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SPIRAL FUNCTIONS AND RELATED CLASSES WITH FIXED SECOND COEFFICIENT

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ABSTRACT. Denote by $G_p(\lambda, \alpha)(|\lambda| < \pi/2,\ 0 \leqslant \alpha \leqslant \cos\lambda,\ 0 \leqslant p \leqslant \cos\lambda - \alpha,\ \cos\lambda \neq \alpha)$ the class of functions $g(z) = 1 + 2a_2z^2 + \cdots$ analytic in $|z| < 1$ for which $\operatorname{Re}(e^{i\lambda}g(z)) > \alpha$ with $|a_2| = p$. We determine the largest disk $|z| < r = r(\lambda, \alpha, \gamma, \beta, p)$ in which functions in $G_p(\lambda, \alpha)$ satisfy $\operatorname{Re}\{e^{i\gamma}g(z)\} > \beta$. By specializing our parameters and our function $g(z)$, we obtain results relating subclasses of spiral functions to subclasses of starlike functions. When $p = 0$ results concerning odd functions are found.

1. **Introduction.** Let S be the class of functions analytic and univalent in the unit disk, with $f(z)$ in S normalized by $f(0) = 0$ and $f'(0) = 1$. A function $f(z)$ is said to be in $S(\lambda, \alpha)$ if

$$\text{(1)} \qquad \operatorname{Re}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > \alpha\left(|z| < 1, |\lambda| < \frac{\pi}{2},\quad 0 \leqq \alpha < \cos\lambda\right).$$

The class $S(\lambda,\alpha)$ of λ-spiral functions of order α was introduced by Libera [4]. For $\alpha = 0$ we have the so called "spiral-like" functions, defined and shown to be in S by Špaček [9].

In [7] Robertson introduced an associated class consisting of those functions $f(z)$ for which $zf'(z)$ is in $S(\lambda, \alpha)$, which we shall denote by $K(\lambda, \alpha)$. In view of (1.1), a function $f(z)$ is in $K(\lambda, \alpha)$ if and only if

$$\text{(2)} \qquad \operatorname{Re}\left\{e^{i\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha\left(|z| < 1, |\lambda| < \frac{\pi}{2},\ 0 \leqq \alpha \leqq \cos\lambda\right).$$

DEFINITION 1. A function $f(z) = z + a_2z^2 + \cdots$ analytic in $|z| < 1$ is said to be in $S_p(\lambda, \alpha)(|\lambda| < \pi/2,\ |a_2| = 2p,\ 0 \leqq p \leqq \cos\lambda - \alpha,\ \cos\lambda \neq \alpha)$ if it satisfies (1).

DEFINITION 2. A function $f(z)$ is in $K_p(\lambda, \alpha)$ if $zf'(z)$ is in $S_p(\lambda, \alpha)$.

Note that functions in $K_p(\lambda, \alpha)$ must satisfy (2). Although $S_p(\lambda, \alpha) \subset$

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S, functions in $K_p(\lambda, \alpha)$ need not be univalent, as is shown in [3].

In this paper we obtain bounds for these two classes, which reduce to those of $S(\lambda, \alpha)$ or $K(\lambda, \alpha)$ when $p = \cos\lambda - \alpha$, and are otherwise an improvement. Results in terms of a fixed second coefficient have been given for various subclasses of S. Finkelstein [2] investigated the classes $S_p(0, 0)$ and $K_p(0, 0)$, the starlike and convex functions with preassigned second coefficient. Extensions of these results can be found in [1] and [8].

2. **Growth Estimates.** The following lemma, proved in [2] and known to Löwner [5], is used in the proof of our main theorem. It gives a growth estimate for analytic mappings of the unit disk into itself in terms of the second coefficient, and thus generalizes Schwarz's lemma.

LEMMA A. *If $\omega(z) = b_1 z + \cdots$ is an analytic map of the unit disk into itself, then $|b_1| \leqq 1$ and*

$$|\omega(z)| \leqq \frac{r(r + |b_1|)}{1 + |b_1| r} (|z| = r).$$

Equality holds at some $z(\neq 0)$ if and only if

$$\omega(z) = \frac{e^{-it} z(z + b_1 e^{it})}{1 + \bar{b}_1 e^{-it} z} (t \geqq 0).$$

To obtain growth estimates for $S_p(\lambda, \alpha)$ and $K_p(\lambda, \alpha)$ it is useful to consider the following class of functions.

DEFINITION 3. A function $g(z) = 1 + 2a_2 z + \cdots$, analytic in the unit disk, is in $G_p(\lambda, \alpha)(|\lambda| < \pi/2,\ |a_2| = p,\ 0 \leqq p \leqq \cos\lambda - \alpha,\ \cos\lambda \neq \alpha)$ if

$$\operatorname{Re}\{e^{i\lambda} g(z)\} > \alpha \, (|z| < 1).$$

Observe that $f(z) \in S_p(\lambda, \alpha)$ if and only if $zf'(z)/f(z) \in G_p(\lambda, \alpha)$ and $f(z) \in K_p(\lambda, \alpha)$ if and only if $1 + zf''(z)/f'(z) \in G_p(\lambda, \alpha)$.

In the proof of Theorem 1 we shall use a result of Robertson [6].

LEMMA B. *For all real μ and ν the following sharp inequality holds*:

$$\frac{(1 - R^2)\cos\mu + 2R\sin\mu\cos\gamma}{1 - 2R\cos\nu + R^2}$$

$$\geqq \frac{(1 + R^2)\cos\mu - 2R}{1 - R^2} (0 \leqq R < 1).$$

THEOREM 1. *Suppose* $g(z) \in G_p(\lambda, \alpha)$, $|\gamma| < \pi/2$, *and*

$$U = \mathrm{pr} + (\cos\lambda - \alpha)$$

$$V = (\cos\lambda - \alpha)r + p.$$

Then

$$\operatorname{Re}\{e^{i\gamma}g(z)\} \geqq \frac{U^2\cos\gamma - 2r(\cos\lambda - \alpha)UV + V^2r^2[\cos(\gamma - 2\lambda) - 2\alpha\cos(\gamma - \lambda)]}{U^2 - r^2V^2} \quad (|z| = r). \tag{3}$$

The result is sharp.

PROOF. If $g(z) \in G_p(\lambda, \alpha)$ we may write

$$\frac{e^{i\lambda}g(z) - (\alpha + i\sin\lambda)}{\cos\lambda - \alpha} = \frac{1 + \omega(z)}{1 - \omega(z)}, \tag{4}$$

where $\omega(z) = b_1 z + \cdots$ ($|b_1| = p/\cos\lambda - \alpha$) satisfies the hypotheses of Lemma A. Thus

$$|\omega(z)| \leqq \frac{rV}{U}(|z| = r). \tag{5}$$

From (4) we have

$$e^{i\gamma}g(z) = e^{i(\gamma-\lambda)}\left\{(\cos\lambda - \alpha)\left[\frac{1 + \omega(z)}{1 - \omega(z)}\right] + \alpha + i\sin\lambda\right\}$$

and

$$\begin{aligned} \operatorname{Re}\{e^{i\gamma}g(z)\} = {} & (\cos\lambda - \alpha)\cos(\gamma - \lambda)\operatorname{Re}\left[\frac{1 + \omega(z)}{1 - \omega(z)}\right] \\ & + (\cos\lambda - \alpha)\sin(\gamma - \lambda)\operatorname{Im}\left[\frac{1 + \omega(z)}{1 - \omega(z)}\right] \\ & + \alpha\cos(\gamma - \lambda) - \sin(\gamma - \lambda)\sin\lambda. \end{aligned} \tag{6}$$

Setting $\omega(z) = \mathrm{R}e^{i\theta}$ in (6) and simplifying, we obtain

$$\begin{aligned} \operatorname{Re}\{e^{i\gamma}g(z)\} & \\ = (\cos\lambda - \alpha) & \left[\frac{(1 - R^2)\cos(\gamma - \lambda) + 2R\sin\theta\sin(\gamma - \lambda)}{1 - 2R\cos\theta + R^2}\right] \\ & + \alpha\cos(\gamma - \lambda) - \sin(\gamma - \lambda)\sin\lambda. \end{aligned} \tag{7}$$

An application of Lemma B to (7) with $\mu = \gamma - \lambda$ and $\nu = \theta$ yields

$$\{\operatorname{Re} e^{i\gamma} g(z)\} \geqq (\cos\lambda - \alpha)\left[\frac{(1+R^2)\cos(\gamma-\lambda) - 2R}{1-R^2}\right] \tag{8}$$
$$+ \alpha\cos(\gamma-\lambda) - \sin(\gamma-\lambda)\sin\lambda.$$

A substitution of (5) into (8) leads to (3). The sharp function may be obtained by combining the sharp functions of Lemmas A and B.

Corollary 1. *If* $g(z) \in G_p(\lambda, \alpha)$, *then* $\operatorname{Re}\{e^{i\gamma} g(z)\} > \beta$ *for* $|z| < \tilde{r}$, *where* $\tilde{r} = \tilde{r}(\lambda, \alpha, \gamma, \beta, p)$ *is the least positive root of*

$$(\cos\gamma - \beta)(U^2 - V^2 r^2) \tag{9}$$
$$- 2Vr(\cos\lambda - \alpha)(U - Vr\cos(\gamma - \lambda) = 0(|z| = r).$$

Proof. By Theorem 1, $\operatorname{Re}\{e^{i\gamma} g(z)\} > \beta$ when the right side of (3) is $\geqq \beta$. This is equivalent to

$$U^2 \cos\gamma - 2r(\cos\lambda - \alpha)UV + V^2 r^2[\cos(\gamma - 2\lambda) - 2\alpha\cos(\gamma-\lambda)]$$
$$\geqq \beta[U^2 - V^2 r^2]$$

or

$$(\cos\gamma - \beta)(U^2 - V^2 r^2) - 2Vr(\cos\lambda - \alpha)(U - Vr\cos(\gamma - \lambda)) \geqq 0.$$

The sharpness follows as in Theorem 1.

Definition 4. If $f(z) \in S$ and $|\gamma| < \pi/2$, then the spiral radius of order γ and type β of $f(z)$, written $R(\gamma, \beta, f(z))$, is given by

$$R(\gamma, \beta, f(z)) = \sup\left[r : \operatorname{Re}\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\} > \beta, |z| = r\right].$$

Definition 5. If F is a subclass of S, then the spiral radius of order γ and type β of F, denoted $R(\gamma, \beta, F)$, is given by

$$R(\gamma, \beta, F) = \inf_{f(z) \in F} R(\gamma, \beta, f(z)). \tag{10}$$

These definitions reduce to those of Libera [4] when $\beta = 0$. If in addition $\gamma = 0$, then the right side of (10) is the radius of starlikeness of the family F.

Setting $g(z) = zf'(z)/f(z)$ in Corollary 1, we see that $\tilde{r} = R(\gamma, \beta, S_p(\lambda, \alpha))$.

Now set $\alpha = \gamma = \beta = 0$ in Corollary 1, so that $\tilde{r}$ depends only on λ and p. For fixed λ we put $\tilde{r} = \tilde{r}(p, \lambda)$.

We can now relate λ-spiral functions to starlike functions. In the sequel set $G_p(\lambda, 0) = G_p(\lambda)$, $S_p(\lambda, 0) = S_p(\lambda)$, and $K_p(\lambda, 0) = K_p(\lambda)$.

COROLLARY 2. *If* $g(z) \in G_p(\lambda)$ *and we set* $C = \cos\lambda + |\sin\lambda|$ *then* $\operatorname{Re}\{g(z)\} > 0$ *for* $|z| < \tilde{r}(p, \lambda)$, *where*

$$\tilde{r}(p,\lambda) = \left[\frac{p(1-C) + \sqrt{p^2(1-C)^2 + 4C\cos^2\lambda}}{2C\cos\lambda}\right]. \tag{11}$$

Furthermore $\tilde{r}(p, \lambda)$ *is decreasing* $(0 \leqq p \leqq \cos\lambda)$ *with*

$$\tilde{r}(0) = \frac{1}{\sqrt{C}}$$

$$\tilde{r}(\cos\lambda) = \frac{1}{C}.$$

PROOF. Set $\alpha = \gamma = \beta = 0$ in (9). The least positive root of this equation is given by (11). The quantities $\partial/\partial p\,[\tilde{r}(p, \lambda)]$ and

$$(1-C)\left[\sqrt{p^2(1-C)^2 + 4C\cos^2\lambda} + p(1-C)\right] \tag{12}$$

have the same sign. Since

$$\sqrt{p^2(1-C)^2 + 4C\cos^2\lambda} \geqq p|1-C|$$

and $C \geqq 1$, (12) is nonpositive. Thus $\tilde{r}(p, \lambda)$ is decreasing in p.

REMARK. The radius of starlikeness of $S_p(\lambda)$ is thus given by (11). Since C is maximized at $|\lambda| = \pi/4$, for all p we have

$$\left[\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right]^{-1} = \frac{1}{\sqrt{2}} \leqq \tilde{r}(p,\lambda) \leqq \frac{1}{\sqrt[4]{2}} = \left[\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right]^{-1/2}. \tag{13}$$

The lower bound in (13) is the radius of starlikeness of λ-spiral functions, found by Robertson in [6]. The upper bound in (13) shows that odd λ-spiral functions are starlike for $|z| < 1/\sqrt[4]{2}$

In the result that follows we relate starlike to spiral functions.

COROLLARY 3. *If* $g(z) \in G_p(0)$ *and we set* $D = \sec\gamma + |\tan\gamma|$ *then* $\operatorname{Re}\{e^{i\gamma}g(z)\} > 0$ *for* $|z| < \tilde{r}(p, \gamma)$, *where*

$$\tilde{r}(p, \gamma) = \left[\frac{p(1-D) + \sqrt{p^2(1-D)^2 + 4D}}{2D}\right]. \tag{14}$$

Furthermore $\tilde{r}(p, \gamma)$ *is decreasing* $(0 \leqq p \leqq 1)$ *with*

$$\tilde{r}(0) = \frac{1}{\sqrt{D}}$$

$$\tilde{r}(1) = \frac{1}{D}.$$

PROOF. Set $\alpha = \lambda = \beta = 0$ in (9). The least positive root is then given by (14). It can easily be shown that $\partial/\partial p[\tilde{r}(p, \gamma)] \leqq 0$.

REMARK 1. Thus if $f(z) \in S_p(0)$, then $\mathrm{Re}\{e^{i\gamma}\ zf'(z)/f(z)\} > 0$ $|z| < \tilde{r}(p, \gamma)$, where $\tilde{r}(p, \gamma)$ is defined by (14).

REMARK 2. We can obtain results about the family $K_p(\lambda)$ from those found for $S_p(\lambda)$ by a simple application of Definition 2.

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