega) \leqq A < +\infty$. Thus, by Theorem 3.7, the solutions u_n to $Mu_n = \mu_n$, $u_n = g$ in $\partial\Omega$ converge to a solution to $Mu = \mu$, $u = g$ in $\partial\Omega$. According to Lemma 3.6, the solution u must be given by (4.2).

Before noting how the special case $u \in P$ follows from Theorem 2.8, we first list some properties of the class $\mathcal{V}(\mu, g)$.

(4.3) $\mathcal{P}(\mu, g)$ is closed in $C(\bar{\Omega})$.

(4.4) $u, v \in \mathcal{P}(\mu g) \Rightarrow \max(u, v) \in \mathcal{P}(\mu, g)$.

(4.5) If $x_0 \in \Omega, \{|x - x_0| \leqq r\} \subset \Omega$ with $\mu(|x - x_0| = r) = 0$, and if $u \in \mathcal{P}(\mu, g)$, then any "Poisson modification" of u,

$$w(x) = \begin{cases} u(x) & x \in \Omega, \quad |x - x_0| \geqq r \\ v(x) & |x - x_0| \leqq r \end{cases}$$

where v is convex on $|x - x_0| \leqq r$, $v(x) = u(x)$ for $|x - x_0| = r$, and $Mv \geqq \mu$ on $|x - x_0| < r$, also belongs to $\mathcal{P}(\mu, g)$.

The assertion (4.3) is a consequence of Proposition 3.1, while (4.4) follows from the inequality valid for all convex functions,

(4.6) $$M(\max(u, v)) \geqq \min (Mu, Mv).$$

To prove (4.6) let ν denote the right hand side of (4.6). First suppose $\nu \{x \in \Omega : u(x) = v(x)\} = 0$. Then on $\{x \in \Omega : u(x) > v(x)\}$, we have $M(\max(u, v)) = Mu \geqq \nu$ and, similarly, $M(\max(u, v)) \geqq \nu$ on the open set where $u < v$. Since $\nu\{u = v\} = 0$, (4.6) follows. In the general case, just replace u by $u + \epsilon$. For almost all $\epsilon > 0$, $\nu \{x \in \Omega : u(x) + \epsilon = v(x)\} = 0$, since the sets $\Gamma_\epsilon = \{x \in \Omega : u(x) + \epsilon = v(x)\}$ are all disjoint. Choosing a sequence of ϵ tending to zero with $\nu(\Gamma_\epsilon) = 0$, we have from Proposition (3.1) that $M(\max(u, v)) = \lim_{\epsilon \to 0} M(\max(u + \epsilon, v)) \geqq \nu$ as asserted.

To prove (4.5), note that $u \geqq v$ in $|x - x_0| \leqq r$, so $w \in C(\bar{\Omega})$ and is convex in Ω. We clearly have $Mw = Mu \geqq \mu$ if $x \in \Omega$ and $|x - x_0| > r$, while $Mw = Mv \geqq \mu$ if $|x - x_0| < r$. Since $\mu(|x - x_0| = r) = 0$, we thus have $Mw \geqq \mu$, so $w \in \mathcal{P}(\mu, g)$.

Note that as a consequence of (4.3) and (4.4), we have $U \in \mathcal{P}(\mu, g)$ so $MU \geqq \mu$.

We shall now assume $\mu = \sum_{i=1}^m a_i \, \delta_{x_i}$, where $a_i \geqq 0$, $x_i \in \Omega$ and δ_{x_i} is the Dirac measure at x_i. Note that if $v(x) = |x|$, then the gradient image of any neighborhood of 0 under v is the unit ball in $\mathbf{R}^n$, so $Mv = \tau_n \, \delta_0$, where τ_n is the volume of the unit ball in $\mathbf{R}^n$. Therefore, if we put $f(x) = 1/\tau_n \sum_{i=1}^m a_i \, |x - x_i|$, and choose a convex function u on Ω with $u(x) = g(x) - f(x)$ for $x \in \partial\Omega$ (for example we may choose u with $Mu = 0$), then the function $v = f + u$ is convex, $v = g$ in $\partial\Omega$, and by Proposition 3.3 satisfies $Mv \geqq \mu$. In particular, $\mathcal{P}(\mu, g)$ is not empty.

We shall now prove that if U is given by (4.2) then $MU = \mu = \sum_{i=1}^m a_i \, \delta_{x_i}$ by showing the equation holds on a neighborhood of each point x_0 of Ω. First assume $x_0 \neq x_i$, $1 \leqq i \leqq n$. Choose $r > 0$ so small that $\{|x - x_0| \leqq r\} \subset \Omega$ and $|x_i - x_0| > r$, $1 \leqq i \leqq m$. According to Theorem 2.8, there exists a unique convex function v on $|x - x_0| \leqq r$ with $Mv \equiv 0$ for $|x - x_0| < r$ and $v = U$ on $|x - x_0| = r$. By (4.5), we have that

$$w = \begin{cases} U & x \in \Omega, |x - x_0| \geqq r \\ v & |x - x_0| \leqq r \end{cases}$$

belongs to $\mathcal{P}(\mu, g)$. But $w \geqq U$, since $v \geqq U$ for $|x - x_0| \leqq r$. Hence, $w \equiv U$, by the definition of U, so for $|x - x_0| < r$, $Mw = Mv \equiv 0$.

We therefore know that MU is a nonnegative Borel measure supported on $\{x_1, \cdots, x_n\}$ with $MU \geqq \mu$. Write

$$MU = \sum_{i=1}^{m} \lambda_i \alpha_i \delta_{x_i}.$$

We have $\lambda_i \geqq 1$ and must show $\lambda_i = 1$, $1 \leqq i \leqq m$. Fix attention on one x_i, which we may assume is 0. Thus, we have $MU = \lambda a \delta_0$ near $x = 0$ and we have to prove $\lambda = 1$. Assume by way of contradiction that $\lambda > 1$. Choose $r > 0$ so small that $\{|x| \leqq r\} \subset \Omega$ and 0 is the only one of the x_i with $|x_i| \leqq r$. The gradient image of 0 under U is convex with positive measure, and so must contain a ball of some positive radius $\epsilon > 0$. By subtracting a linear function from U (and g), we may assume the gradient image of 0 under U contains the ball of radius $\epsilon > 0$ about the origin, so that $U(x) - U(0) \geqq \epsilon|x|$. Then, by subtracting a constant from U (and g) we may assume $U(0) < 0$ while $U(x) \geqq 0$ for all x with $|x| \geqq r$. Then define a convex function w by

$$w(x) = \begin{cases} U(x) & \text{if } U(x) \geqq 0 \\ \lambda^{-1/n} U(x) & \text{if } U(x) < 0. \end{cases}$$

On the set $\{U < 0\}$, which contains a neighborhood of zero, we have $Mw = (1/\lambda)MU = a\,\delta_0$. The function w is equal to U outside $|x| < r$, so $w \in \mathcal{P}(\mu, g)$. However, $w(0) > U(0)$, which contradicts the definition of U. Therefore, $\lambda = 1$, which completes the proof.

5. **Comparison Theorem for Nonconvex *u*.** Let Ω be a bounded, strictly convex open set in $\mathbf{R}^n$, and u a real-valued function in Ω whose second partial derivatives are in $L^n(\Omega)$. Then

$$\det \left[\frac{\partial^2 u}{\partial x_i \partial x_j} \right]$$

is in $L^1(\Omega)$, so $Mu = \det[\partial^2 u/(\partial x_i \partial x_j)]\, dx_1 \cdots dx_n$ is a welldefined absolutely continuous finite measure in Ω. Also, by the Sobolev embedding theorem [4], $u \in C(\bar{\Omega})$. The goal of this section is to compare u to convex functions, w, with Mw comparable to Mu. The basic result is

THEOREM 5.1. *Let Ω be a bounded, strictly convex set in $\mathbf{R}^n$. Suppose that u is a real-valued function in Ω whose second partial derivatives belong to $L^n(\Omega)$. If $w \in C(\bar{\Omega})$ is convex in Ω and*

$$Mw \geqq [Mu]^+ \equiv \max(Mu, 0)$$

then

$$\min\{u(x) - w(x) \mid x \in \Omega\} = \min\{u(x) - w(x) \mid x \in \partial\Omega\}.$$

In particular, if $w \leqq u$ on $\partial\Omega$ then $w \leqq u$ in Ω.

We will first prove a slightly stronger minimum principle when u is smooth. The general case will then follow by an approximation argument.

LEMMA 5.2. *Suppose Ω is a bounded open set in $\mathbf{R}^n$ and $u \in C^2(\bar{\Omega})$. If $v \in C(\bar{\Omega})$ is convex in Ω with $Mv \geqq Mu$ on the (possibly empty)* open subset of *Ω where the Hessian matrix $[\partial^2 u/(\partial x_i \partial x_j)]$ is positive definite. Then*

$$\min\{u(x) - v(x) : x \in \bar{\Omega}\} = \min\{u(x) - v(x) : x \in \partial\Omega\}.$$

PROOF. Assume by way of contradiction that for some $x_0 \in \Omega$ we have

$$\begin{aligned} u(x_0) - v(x_0) &= \min\{u(x) - v(x) \mid x \in \bar{\Omega}\} \\ &< \min\{u(x) - v(x) \mid x \in \partial\Omega\}. \end{aligned}$$

For $\epsilon > 0$, let $v_\epsilon(x) = v(x) + \epsilon|x - x_0|^2$. If ϵ is sufficiently small, then the function $u(x) - v_\epsilon(x)$ does not attain its minimum on $\partial\Omega$, so there exists $x_1 \in \Omega$ with

$$u(x_1) - v_\epsilon(x_1) = \min\{u(x) - v_\epsilon(x) \mid x \in \bar{\Omega}\}. \tag{5.1}$$

Now set $\Omega_1 = \{x \in \Omega : \text{the Hessian matrix of } u \text{ at } x \text{ is positive}\}$. Then u is convex on Ω_1, and $M(v_\epsilon) \geqq M(v) \geqq Mu$ on Ω_1. From Lemma 3.6, it follows that the minimum of $u - v_\epsilon$ over $\bar{\Omega}_1$ is assumed on the boundary of Ω_1. Therefore, we may assume that the point x_1 in (5.1) does not belong to Ω_1.

Since $x_1 \notin \Omega_1$, at least one eigenvalue λ of the Hessian of u at x_1

is nonpositive. Let $\alpha \in \mathbf{R}^n$, $|\alpha| = 1$, be an eigenvector for this eigenvalue. Then for small values of $t \in \mathbf{R}$, we have

$$u(x_1 + t\alpha) = u(x_1) + \ell(t) + \lambda t^2 + o(t^2) \tag{5.2}$$

where $\ell(t)$ is a linear function of t. Also,

$$\begin{aligned} v_\epsilon(x_1 + t\alpha) - v_\epsilon(x_1) &= v(x_1 + t\alpha) - v(x_1) \\ &\quad + \epsilon[|x_1 + t\alpha - x_0|^2 - |x_1 - x_0|^2] \\ &\geqq \ell_1(t) + \epsilon[|x_1 + t\alpha - x_0|^2 - |x_1 - x_0|^2] \\ &= \ell_2(t) + \epsilon \cdot t^2 \end{aligned}$$

for some linear function $\ell_2(t)$. Combining this inequality with (5.2) yields

$$\begin{aligned} u(x_1 + t\alpha) - v_\epsilon(x_1 + t\alpha) &\leqq u(x_1) - v_\epsilon(x_1) \\ &\quad + \ell_3(t) + (\lambda - \epsilon)t^2 + o(t^2). \end{aligned}$$

However, since $\lambda \leqq 0$ and $\epsilon > 0$, this means that $t \to u(x_1 + t\alpha) - v_\epsilon(x_1 + t\alpha)$ cannot have a local minimum at $t = 0$, and the lemma is proved.

Proof of Theorem 5.1. First of all, since Ω is convex, u is the restriction to Ω of a compactly supported function defined on all of $\mathbf{R}^n$ with its second partial derivatives in $L^n(\mathbf{R}^n)$ ([4], Thm. 12, p. 45). Then there exists a sequence of functions $u_j \in C^\infty(\mathbf{R}^n)$ such that u_j vanishes for $|x| > R$ and the second partial derivatives of u_j converge to the corresponding derivatives of u in $L^n(\Omega)$. Then, by a Sobolev embedding lemma [4], we have also that $u_j \to u$ in $C(\bar{\Omega})$.

Let μ_j be the finite nonnegative Borel measure in Ω, $\mu_j = [Mu_j]^+$ and $\mu = [Mu]^+$, so that $\mu_j \to \mu$ in $L^1(\Omega)$. Let v_j be the unique convex function in $C(\bar{\Omega})$ such that $v_j = u_j$ on $\partial\Omega$ and $Mv_j = \mu_j$ in Ω. The existence and uniqueness of v_j is given by Theorem 4.1. By Lemma 5.2, we have that $v_j \leqq u_j$ in $\bar{\Omega}$. Furthermore, since $u_j \to u$ in $C(\partial\Omega)$ and $Mv_j \to \mu$ in $L^1(\Omega)$, the functions v_j converge in $C(\bar{\Omega})$ to the unique convex function in $C(\bar{\Omega})$ such that $v = u$ on $\partial\Omega$ and $Mv = \mu$ in Ω, by Proposition 3.7. We have just proved that $v \leqq u$ in Ω.

To complete the proof, note that for $x \in \Omega$,

$$\begin{aligned} u(x) - w(x) &= \{u(x) - v(x)\} + \{v(x) - w(x)\} \\ &\geqq v(x) - w(x) \\ &\geqq \min\{v(x) - w(x) : x \in \partial\Omega\} \\ &= \min\{u(x) - w(x) : x \in \partial\Omega\}, \end{aligned}$$

where the last inequality follows from Lemma 3.6.

EXAMPLE. If $u \in W_{2,n}(\Omega)$ and $Mu = 0$ in Ω then $u \geqq \underline{u}$ where $\underline{u}$ is the convex solution of $Mu = 0$ such that $\underline{u} = u$ on $\partial\Omega$. Similarly $-u \geqq v$ where v is the convex solution of $M\bar{v} = 0$ with $v = -u$ on $\partial\Omega$. If $\bar{u}$ is the concave solution of $M\bar{u} = 0$ with $\bar{u} = u$ on $\partial\Omega$ then $v = -\bar{u}$ and we have proved that $\underline{u} \leqq u \leqq \bar{u}$. Now if $u = 0$ on $\partial\Omega$ then $\underline{u} = \bar{u} = 0$ and we have shown that u must vanish identically in Ω. It is this assertion when $n = 2$ that Rabinowitz requires in remark 2.26 of [9].

REFERENCES

1. A. D. Alexandrov, *Die Innere Geometrie der Konvexen Flächen*, Akademie-Verlag, Berlin, 1955.

2. A. D. Alexandrov and V. A. Zalgaller, *Intrinsic Geometry of Surfaces*, Translations of Mathematical Monographs **15**, Amer. Math. Soc., 1967.

3. E. Bedford and B. A. Taylor, *The minimum principle for a complex Monge-Ampère equation*, Inv. Math. **37** (1976), 1–44.

4. A. P. Calderón, *Lebesgue spaces of differentiable functions and distributions*, Proc. Symp. Pure Math. **4** (1961), 33–50.

5. S. S. Chern, H. I. Levine, and L. Nirenberg, *Intrinsic norms on a complex manifold, Global Analysis (Papers in Honor of K. Kodaira)*, Univ. Tokyo Press, Tokyo, 1969, 119–139.

6. A. V. Pogorelov, *Monge-Ampère Equations of Elliptic Type*, P. Noordhoff, Ltd., Groningen, 1964.

7. ——, *Extrinsic Geometry of Convex Surfaces*, Translations of Mathematical Monographs, Vol. **35** Amer. Math. Soc., 1973.

8. ——, *The Dirchlet problem for the n-dimensional analogue of the Monge-Ampère equation*, Soviet Math. Dokl. vol. **12** (1971), 1727–1731.

9. P. Rabinowitz, *A note on topological degree for potential operators*, J. Math. Anal. Appl. **51** (1975), 483–492.

10. H. Rademacher, *Über partielle und totale differenzier-barkeit von funktionen mehrerer variablen, und über die transformation der doppelintegrale*, Math. Ann. **79** (1918), 340–359.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48104