ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 6, Number 3, Summer 1976

A GREEN'S FUNCTION CONVERGENCE PRINCIPLE, WITH APPLICATIONS TO COMPUTATION AND NORM ESTIMATES

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ABSTRACT. A new tool is developed for the study of Green's functions of multipoint boundary value problems for kth order linear ordinary scalar equations. The tool takes the form of a convergence principle for Green's functions, and is the proper version of a continuity theorem for boundary value problems. Two types of applications are considered: (1) practical computation of Green's functions and (2) norm estimates for Green's functions in the spaces $C^n[a, b]$ and $L^1[a, b]$. Special attention is given to frequently used equations and boundary conditions, in particular, the equation $y^{(k)} = 0$ is studied in detail for 2-point and multipoint boundary conditions.

1. **Introduction.** The purpose of this paper is to develop new techniques for the study of Green's function and the associated integral operator for the kth order linear ordinary scalar equation with continuous coefficients

$$(1.1) \qquad Ku \equiv u^{(k)} + \sum_{s=0}^{k-1} q_s(t)u^{(s)} = 0$$

subject to the *Niccoletti boundary conditions* [13]:

$$(1.2) \qquad u \text{ has } \alpha \text{ zeros at } T, \ |\alpha| = k.$$

In relation (1.2), $\alpha = (n_0, \cdots, n_\nu)$, $T = \{a = s_0 < s_1 < \cdots < s_\nu = b\}$, $|\alpha| = \sum_{i=0}^{\nu} n_i$, the symbols $n_0, \cdots, n_\nu$ are positive integers. Relation (1.2) shall abbreviate the conditions $u^{(i)}(s_j) = 0 \quad (0 \leqq i < n_j, 0 \leqq j \leqq \nu)$, following the usage of the author [8].

The tool developed here is a *convergence principle* for the Green's function $G(t, s; \alpha, T)$ of problem (1.1)–(1.2). *Roughly speaking, the principle says that inequalities, identities, etc. for Green's functions can be obtained in the general case by limiting on the special case of k-point Green's functions.*

Received by the editors on May 31, 1974, and in revised form on June 27, 1975.
AMS 1970 subject classifications: primary 34B05, secondary 34B10.
Key words and phrases. Green's function, Niccoletti problem, convergence properties of solutions, convergence of Green's functions, norm estimates, computation of Green's functions.
*This research was supported by the U.S. Army under grant number ARO-D-31-124-72-G56.

To illustrate the convergence principle, consider the Beesack inequality [3]:

$$|G(t, s; \alpha_0, T_0)| \leqq \frac{\left| \prod_{i=0}^{k-1} (t - t_i) \right|}{(b-a)(k-1)!}, \tag{1.3}$$

valid for the equation $y^{(k)} = 0$, $\alpha_0 = (1, 1, \cdots, 1)$, $T_0 = \{a = t_0 < \cdots < t_{k-1} = b\}$. The convergence principle obtained here allows us to formally limit across this inequality and obtain the general inequality

$$|G(t, s; \alpha, T)| \leqq \frac{\left| \prod_{i=0}^{\nu} (t - s_i)^{n_i} \right|}{(b-a)(k-1)!}, \tag{1.4}$$

valid for $y^{(k)} = 0$ and boundary conditions (1.2).

An elementary proof of (1.3) has been obtained by Nehari [12], therefore the convergence principle results in a conceptually simple proof of (1.4).

The convergence principle can be written as

$$\lim_{p \to \infty} G(t, s; \alpha_p, T_p) = G(t, s; \alpha, T). \tag{1.5}$$

The limit is to be taken in an appropriate Banach Space.

The difficulties associated with the demonstration of equality (1.5) are as follows. First, a suitable representation of G must be obtained (§ 2). Secondly, it is necessary to define the notion of convergence of boundary data $(\alpha_p, T_p) \to (\alpha_0, T_0)$ in a setting general enough to include geometrical intuition. This is done in § 4, and it is shown that this notion can be reformulated in terms of convergence of normalized boundary operators in an appropriate Banach space of vector-valued linear operators. Finally, the derivation of (1.5) requires the development of certain calculus identities and inequalities for multivariate determinants (§ 3).

The technical results on determinants in § 3 are developed for use in §§ 4 and 5. However, these identities are perhaps of independent interest, because of the role of such determinants in the oscillation properties of solutions of (1.1). For example, see the author's work [8], [9], Peterson [15], and the references therein.

The various forms of the convergence principle for Green's functions are given in § 5. Applications of the principle appear in §§ 6–9.

Practical computation of Green's function is considered in § 6. Scalar formulas for G are obtained for equations of special interest. In particular, $y^{(k)} = 0$ is treated for k-point and 2-point problems, and constant coefficient equations are treated for k-point problems. A method is given for practical computation of G, using the convergence principle.

Sign properties of G, under the hypothesis of disconjugacy of K, are recorded in § 7.

Norm estimates for G in the spaces $C^n[a, b]$ and $L^1[a, b]$ are obtained in §§ 8 and 9. In § 8 we treat the special equation $u^{(k)} = 0$, and in § 9 the general equation $Ku = 0$.

DEFINITION 1.1. The equation $Ku = 0$ is *disconjugate* on $[a, b]$ iff the only solution of $Ku = 0$ with k zeros in $[a, b]$ counting multiplicities is $u \equiv 0$.

POLYA'S DISCONJUGACY CRITERION

A well-known criteria of Polya [17] states that K is disconjugate on $[a, b]$ iff there exists $k + 1$ positive functions $b_0, \cdots, b_k$ such that

$$Ku \equiv b_k^{-1}(\cdots (b_1^{-1}(b_0^{-1}u)')' \cdots)' \quad (a \leqq t \leqq b)$$

for every $u \in C^k[a, b]$. The nonspecialist may find Coppel's notes [6] a convenient reference for this result.

If the Polya factorization is valid, then K can be treated as $(d/dt)^k$, in the sense that $Ku \equiv b_k^{-1}u^{[k]}$, the symbol $u^{[k]}$ being the k-th *generalized derivative* given inductively by the relations

$$u^{[o]} = u, u^{[i+1]} = (b_i^{-1}u^{[i]})', \ (0 \leqq i \leqq k - 1).$$

DEFINITION 1.2. The symbol $H(a, b; \alpha, T)$ shall abbreviate the hypothesis that the Niccoletti problem (1.1)–(1.2) has only the zero solution.

DEFINITION 1.3. Let $\{\lambda_p\}_{p=1}^{\infty}$ be a real-valued sequence. The statement $\mu_p = O(\lambda_p)$ shall mean that a constant $M \geqq 0$ exists satisfying $|\mu_p| \leqq M|\lambda_p|$ for $p \geqq 1$. The statement $f(x) = O(g(x))$ [as $x \to c$] shall mean that a constant $N \geqq 0$ exists satisfying $|f(x)| \leqq N|g(x)|$ in some deleted neighborhood of $x = c$.

DEFINITION 1.4. The *usual norm* in $C^m(I \to R^n)$ is defined by

$$\|f\| = \max \{\sup\{|f^{(i)}(t)| : t \in I\} : 0 \leqq i \leqq m\},$$

where $| \cdot |$ is the Euclidean norm in R^n.

DEFINITION 1.5. The *adjoint operator* K^* corresponding to the operator K of (1.1) is defined only in case $q_k \in C^k$, and in this case

$$K^*y \equiv (-1)^k y^{(k)} + \sum_{i=0}^{k-1} (-1)^i (q_i(t)y)^{(i)}.$$

REMARK 1.6. The problem referred to above as the *Niccoletti boundary value problem* is called by many the *de la Valleé Poussin boundary value problem*. The ideas of Niccoletti [13] focus on first-order systems of linear differential equations with boundary conditions imposed at points $t_1, t_2, \cdots, t_n$.

2. **The Green's function.** Consider the linear ordinary differential equation

$$Ku = 0;\ Ku \equiv u^{(k)} + \sum_{s=0}^{k-1} q_s(t)u^{(s)}. \tag{2.1}$$

It is assumed that each coefficient $q_s(t)$ belongs to $C[a, b]$. The problem considered here is the inversion of the operator equation

$$Ku = f \tag{2.2}$$

with $f \in C[a, b]$ fixed, subject to the boundary condition

$$u \text{ has } \alpha_0 \text{ zeros at } T_0. \tag{2.3}$$

Under appropriate assumptions, the inverse is a linear integral operator with continuous kernel $G(t, s; \alpha_0, T_0)$, called the *Green's function.*

CONSTRUCTION OF $G(t, s; \alpha_0, T_0)$.

Let $U = (u_1, \cdots, u_k)$ be a fixed but otherwise arbitrary basis for $Ku = 0$. Denote by Z the $k \times k$ matrix whose rows are $U^{(i)}(s_j)$ $(0 \leqq i \leqq n_j - 1, j = 0, 1, \cdots, \nu)$, in natural order.

LEMMA 2.1. det $Z \neq 0$ *iff* $H(a, b; \alpha_0, T_0)$.

LEMMA 2.2. *If* $H(a, b; \alpha_0, T_0)$, *then problem* (2.2)–(2.3) *is invertible.*

Therefore, the problem has an inverse under the uniqueness assumption that (2.1), (2.3) has no nontrivial solution. We proceed now to write down the inverse operator, under this assumption.

Let $W(s)$ be the Wronskian matrix of $u_1, \cdots, u_k$ and put

$$e = (0, \cdots, 0, 1)^T \in R^k. \tag{2.4}$$

Define $c(s) \in C([a, b] \to \mathbf{R}^k)$, $b(s) \in PC([a, b] \to \mathbf{R}^k)$, as follows:

$$W(s)c(s) = e, \tag{2.5}$$

$$\begin{aligned} U^{(i)}(s_j)[b(s) + c(s)] &= 0, 0 \leqq i < n_j, s \leqq s_j, \\ U^{(i)}(s_j)\, b(s) &= 0,\ 0 \leqq i < n_j, s \geqq s_j, \end{aligned} \tag{2.6}$$

for $j = 0, 1, \cdots, \nu$.

Let $V(s) = \operatorname{diag}(\chi_{E_0}(s)\, I_{n_0}, \chi_{E_1}(s)\, I_{n_1}, \cdots, \chi_{E_\nu}(s)\, I_{n_\nu})$. Here, $E_0 = \{s_0\}$, $E_i = [s_0, s_i]$ (for $1 \leqq i \leqq \nu$), I_{n_i} is the $n_i \times n_i$ identity matrix, and χ_{E_i} stands for the characteristic function of E_i.

System (2.5), (2.6) can be converted to vector-matrix form:

$$c(s) = W^{-1}(s)e, b(s) = -(Z^{-1}V(s)Z)W^{-1}(s)e. \tag{2.7}$$

Let us *define* $G(t, s; \alpha_0, T_0)$ as follows:

$$G(t, s; \alpha_0, T_0) = \begin{cases} U(t)[b(s) + c(s)], a \leqq s < t \leqq b, \\ U(t)b(s), a \leqq t \leqq s \leqq b. \end{cases} \tag{2.8}$$

A convenient matrix formulation of (2.8) is obtained from (2.7) as follows: let $\epsilon(u) = 1$ or 0 accordingly as $u > 0$ or $u \leqq 0$, then

$$G(t, s; \alpha_0, T_0) = U(t)Z^{-1}[\epsilon(t - s)I - V(s)]\, ZW^{-1}(s)e. \tag{2.9}$$

For purposes of calculation, the most substantial reduction in the number of terms in relation (2.9) is witnessed by introducing the new basis $U^* = UZ^{-1}$. Indeed, the Wronskian matrix W^* of the new basis U^* is given by $W^* = WZ^{-1}$, therefore $W^{*-1} = ZW^{-1}$. With this notation, relation (2.9) becomes

$$G(t, s; \alpha_0, T_0) = U^*(t)[\epsilon(t - s)I - V(s)]\, W^{*-1}(s)e. \tag{2.10}$$

The basis $U^* = (u_1^*, \cdots, u_k^*)$ is given in terms of the basis U by $u_j^* = \det Y_j(t)/\det Z$, where $Y_j(t)$ is the matrix Z with row j replaced by $U(t)$, $1 \leqq j \leqq k$.

Properties of the Green's Function

Lemma 2.3. *Assume $H(a, b; \alpha_0, T_0)$. Then $G(t, s; \alpha_0, T_0)$ does not depend on the basis U selected for its construction.*

Proof. Let U and U^+ be two bases, Z and Z^+ the corresponding Z-matrices, W and W^+ the respective Wronskian matrices. Then $U = U^+D$ with D nonsingular, hence $Z = Z^+D$, $W = W^+D$ and

$$\begin{aligned} &U(t)Z^{-1}[\epsilon(t-s)I - V(s)]\,ZW^{-1}(s)e \\ &= U^{+}(t)D[Z^{+}D]^{-1}[\epsilon(t-s)I - V(s)]\,Z^{+}D[W^{+}(s)D]^{-1}e \\ &= U^{+}(t)Z^{+-1}[\epsilon(t-s)I - V(s)]\,Z^{+}W^{+-1}(s)e. \end{aligned}$$

Therefore (2.9) does not depend on the basis, and the lemma is proved.

LEMMA 2.4. *Assume* $H(a, b; \alpha_0, T_0)$. *Let* $G(t, s; \alpha_0, T_0)$ *be given by* (2.9). *Then the unique solution of* (2.2)–(2.3) *guaranteed by Lemma* 2.2 *is given by*

$$u(t) = \int_a^b G(t, s; \alpha_0, T_0)f(s)\,ds. \tag{2.11}$$

PROOF. By Lemma 2.3 we can use (2.10) for the definition of G. Therefore,

$$u(t) = U^*(t)\left[\int_a^t [I - V(s)]\,W^{*-1}(s)f(s)\,ds - \int_t^b V(s)W^{*-1}(s)f(s)\,ds\right]e.$$

The points $s_0, \cdots, s_\nu$ cause trouble for the differentiation process, but one can show directly that $u \in C^k[a, b]$ (use (2.5), (2.6)), and $u^{(i)}(t) = [(d/dt)^i U^*(t)]\,[\int_a^b [\epsilon(t-s)\cdot I - V(s)]\,W^{*-1}(s)f(s)\,ds]\,e$ for $0 \leqq i \leqq k-1$. However, we must add for $i = k$ the term $[(d/dt)^{k-1}U^*(t)]\,[W^{*-1}(t)e]\,f(t)$; by cofactor expansion, this is $f(t)$. Therefore, $Ku = f$. The boundary conditions (2.3) are an immediate consequence of (2.6), (2.7).

REMARK 2.5. Two-point problems are discussed in Naimark [**11**], but the formulas recorded there are not useful for the purposes here. A literature search reveals several different viewpoints for proving the *existence* of G, but few seem to consider the question of *computation*; see [**3**], [**5**], [**6**], [**7**], [**10**], [**11**], [**13**], [**18**], [**20**] and the references therein. One exception here is the work of Pokornyi [**16**], where a formula equivalent to (6.2) *infra* is used in connection with lower estimates for G.

The Green's function G defined above satisfies

$$G^{(k-1)}(s^+, s; \alpha, T) - G^{(k-1)}(s^-, s; \alpha, T) = 1, \tag{2.12}$$

in agreement with Coppel [**6**] ($G^{(k-1)} = (\partial/\partial t)^{k-1}G$). Some authors arrive at -1 for the RHS of (2.12), because they consider (2.11) to be the solution of $Ku = -f$.

3. Determinant identities and inequalities. The identities and inequalities developed in this section will be used in §§ 4 and 5.

To illustrate what needs to be done, consider a function $u \in C^3[0, 1]$, points $0 \leqq x_1 < x_2 < x_3 \leqq 1$, and put $x = (x_1, x_2, x_3)$,

$$A(x) = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}, \quad B(x) = \begin{bmatrix} 1 & x_1 & u(x_1) \\ 1 & x_2 & u(x_2) \\ 1 & x_3 & u(x_3) \end{bmatrix}.$$

The problem is to determine the limiting value of the quotient $F(x) \equiv [\det B(x)]/[\det A(x)]$ as $x \to 0$.

An intuitive notion of what should be true can be gained by setting $x = x(t) \equiv (t, 2t, 3t)$, $0 < t \leqq 1/3$. This procedure produces a one-variable problem to which L'Hospital's rule is applicable, and one finds by the rule for differentiation of determinants that

$$\lim F(x) = \frac{\begin{vmatrix} 1 & 0 & u(0) \\ 0 & 1 & u'(0) \\ 0 & 0 & u''(0) \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix}} = \frac{u''(0)}{2!}, \text{ as } x \to 0.$$

Therefore, the correct answer is known for the limit, provided it exists, and one is led to seek a relation

$$\det B(x) = \det A(x)[\det W(0) + O(|x|)],$$

where $W(t)$ is the Wronskian matrix of 1, t, $u(t)$, except for some constant factor.

In the lemma below, we consider $n \times n$ matrices whose rows have the form $V(x_i)$, $1 \leqq i \leqq n$, for some C^n function V. In the application later on, V will be obtained from the row vector $[1, t, t^2/2!, \cdots, t^{n-1}/(n-1)!]$ be replacing one of the elements by $u(t)$, $u \in C^n$.

Throughout this section, $|\cdot|$ is the maximum norm in Euclidean space of any dimension.

LEMMA 3.1. *Let* $V(t) = (v_1(t), \cdots, v_n(t)) \in C^n([a, b] \to R^n)$, $0 \in [a, b]$, $a \leqq x_1 < \cdots < x_n \leqq b$, *and define*

$$A(x) = \begin{bmatrix} v_1(x_1) & \cdots v_n(x_1) \\ v_1(x_2) & \cdots v_n(x_2) \\ \vdots & \vdots \\ v_1(x_n) & \cdots v_n(x_n) \end{bmatrix}, \quad W(t) = \begin{bmatrix} v_1(t) & \cdots v_n(t) \\ v_1'(t) & \cdots v_n'(t) \\ \vdots & \vdots \\ v_1^{(n-1)}(t) & \cdots v_n^{(n-1)}(t) \end{bmatrix},$$

$P(x) = \operatorname{diag}(1, (1/1!)(x_2 - x_1), \cdots, (1/(n-1)!\prod_{i=1}^{n-1}(x_n - x_i))$. *Then there exists continuous functions*

$$r_{i,j}(x_1, \cdots, x_i) : [a, b] \to R, \quad 1 \leqq i, j \leqq n,$$

such that the $n \times n$ *remainder matrix* $R(x) = [r_{i,j}(x_1, \cdots, x_i)]$ *satisfies*

$$\det A(x) = \det \{P(x)[W(0) + R(x)]\}. \tag{3.1}$$

The functions $\{r_{i,j}\}$ *satisfy the error estimate*

$$|r_{i,j}(x_1, \cdots, x_i)| \leqq \left(\prod_{s=0}^{i-1} s!\right) \times \tag{3.2}$$

$$\max\{|v_j^{(i)}(x)| : x \in Q_i\} \max\{|x_1|, \cdots, |x_i|\}, 1 \leqq i, j \leqq n,$$

where

$$Q_i = \bigcap \{[c, d] : 0, x_1, x_i \in [c, d]\}.$$

Proof. Identity (3.1) is proved by using elementary row operations on $\det A(x)$. Let us show that the m-th row $V(x_m)$ of $\det A(x)$ can be replaced by

$$[V^{(m-1)}(0) + R_m(x_1, \cdots, x_m)](1/(m-1)!)\prod_{i=1}^{m-1}(x_m - x_i),$$

where the components of $R_m = (r_{m1}, \cdots, r_{mn})$ satisfy (3.2). This will be done by using elementary row operations on the first $m - 1$ rows of $\det A(x)$. Therefore, the claimed identity (3.1) follows by successive application of this special result to rows $n, n-1, \cdots, 1$ of $\det A(x)$.

The first step is to expand $V(t)$ in a vector Maclaurin expansion with integral remainder:

$$V(t) = \sum_{i=0}^{m-1} \frac{1}{i!} V^{(i)}(0)t^i + \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} V^{(m)}(s)\, ds. \tag{3.3}$$

The integral remainder in (3.3) will be abbreviated by $\phi(t)$ hereafter.

Let us put $t = x_j$ $(1 \leqq j \leqq m)$ into relation (3.3) to obtain the identities.

$$V(x_j) = \sum_{i=0}^{m-1} \frac{1}{i!} V^{(i)}(0)x_j^i + \phi(x_j) \quad (1 \leqq j \leqq m). \tag{3.4}$$

Define $C_{i,j}$ to be the cofactor of element i, j in the *Vandermonde determinant*

$$\mathcal{V}_m \equiv \begin{vmatrix} 1 & x_1 & \cdots & x_1^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_m & \cdots & x_m^{m-1} \end{vmatrix} = \prod \{(x_j - x_i): 1 \leqq i < j \leqq m\}.$$

The dependence of $C_{i,j}$ on m has been suppressed, for brevity.

Consider the row $V(x_m)$ in det $A(x)$. Add to this row the row combination

$$\sum_{j=1}^{m-1} \frac{C_{j,m}}{C_{m,m}} V(x_j);$$

this will not alter the value of det $A(x)$. The new row $V_m{}^*$ obtained in this way is given, because of (3.4) and the identity $C_{m,m} = \mathcal{V}_{m-1}$, by

$$\begin{aligned} V_m{}^* &\equiv V(x_m) + \sum_{j=1}^{m-1} \frac{C_{j,m}}{C_{m,m}} V(x_j) \\ &= \frac{1}{\mathcal{V}_{m-1}} \sum_{j=1}^{m} C_{j,m} \left\{ \sum_{i=0}^{m-1} \frac{1}{i!} V^{(i)}(0) x_j{}^i + \phi(x_j) \right\} \\ &= \frac{1}{\mathcal{V}_{m-1}} \sum_{i=0}^{m-1} \left(\sum_{j=1}^{m} C_{j,m} x_j{}^i \right) \frac{1}{i!} V^{(i)}(0) \\ &\quad + \frac{1}{\mathcal{V}_{m-1}} \sum_{j=1}^{m} C_{j,m} \phi(x_j). \end{aligned}$$

The cofactor expansion identity

$$\sum_{j=1}^{m} x_j{}^i C_{j,m} = \begin{vmatrix} 1 & x_1 & \cdots x_1{}^{m-2} & x_1{}^i \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & \cdots x_m{}^{m-2} & x_m{}^i \end{vmatrix} = \mathcal{V}_m \delta_{i,m-1} (0 \leqq i \leqq m-1)$$

(where $\delta_{i,k}$ is Kronecker's delta) gives

$$\begin{aligned} V_m{}^* &= \frac{\mathcal{V}_m}{\mathcal{V}_{m-1}} \frac{V^{(m-1)}(0)}{(m-1)!} + \psi(x_1, \cdots, x_m) \\ &= \prod_{i=1}^{m-1} (x_m - x_i) \frac{V^{(m-1)}(0)}{(m-1)!} + \psi(x_1, \cdots, x_m), \end{aligned}$$

where

$$\psi(x_1, \cdots, x_m) \equiv \frac{1}{\mathcal{V}_{m-1}} \sum_{j=1}^{m} C_{j,m}\phi(x_j).$$

To complete the proof, it suffices to show that

$$\psi = \frac{1}{(m-1)!} R_m(x_1, \cdots, x_m) \prod_{i=1}^{m-1} (x_m - x_i),$$

where $R_m = (r_{m1}, \cdots, r_{mn})$ has components satisfying (3.2). To do this let $\psi = (\psi_1, \cdots, \psi_n)$, $\phi = (\phi_1, \cdots, \phi_n)$, so that

$$\psi_i = \frac{1}{\mathcal{V}_{m-1}} \cdot \sum_{j=1}^{m} C_{j,m}\phi_i(x_j),$$

$$\phi_i(t) = \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} v_i^{(m)}(s)\, ds, \quad 1 \leqq i \leqq n.$$

Then, for $1 \leqq i \leqq n$,

$$\begin{aligned} \mathcal{V}_{m-1}\psi_i &= \sum_{j=1}^{m} C_{jm}\phi_i(x_j) \\ &= \begin{vmatrix} 1 & x_1 & \cdots x_1^{m-2} & \phi_i(x_1) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x_m & \cdots x_m^{m-2} & \phi_i(x_m) \end{vmatrix}, \end{aligned}$$

and we denote this determinant by $Y_i\,(x_1, \cdots, x_m)$, $1 \leqq i \leqq n$. The dependence of Y_i on v_i has been deleted for brevity.

Let $\mathcal{P}_k$ be the proposition that for all possible choices of $v_i(t) \in C^k[a, b]$ and each choice of distinct points $x_1 < x_2 < \cdots < x_k$ in $[a, b]$,

$$\text{(3.5)} \quad |Y_i(x_1, \cdots, x_k)| \leqq \left(\prod_{j=0}^{k-2} j! \right) \mathcal{V}_k \|v_i^{(k)}\| \max_{1 \leqq j \leqq k} |x_j|, \ (1 \leqq i \leqq n),$$

where $\|\cdot\|$ is the max norm on $[x_1, x_k]$, $k = 2, 3, \cdots$. It will be shown that $\mathcal{P}_k$ is true for each $k \geqq 2$.

Consider first the proposition $\mathcal{P}_2$. Then $|Y_i| = |\phi_i(x_2) - \phi_i(x_1)| = |\int_{x_1}^{x_2} \phi_i{}'(t)\, dt| = |\int_{x_1}^{x_2} (d/dt) (\int_0^t (t-s)v_i''(s)\, ds)\, dt| = |\int_{x_1}^{x_2} \int_0^t v_i''(s)\, ds\, dt| \leqq |x_2 - x_1|\ \|v_i''\|\ \max_{x_1 \leqq t \leqq x_2} |t|$, hence (3.5) holds and $\mathcal{P}_2$ is true.

Suppose proposition $\mathcal{P}_k$ is true for some $k \geqq 2$. Let us verify that $\mathcal{P}_{k+1}$ is true. Let v_i be any function of class C^{k+1}, then by the fundamental theorem of calculus, applied to rows 2 through $k+1$ of Y_i, and the zero properties of Y_i, it follows that

$$Y_i(x_1, \cdots, x_{k+1}) = \int_{x_k}^{x_{k+1}} \cdots \int_{x_1}^{x_2} Z_i(t_1, \cdots, t_k)\, dt_1 \cdots dt_k,$$

where

$$Z_i = (k-1)! \begin{vmatrix} 1 & x_1/1 & x_1^2/2 & \cdots x_1^{k-1}/k-1 & \phi_i(x_1) \\ 0 & 1 & t_1 & \cdots t_1^{k-2} & \phi_i'(t_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & t_k & \cdots t_k^{k-2} & \phi_i'(t_k) \end{vmatrix}$$

$$= \frac{\partial^k Y_i}{\partial x_{k+1} \cdots \partial x_2}(x_1\, t_1, \cdots, t_k),$$

$t_j \in [x_j, x_{j+1}]$, $1 \leqq j \leqq k$. A cofactor expansion along column 1 allows Z_i to be rewritten in the form

$$Z_i = (k-1)! \begin{vmatrix} 1 & t_1 & \cdots \phi_i'(t_1) \\ \vdots & \vdots & \vdots \\ 1 & t_k & \cdots \phi_i'(t_k) \end{vmatrix}.$$

If $x_j < t_j < x_{j+1}$ $(1 \leqq j \leqq k)$, then by relation (3.5) and the induction hypothesis $\mathcal{P}_k$,

$$|Z_i| \leqq (k-1)! \left\{ \prod_{j=0}^{k-2} j! \right\} \left\{ \prod_{1 \leqq i < j \leqq k} (t_j - t_i) \right\} \|v_i^{(k+1)}\| \max_{1 \leqq i \leqq k} |t_i|,$$

where $\| \cdot \|$ is the max norm on $[x_1, x_{k+1}]$. Indeed,

$$\phi_i(t) = \int_0^t \frac{(t-s)^k}{k!} v_i^{(k+1)}(s)\, ds,$$

so

$$\phi_i'(t) = \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} [v_i'(s)]^{(k)}\, ds.$$

Therefore (3.5) can be applied; $(Z_i(t_1, \cdots, t_k)/(k-1)!$ replaces $Y_i(x_1, \cdots, x_k))$, by virtue of the induction hypotheses $\mathcal{P}_k$.

Moreover, $t_j \in [x_j, x_{j+1}]$ $(1 \leqq j \leqq k)$ implies $0 \leqq t_j - t_i \leqq x_{j+1} - x_i$, $(j > i)$, therefore

$$0 \leqq \prod_{1 \leqq i < j \leqq k} (t_j - t_i) \leqq \prod_{1 \leqq i < j \leqq k} (x_{j+1} - x_i).$$

Combining these inequalities gives

$$|Y_i(x_1, \cdots, x_{k+1})| \int_{x_k}^{x_{k+1}} \cdots \int_{x_1}^{x_2} |Z_i|\, dt_k \cdots dt_1$$

$$\leqq \left\{ \prod_{i=1}^{k} (x_{i+1} - x_i) \right\} \left\{ \prod_{r=0}^{k-1} r! \right\}$$
$$\left\{ \prod_{1 \leqq i < j \leqq k} (x_{j+1} - x_i) \right\} \|v_i^{(k+1)}\| \max_{1 \leqq i \leqq k+1} |x_i|$$

$$= \mathcal{V}_{k+1} \left\{ \prod_{j=0}^{k-1} j! \right\} \|v_i^{(k+1)}\| \max_{1 \leqq i \leqq k+1} |x_i|.$$

Therefore, $\mathcal{P}_{k+1}$ is true, and the induction is complete.

Proposition $\mathcal{P}_m$ is applied to $Y_i(x_1, \cdots, x_m) \equiv \mathcal{V}_{m-1}\psi_i$, giving the inequality

$$|\psi_i| \leqq \left\{ \prod_{j=0}^{m-2} j! \right\} \frac{\mathcal{V}_m}{\mathcal{V}_{m-1}} \cdot \|v_i^{(m)}\| \max_{1 \leqq i \leqq m} |x_i|.$$

Define $R_m(x_1, \cdots, x_m) = (r_{m1}, \cdots, r_{mn})$, where

$$\frac{1}{(m-1)!} \left\{ \prod_{i=1}^{m-1} (x_m - x_i) \right\} r_{mj} = \psi_j \quad (1 \leqq j \leqq n).$$

Then r_{mj} satisfies (3.2), as claimed. If $m = 1$, then write $V(t) = V(0) + \int_0^t V'(s)\, ds$, and take $R_1 = (r_{11}, \cdots, r_{1n}) = \int_0^{x_1} V'(s)\, ds$. The proof is complete.

Remark. The technique of proof in Lemma 3.1 applies to any matrix $A(x)$ which contains rows $V(x_1), \cdots, V(x_m)$. In particular, results like (3.1) can be formulated for square matrices which have several constant rows (i.e., they do not involve $x_1, \cdots, x_n$).

Lemma 3.2. *Let W_i, R_i, $1 \leqq i \leqq n$, be row vectors in R^n, and put*

$$M = \max\{\|W_i\| + \|R_i\| : 1 \leqq i \leqq n\},$$

where $\|\cdot\|$ is the Euclidean norm in R^n. Then

$$\det \begin{bmatrix} W_1 + R_1 \\ \vdots \\ W_n + R_n \end{bmatrix} = \det \begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix} + \sum_{i=2}^{2^n} \det E_i \tag{3.6}$$

where E_i is an $n \times n$ matrix, $2 \leqq i \leqq 2_n$, satisfying

$$|\det E_i| \leqq M^{n-1} \max \{\|R_1\|, \cdots, \|R_n\|\}. \tag{3.7}$$

PROOF. Relation (3.6) follows from the row sum rule for determinants. The rows E_{ij} of the matrix E_i satisfy $\|E_{ij}\| \leqq \|W_j\| + \|R_j\| \leqq M$, and at least one $E_{ij} = R_j$. Therefore, (3.7) follows from the classical Hadamard inequality for determinants.

4. **Convergence of boundary operators.** The purpose of this section is to study *the notion of convergence* of a sequence $\{(\alpha_p, T_p)\}_{p=0}^{\infty}$ of Niccoletti boundary conditions to a given condition (α_0, T_0) [defined in 4.2], written hereafter as

$$(\alpha_p, T_p) \to (\alpha_0, T_0), p \to \infty.$$

The principle motivation for the definition comes from the geometrical properties of the sequence $\{u_p\}_{p=0}^{\infty}$ given by $u_p(t) = \int_a^b G(t, s;\ \alpha_p, T_p) f(s)\, ds$, where $f > 0$, $f \in C[a, b]$. In fact, to require that $\|u_p - u_0\| \to 0$ as $p \to \infty$ in the usual norm of $C^{k-1}[a, b]$, means that the sequence of point sets R_p must cluster at T_0 and the nearby multiplicities assigned by α_p must correctly add to the multiplicities assigned by α_0.

Although this intuitive notion of convergence can be formalized, it is tedious and extremely complicated to use in proofs. The contribution of this section is to restate the definition of $(\alpha_p, T_p) \to (\alpha_0, T_0)$ in operator-theoretic terminology, which is more suitable for the purposes of calculation.

To illustrate the central ideas, consider the operator $K \equiv (d/dt)^4$ and the Niccoletti conditions

$$\alpha_p = (2, 1, 1), T_p = \{0 < t_p < 1\} \quad (p \geqq 1), \lim_{p \to \infty} t_p = 1,$$

$$\alpha_0 = (2, 2), T_0 = \{0 < 1\}.$$

Consider the two boundary-value operators

$$\mathcal{L}_p u = (u(0), u'(0), u(t_p), u(1))^T, p \geqq 1,$$

$$\mathcal{L}_0 u = (u(0), u'(0), u(1), u'(1))^T.$$

If $u \in C^3[0, 1]$, then it is simply false that one has $\|\mathcal{L}_p u - \mathcal{L}_0 u\| \to 0$ as $p \to \infty$. However, it turns out that the solution of $Ku_p = f$, $\mathcal{L}_p u_p = 0$ tends to the solution of $Ku_0 = f$, $\mathcal{L}_0 u_0 = 0$ (as $p \to \infty$) in $C^3[0, 1]$. To remedy this problem, consider the matrix

$$\mathcal{N}_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_p - 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (p \geqq 1)$$

and the new boundary operator

$$\mathcal{L}_p^* \equiv \mathcal{N}_p^{-1}\mathcal{L}_p = \left(u(0), u'(0), u(1), \frac{u(t_p) - u(1)}{t_p - 1} \right)^T.$$

Then $\|\mathcal{L}_p^* u - \mathcal{L}_0 u\| \to 0$ as $p \to \infty$ in the space R^k, and furthermore,

$$\mathcal{L}_p^* u = 0 \text{ iff } \mathcal{L}_p u = 0,$$

due to the fact that $\mathcal{N}_p$ is invertible. This means that

$$Ku = f, \mathcal{L}_p^* u = 0 \ \text{ iff } \ Ku = f, \mathcal{L}_p u = 0.$$

The matrix $\mathcal{N}_p$, called *the normalizing matrix* in the sequel, is certainly not unique, because if $\{\psi_p\}_{p=1}^{\infty}$ is any sequence of invertible matrices such that $\psi_p \to I$ as $p \to \infty$, then $\mathcal{L}_p^{**} \equiv (\mathcal{N}_p\psi_p)^{-1}\mathcal{L}_p = \psi_p^{-1}\mathcal{L}_p^*$ will also converge to $\mathcal{L}_0$ in the uniform operator topology. In this sense, $\{\mathcal{N}_p\}_{p=1}^{\infty}$ represents a *normal form* for the premultiplication factors in the sequence $\{\mathcal{L}_p^*\}_{p=1}^{\infty}$.

LEMMA 4.1. *Let $Ku = 0$ satisfy the uniqueness condition $H(a, b; \alpha, T)$. Then there exist disjoint open intervals $I_0, \cdots, I_\nu$ with $s_j \in I_j$ $(0 \leqq j \leqq \nu)$, having the following property: If $Ku = 0$ and u has n_j zeros in I_j counting multiplicities $(0 \leqq j \leqq \nu)$, then $u \equiv 0$.*

PROOF. Let $[c, d]$ be a compact interval whose interior contains $[a, b]$ and let $E \equiv C^{k-1}([c, d] \to R)$, equipped with the usual norm $\|\cdot\|$.

Supose the lemma is false, then there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of solutions of $Ku = 0$ satisfying $\|u_n\| = 1$ such that u_n has n_j zeros in $(s_j - 1/n, s_j + 1/n)$, $0 \leqq j \leqq \nu$. Since $\{u : Ku = 0\}$ is a finite dimensional subspace of E, the sequence $\{u_n\}_{n=1}^{\infty}$ is precompact in E. A subsequence of $\{u_n\}_{n=1}^{\infty}$ will converge to a solution u of $Ku = 0$, $\|u\| = 1$, having α zeros at T, a contradiction to the uniqueness condition. The proof is complete.

DEFINITION 4.2. Let $[a, b]$ be a fixed compact interval, and let

$$\alpha_0 = (n_0, \ \cdots, \ n_\nu), \qquad T_0 = \{s_0 < \cdots < s_\nu\},$$

$$\alpha_p = (n_{0,p}, \ \cdots, \ n_{\nu_p,p}), \quad T_p = \{s_{0p} < \cdots < s_{\nu_p p}\},$$

with $a = s_0 = s_{0p}$, $b = s_\nu = s_{\nu_p p}$, $p \geqq 1$.

The sequence $\{(\alpha_p, T_p)\}_{p=1}^{\infty}$ *is said to converge to* (α_0, T_0) *as* $p \to \infty$, written $(\alpha_p, T_p) \to (\alpha_0, T_0)$, $p \to \infty$, iff for each $\epsilon > 0$ there exist disjoint relatively open intervals $I_0, \cdots, I_\nu$ in $[a, b]$ and an integer $M \equiv M(\epsilon)$ such that for $p \geqq M$:

$$T_p \subseteq \bigcup \{I_j : 0 \leqq j \leqq \nu\}, \tag{4.1}$$

$$n_j = \sum \{n_{ip} : s_{ip} \in I_j\},\ 0 \leqq j \leqq \nu, \tag{4.2}$$

$$\text{Each } I_j \text{ has length less than } \epsilon,\ 0 \leqq j \leqq \nu. \tag{4.3}$$

The sequence $\{(\alpha_p, T_p)\}_{p=1}^{\infty}$ *is said to converge simply to* (α_0, T_0) *as* $p \to \infty$, *written* $(\alpha_p, T_p) \xrightarrow{s} (\alpha_0, T_0)$, $p \to \infty$, iff (4.1)–(4.3) hold, and in addition, for $0 \leqq j \leqq \nu$ and $p \geqq M$, either

(4.4) The finite set $T_p \cap I_j$ consists of either one point $t_j{}^p$, to which α_p assigns n_j zeros, or

(4.5) The finite set $T_p \cap I_j$ consists of points $t^p_{j,1} < t^p_{j,2} < \cdots < t^p_{j,n_j}$, and α_p assigns a simple zero at each of these points.

By Lemma 4.1, one is at liberty to construct sequences $\{(\alpha_p, T_p)\}_{p=1}^{\infty}$ which converge to (α_0, T_0), having the property that $G(t, s; \alpha_p, T_p)$ exists. Furthermore, due to Lemma 4.1, a statement such as

$$\lim_{p\to\infty} G(t, s; \alpha_p, T_p) = G(t, s; \alpha_0, T_0)$$

has the possibility of making sense, provided $(\alpha_p, T_p) \to (\alpha_0, T_0)$.

DEFINITION 4.3. The *Niccoletti boundary operator* $\mathcal{L}[\alpha, T]$ is the linear operator $\mathcal{L} : C^{k-2}[a, b] \to R^k$ defined by the identity

$$\mathcal{L}u \equiv [u(s_0), \cdots, u^{(n_0-1)}(s_0), \cdots, u(s_\nu), \cdots, u^{(n_\nu-1)}(s_\nu)]^T.$$

In particular, $\mathcal{L}u = 0$ means that u has α zeros at T.

If $\{\alpha_p, T_p)\}_{p=0}^{\infty}$ is a sequence of Niccoletti boundary conditions for $Ku = 0$ on $[a, b]$, then define $\mathcal{L}_p \equiv \mathcal{L}[\alpha_p, T_p]$, $p \geqq 0$.

DEFINITION 4.4. Let $u_1, \cdots, u_k$ be an arbitrary basis for the solution space of $Ku = 0$, and put $U = (u_1, \cdots, u_k)$. Define $Z[U; \alpha, T]$ to be the $k \times k$ matrix whose successive columns are $\mathcal{L}u_1, \cdots, \mathcal{L}u_k$, $\mathcal{L} \equiv \mathcal{L}[\alpha, T]$.

If $\{\alpha_p, T_p)\}_{p=0}^{\infty}$ is a sequence of Niccoletti boundary conditions for $Ku = 0$ on $[a, b]$, then define $Z_p(U) \equiv Z[U; \alpha_p, T_p]$, $p \geqq 0$.

LEMMA 4.5. *The matrix* $Z[U; \alpha, T]$ *is nonsingular iff* $H(a, b; \alpha, T)$. *Therefore, the uniqueness assumption* $H(a, b; \alpha, T)$ *plus the value of* $Z[U; \alpha, T]$ *uniquely determines the basis* U.

PROOF. If $C \in R^k$, then $u = UC^T$ satisfies $\mathcal{L}u = 0$ iff $Z[U; \alpha, T]C^T = 0$. By linear algebra, the first statement holds. The second statement follows from the easily proved relation $Z[U\psi; \alpha, T] = Z[U; \alpha, T] \cdot \psi$, valid for any nonsingular $k \times k$ matrix ψ.

LEMMA 4.6. *If* $H(a, b; \alpha_0, T_0)$ and $(\alpha_p, T_p) \to (\alpha_0, T_0)$ *as* $p \to \infty$, *then* $\det Z_p(U) \neq 0$ *for all large* p.

PROOF. By Lemma 4.1, $H(a, b; \alpha_p, T_p)$ for all large p, therefore the result follows from Lemma 4.5.

DEFINITION 4.7. Let $\{(\alpha_p, T_p)\}_{p=1}^{\infty}$ *converge simply* to (α_0, T_0) [see definition 4.2]. Define the $k \times k$ *normalizing matrix* $\mathcal{N}_p$ by the relations $(p \geqq 1, 0 \leqq j \leqq \nu)$

$$\mathcal{N}_p = \operatorname{diag}(B_0, \cdots, B_\nu),$$

$$B_j = I \quad (n_j \times n_j \text{ identity}) \text{ if (4.4) holds,} \tag{4.6}$$

$$B_j = \begin{bmatrix} 1 & x_1/1! & \cdots & x_1^{n-1}/(n-1)! \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & x_n/1! & \cdots & x_n^{n-1}/(n-1)! \end{bmatrix} \text{ if (4.5) holds,} \tag{4.7}$$

where in relation (4.7) the symbols are defined by

$$n \equiv n_j, x_q \equiv t_{j,q}^p - s_j \quad (1 \leqq q \leqq n_j), \tag{4.8}$$

the notation in (4.8) being taken from definition 4.2.

REMARK 4.8. It is certainly possible to define $\mathcal{N}_p$ in case the convergence is not simple; however it turns out that, for the purposes of this paper, this is unnecessary. The vehicle for avoiding the complications of convergence which is not simple appears in the proof of Theorem 5.1 *infra*.

LEMMA 4.9. *Let* $\{(\alpha_p, T_p)\}_{p=1}^{\infty}$ *converge simple to* (α_0, T_0). *Then the normalizing matrix* $\mathcal{N}_p$ *is invertible.*

PROOF. Since $\det \mathcal{N}_p = \prod_{j=0}^{\nu} \det B_j$ [see definition 4.7 for notation], it suffices to prove that $\det B_j \neq 0$ in (4.7). But in this case,

$$\det B_j = \left(\prod_{i=1}^{n_j-1} \frac{1}{i!} \right) \mathcal{V}_{n_j},$$

where $\mathcal{V}_{n_j}$ is a *Vandermonde determinant*. The value of $\mathcal{V}_{n_j}$ is known to be $\mathcal{V}_{n_j} = \prod \{(x_s - x_r) : 1 \leqq r < s \leqq n_j\}$, and since $x_1 < x_2 < \cdots < x_{n_j}$ [see (4.5) and (4.8)], it follows that $\mathcal{V}_{n_j} \neq 0$, therefore, $\det B_j \neq 0$ in (4.7).

THEOREM 4.10. *Let $\{\alpha_p, T_p)\}_{p=1}^{\infty}$ converge simply to (α_0, T_0). If $\{\mathcal{L}_p\}_{p=0}^{\infty}$ is the sequence of Niccoletti boundary operators of Definition 4.3, $\{\mathcal{N}_p\}_{p=1}^{\infty}$ is the sequence of $k \times k$ normalizing matrices of Definition 4.7, then*

$$\lim_{p\to\infty} \|\mathcal{N}_p^{-1}\mathcal{L}_p - \mathcal{L}_0\| = 0. \tag{4.9}$$

The norm $\|\cdot\|$ in (4.9) is given in terms of the usual norm $\|\cdot\|_X$ in $X \equiv C^{k-2}([a,b] \to R)$, and the Euclidean norm $|\cdot|$ of R^k is given by $\|f\| = \sup\{|f(x)| : x \in X, \|x\|_X = 1\}$.

PROOF. Consider a fixed integer $p \geqq 1$, and put

$$\mathfrak{L}_j^p u = [u(t_j^p), u'(t_j^p), \cdots, u^{(n_j-1)}(t_j^p)]^T \text{ or} \tag{4.10}$$

$$\mathfrak{L}_j^p u = [u(t_{j,1}^p), \cdots, u(t_{j,n_j}^p)]^T, \tag{4.11}$$

accordingly as (4.4) or (4.5) holds, $0 \leqq j \leqq \nu$. Then

$$\mathcal{L}_p u = \begin{bmatrix} \mathfrak{L}_0^p u \\ \vdots \\ \mathfrak{L}_\nu^p u \end{bmatrix}.$$

To prove (4.9), it will be shown that for $\|u\|_X = 1$,

$$|B_j^{-1}\mathfrak{L}_j^p u - \mathfrak{L}_j^0 u| \leqq L\lambda_j^p, 0 \leqq j \leqq \nu, p \geqq 1, \tag{4.12}$$

where $L > 0$ is a constant independent of j, p, and u. Further,

$$\lambda_j^p \equiv \begin{cases} |t_j^p - s_j|, & \text{if (4.4) holds,} \\ \max\limits_{1\leqq i\leqq n_j} |t_{ji}^p - s_j|, & \text{if (4.5) holds.} \end{cases} \tag{4.13}$$

The norm in (4.12) is the Euclidean norm in R_{n_j}, and the matrix B_j in (4.12) is defined by (4.6), (4.7).

Suppose first (4.4) holds, then $B_j = I$, and $\mathfrak{L}_j^p u$ is given by (4.10). By Taylor's theorem, $\mathfrak{L}_j^p u = \mathfrak{L}_j^0 u + r_j$, where the remainder term r_j has components of the form $\int_c^d u^{(i)}(s)\,ds$ $(1 \leqq i \leqq n_j)$, $c = s_j$, $d = t_j^p$. Since $\|u\|_X = 1$ and $|r_j| \leqq \sqrt{n_j}\,\lambda_j^p$, the estimate (4.12) follows easily.

Secondly, suppose (4.5) holds. Then B_j is a Vandermonde matrix and $\mathfrak{L}_j^p u$ is given by (4.11). Write $B_j^{-1}\mathfrak{L}_j^p u \equiv [a_1, \cdots, a_{n_j}]^T$. Then for $1 \leqq s \leqq n_j$,

$$a_s = [\det B_j]^{-1} \sum_{r=1}^{n_j} u(t_{j,r}^p) C_{r,s}, \tag{4.14}$$

where $C_{r,s}$ is the cofactor of row r and column s of the matrix B_j.

By cofactor expansion, the RHS of (4.14) can be written as an $n \times n$ determinant, $n \equiv n_j$, whose rows are $V(x_1), \cdots, V(x_n)$,

$$V(t) \equiv \left(1, t, \cdots, \frac{t^{s-2}}{(s-2)!}, u(t+s_j), \frac{t^s}{s!}, \cdots, \frac{t^{n-1}}{(n-1)!}\right) \cdot [\det B_j]^{-1},$$

with $x_1, \cdots, x_n$ given as in (4.8).

Application of Lemma 3.1 gives

$$a_s = \frac{\det P(x)}{\det B_j} \det[W(0) + R(x)], \tag{4.15}$$

notation being borrowed from Lemma 3.1. Observe that in Lemma 3.1, the factor $\det P(x)$ is exactly $\det B_j$, therefore the RHS of (4.15) reduces to $\det[W(0) + R(x)]$.

Since

$$\begin{aligned} V(0) &= (1, 0, \cdots, 0, u(s_j), 0, \cdots, 0), \\ V'(0) &= (0, 1, 0, \cdots, 0, u'(s_j), 0, \cdots, 0), \\ &\vdots \\ V^{(n-1)}(0) &= (0, \cdots, 0, u^{(n-1)}(s_j), 0, \cdots, 0, 1), \end{aligned}$$

the value of $\det W(0)$ is precisely the element of $\mathfrak{L}_j{}^0 u$ located in position s. Therefore, relation (4.15) and Lemma 3.2 allow us to estimate the difference between corresponding components of $B_j{}^{-1}\mathfrak{L}_j{}^p u$ and $\mathfrak{L}_j{}^0 u$. This difference will be bounded by an absolute constant times the number $\lambda_j{}^p$ of relation (4.13), because of estimate (3.2), Lemmas 3.1, 3.2 and the fact that $\|u\|_X = 1$. Therefore, relation (4.12) holds in the second case.

The proof is complete.

COROLLARY 4.11. *Under the hypotheses of Theorem* 4.10,

$$\|\mathcal{N}_p{}^{-1}\mathcal{L}_p - \mathcal{L}_0\| \leqq N\lambda_p, p \geqq 1, \tag{4.16}$$

where $N > 0$ *is a constant independent of* p, *and* $\lambda_p = \max\{\lambda_j{}^p : 0 \leqq j \leqq \nu\}$ (*see* (4.13) *for the definition of* $\lambda_j{}^p$).

DEFINITION 4.12. Let $\{(\alpha_p, T_p)\}_{p=1}^{\infty}$ converge simply to (α_0, T_0), and let $\mathcal{L}_p$ and $\mathcal{N}_p$ be defined as in Definitions 4.3, 4.7. The sequence $\{\mathcal{L}_p{}^*\}_{p=1}^{\infty}$,

$$\mathcal{L}_p{}^* \equiv \mathcal{N}_p{}^{-1}\mathcal{L}_p, p \geqq 1, \tag{4.17}$$

is called *the sequence of normalized Niccoletti boundary operators* for $\{(\alpha_p, T_p)\}_{p=0}^{\infty}$.

Convergence Properties of Solutions

Suppose $\{(\alpha_p, T_p)\}_{p=1}^{\infty}$ converges simply to (α_0, T_0). Then the Green's function $G(t, s; \alpha_p, T_p)$ can be represented in many ways by a particular choice of basis $U = (u_1, \cdots, u_k)$, in relation (2.9). It was noted earlier that the condition $Z[U; \alpha_p, T_p] = I$ makes for a considerable reduction of terms in the expression for $G(t, s; \alpha_p, T_p)$, and therefore this choice of basis is quite suitable for explicit computation. On the other hand, if a limiting process is being carried out, the condition $Z[U; \alpha_p, T_p] = \mathscr{N}_p$ makes the limiting process as simple as possible. Therefore, this choice of basis is to be preferred in computations involving a limit process.

Definition 4.13. Let $\{(\alpha_p, T_p)\}_{p=1}^{\infty} \xrightarrow{s} (\alpha_0, T_0)$. The sequence $\{U_p\}_{p=0}^{\infty}$ determined by the condition

$$Z_p(U_p) \equiv Z[U_p; \alpha_p, T_p] = \mathscr{N}_p, p \geqq 1,$$

is called the *sequence of fundamental Niccoletti solutions* associated with $\{(\alpha_p, T_p)\}_{p=1}^{\infty}$; the basis U_0 is determined by the condition $Z[U_0; \alpha_0, T_0] = I$ (see Lemma 4.5).

Theorem 4.14. *Let* $\{(\alpha_p, T_p)\}_{p=1}^{\infty} \xrightarrow{s} (\alpha_0, T_0)$, *and put* $\Phi_p \equiv \mathscr{N}_p^{-1} Z_p(U_0)$. *Then the sequence* $\{U_p\}_{p=0}^{\infty}$ *of fundamental Niccoletti solutions satisfies*

$$\lim_{p\to\infty} \Phi_p = I, \tag{4.18}$$

$$U_p = U_0 \Phi_p^{-1}, \tag{4.19}$$

(4.20) $\|U_p - U_0\| \to 0$ *as* $p \to \infty$ *in the usual norm of* $C^k([a, b] \to R^k)$.

Proof. The columns of Φ_p are obtained by applying $\mathscr{N}_p^{-1} \mathscr{L}_p \equiv \mathscr{L}_p^*$ to the elements of U_0, therefore (4.18) holds; indeed, Theorem 4.10 says that the columns of Φ_p converge to the columns of $Z_0(U_0)$, and $Z_0(U_0) = I$.

Relation (4.19) holds because of the identity $Z_p(U_0\Phi_p^{-1}) = Z_p(U_0)\Phi_p^{-1} = \mathscr{N}_p = Z_p(U_p)$ and Lemma 4.5.

Finally, (4.20) holds, because (4.19) implies

$$\|U_p - U_0\| \leqq \|U_0\| \, \|\Phi_p^{-1} - I\|, \tag{4.21}$$

the second norm in (4.21) being the matrix operator norm in R^k.

Corollary 4.15. *Under the hypotheses of Theorem* 4.14,

$$\|U_p - U_0\| \leqq N_1\lambda_p, p \geqq 1,$$

where $N_1 > 0$ is a positive constant and λ_p is defined as in Corollary 4.11.

PROOF. By (4.21), $\|U_p - U_0\| \leqq \|U_0\| \ \|\Phi_p{}^{-1}\| \ \|I - \Phi_p\|$, and by Corollary 4.11, $\|I - \Phi_p\| \leqq k\|U_0\|N\lambda_p$, $p \geqq 1$. Therefore, we may take $N_1 = k\|U_0\|^2 N \sup\{\|\Phi_p{}^{-1}\| : p \geqq 1\}$, by virtue of (4.18).

5. **Convergence theorems for Green's function.** The purpose of this section is to establish the following convergence theorems.

THEOREM 5.1. *Assume the uniqueness condition $H(a, b; \alpha_0, T_0)$, and let $(\alpha_p, T_p) \to (\alpha_0, T_0)$ as $p \to \infty$. Then for $0 \leqq r \leqq k$,*

$$\lim_{p\to\infty} (\partial/\partial t)^r [G(t, s; \alpha_p, T_p) - G(t, s; \alpha_0, T_0)] = 0 \tag{5.1}$$

uniformly in t, s, $a \leqq t, s, \leqq b$.

Furthermore, $|(\partial/\partial t)^r G(t, s; \alpha_p, T_p)| \leqq H$, $p \geqq 0$, $0 \leqq r \leqq k-2$, $a \leqq t, s \leqq b$, for some constant $H > 0$.

THEOREM 5.2. *Assume the uniqueness condition $H(a, b; \alpha_0, T_0)$, and let $(\alpha_p, T_p) \to (\alpha_0, T_0)$ as $p \to \infty$. If $f \in L^1[a, b]$, then for $0 \leqq r \leqq k$,*

$$\lim_{p\to\infty} \int_a^b |(\partial/\partial t)^r [G(t, s; \alpha_p, T_p) - G(t, s; \alpha_0, T_0)]| \ |f(s)| \, ds = 0, \tag{5.2}$$

uniformly on $a \leqq t \leqq b$.

THEOREM 5.3. *Assume the uniqueness condition $H(a, b; \alpha_0, T_0)$, and let $(\alpha_p, T_p) \to (\alpha_0, T_0)$ as $p \to \infty$. If $u_p(t)$ is the solution of $Ku = f$ with α_p zeros at $T_p (p \geqq 0)$ and $f \in C[a, b]$, then*

$$\lim_{p\to\infty} \|u_p - u_0\| = 0 \tag{5.3}$$

in the usual norm $\| \cdot \|$ of $C^k[a, b]$.

The intuitive ideas communicated at the start of § 4 are shown to be correct by Theorem 5.3. In particular, if one writes $u_p = \mathcal{G}_p f$, then $\mathcal{G}_p f \to \mathcal{G}_0 f$, $p \to \infty$, in the space $C^k[a, b]$. Therefore, Theorem 5.3 is a *continuity theorem for boundary value problems.*

The following scheme presents itself for the verification of Theorems 5.1–5.3: First, in Theorem 5.3, show that $\{u_p\}$ is bounded in $C^{k-1}[a, b]$, then use the fact that the normalized Niccoletti boundary operators $\mathcal{L}_p{}^*$ converge in the C^{k-1} operator topology to $\mathcal{L}_0$ to establish that $\|u_p - u_0\| \to 0$. Uniqueness comes into play in proving that any subsequence $\{u_{p_i}\}$ must converge to u_0. Now show that Theorem 5.1 can be recovered from Theorem 5.3, then prove Theorem 5.2.

However, a direct attack, in which Theorem 5.1 is proved first, seems to require less energy. Furthermore, this direct attack exhibits the role of the operator sequence $\{\mathcal{L}_p^*\}_{p=1}^{\infty}$ in an illuminating way. It is this method of attack which is carried out below.

In the lemmas and proofs below, $\|\cdot\|$ denotes either the usual norm in $C([a, b] \to R^k)$ or the matrix operator norm (no confusion should result by this abuse), and $|\cdot|$ stands for the Euclidean norm in R^k.

Let $h(t, s) = U(t)W^{-1}(s)e$ (notation as in (2.9)). Then h does not depend on the particular basis U, and furthermore, h satisfies $h^{(j)}(s, s) \equiv [\partial/\partial t]^j h(s, s) = \delta_{j,k-1}$ (*Kronecker's delta*). Therefore, h is the usual *Cauchy function*.

In proofs it is useful to rearrange the basic formula (2.9) in such a way that the dependence on the boundary operator $\mathcal{L}$ is more explicit. To do so, it is convenient to introduce the auxiliary function.

$$\mathcal{H}(t, s) \equiv \epsilon(t - s)h(t, s),\ a \leqq t, s \leqq b. \tag{5.4}$$

Lemma 5.4. *Notation as in Definitions* 4.3, 4.4. *The Green's function* (2.9) *satisfies for any basis U of $Ku = 0$ the identity*

$$G(t, s; \alpha_0, T_0) = \mathcal{H}(t, s) - U(t)Z[U; \alpha_0, T_0]^{-1}\mathcal{L}[\mathcal{H}(\cdot, s)]. \tag{5.5}$$

Proof. In view of (2.9) and (5.4), it suffices to verify the identity

$$\mathcal{L}[\mathcal{H}(\cdot, s)] = V(s)ZW^{-1}(s)e. \tag{5.6}$$

The matrix $ZW^{-1}(s)$ has rows $U^{(j)}(s_i)W^{-1}(s)$, therefore $U^{(j)}(s_i)W^{-1}(s)e = h^{(j)}(s_i, s)$ and $ZW^{-1}(s)e = \mathcal{L}[h(\cdot, s)]$. Relation (5.6) claims that $\mathcal{L}[\mathcal{H}(\cdot, s)] = V(s)\mathcal{L}[h(\cdot, s)]$, which is easily verified using the identity $h^{(j)}(x, x) = \delta_{j,k-1}$. The proof is complete.

Lemma 5.5. *Let $(\alpha_p, T_p) \xrightarrow{s} (\alpha_0, T_0)$ as $p \to \infty$, and let $\mathcal{N}_p$ be the normalizing matrix of definition* 4.7, *$Z_p(U_0)$ the matrix of Definition* 4.4.

If $Z_0(U_0) = I$, $\Phi_p \equiv \mathcal{N}_p^{-1} Z_p(U_0)$, then for $a \leqq t,\ s \leqq b,\ 0 \leqq r \leqq k$,

$$\begin{aligned}|(\partial/\partial t)^r[G(t, s; \alpha_p, T_p) - G(t, s; \alpha_0, T_0)]| &\leqq \\ \|U_0^{(r)}(\cdot)\Phi_p^{-1}\| \cdot |\mathcal{L}_p^*[\mathcal{H}(\cdot, s)] &- \mathcal{L}_0[\mathcal{H}(\cdot, s)]| \\ + \|U_0^{(r)}(\cdot)\Phi_p^{-1}\| \cdot \|I - \Phi_0\| &\cdot |\mathcal{L}_0[\mathcal{H}(\cdot, s)]|,\end{aligned} \tag{5.7}$$

where $\mathcal{H}$ and $\mathcal{L}_p^*$ are defined by (5.4) *and* (4.17), *respectively,*

Proof. This is a consequence of (5.5), the identity

$$G(t, s; \alpha_0, T_0) - G(t, s; \alpha_p, T_p) =$$
$$U_0(t)\Phi_p^{-1}\{\mathcal{L}_p^*[\mathcal{H}(\cdot, s)] - \Phi_p\mathcal{L}_0[\mathcal{H}(\cdot, s)]\},$$

and routine norm estimates.

LEMMA 5.6. *Notation and assumptions as in Lemma* 5.5. *There is a constant* $C > 0$ *such that for* $0 \leqq r \leqq k, a \leqq t, s \leqq b, p \geqq 1$,

$$(5.8) \qquad |(\partial/\partial t)^r[G(t, s; \alpha_p, T_p) - G(t, s; \alpha_0, T_0)]| \leqq C\lambda_p$$

(*see Corollary* 4.11 *for the definition of* λ_p).

PROOF. By (4.19) and (4.20), $\|U_0^{(r)}(\cdot)\Phi_p^{-1}\|$ is bounded for $p \geqq 1$, $0 \leqq r \leqq k$, by some constant M_1. Since $\|\Phi_p - I\| \leqq \sqrt{k} \cdot \max\{|\mathcal{L}_p^*(u_i) - \mathcal{L}_0(u_i)| : 1 \leqq i \leqq k\}$ where $U_0 = (u_1, \cdots, u_k)$, it follows from (4.16) that a constant M_2 exists satisfying $\|\Phi_p - I\| \leqq M_2\lambda_p$. Therefore, from (4.16) and (5.7), one can take, in the notation of Theorem 4.10,

$$C = M_1 \cdot N \cdot \sup\{\|\mathcal{H}(\cdot, s)\|_X : a \leqq s \leqq b\}$$
$$+ M_1 \cdot M_2 \cdot \sup\{|\mathcal{L}_0[\mathcal{H}(\cdot, s)] : a \leqq s \leqq b\}.$$

The proof is complete.

REMARK 5.7. In practice, such as computing the Green's function of $y^{\text{iv}} = 0$, $y(s_i) = 0$ $(0 \leqq i \leqq 3)$, the use of (5.5) reduces to the use of (2.10). See § 6.

PROOF OF THEOREM 5.1. By Lemma 5.6, the theorem is correct if $(\alpha_p, T_p) \to (\alpha_0, T_0)$.

Suppose now that $(\alpha_p, T_p) \to (\alpha_0, T_0)$. Hereafter, use the notation of Definition 4.2.

Construct for each $p \geqq M$ a finite set of Niccoletti conditions $\{(\alpha_p^i, T_p^i)\}_{i=0}^{\nu_p}$ having the following properties:

(5.9) The set T_p^0 consists of the points of T_p together with $n_{0,p} - 1$ distinct points just to the right of $s_{0,p} \equiv a$.

(5.10) For $1 \leqq i \leqq \nu_p$, the set T_p^i consists of the points of T_p^{i-1} together with $n_{i,p} - 1$ distinct points just to the right (or left, if $i = \nu_p$) of the point $s_{i,p}$.

(5.11) For $0 \leqq i < \nu_p$, the symbol α_p^i assigns simple zeros to the points of T_p^i which are less than $s_{i+1,p}$, and it assigns $n_{j,p}$ zeros at $s_{j,p}(i + 1 \leqq j \leqq \nu_p)$. For $i = \nu_p$, α_p^i assigns simple zeros to all points of T_p^i.

Since the theorem is true for simple convergence, one can select $\{\alpha_p{}^i, T_p{}^i)\}_{i=0}^{\nu_p}$ such that

(5.12) $|(\partial/\partial t)^r[G(t, s; \alpha_p{}^{i+1}, T_p{}^{i+1}) - G(t, s; \alpha_p{}^i, T_p{}^i)]| \leqq 1/(1 + \nu_p)p$, for $a \leqq t$, $s \leqq b$, $0 \leqq r \leqq k$, $-1 \leqq i \leqq \nu_p - 1$, where by definition $\alpha_p{}^{-1} \equiv \alpha_p$, $T_p{}^{-1} \equiv T_p$.

Put $\beta_p = \alpha_p{}^{\nu_p}$, $S_p = T_p{}^{\nu_p}$. By the triangle inequality and (5.12) it follows that

$$\begin{aligned} &|(\partial/\partial t)^r[G(t, s; \alpha_0, T_0) - G(t, s; \alpha_p, T_p)]| \leqq 1/p + \\ &\qquad |(\partial/\partial t)^r[G(t, s; \alpha_0, T_0) - G(t, s; \beta_p, S_p)]|. \end{aligned} \tag{5.13}$$

for $0 \leqq r \leqq k$, $a \leqq t \leqq b$. However, one can retain (5.9)–(5.11) and still have $(\beta_p, S_p) \xrightarrow{s} (\alpha_0, T_0)$, therefore by the special case already established, the right side of (5.13) tends to zero as $p \to \infty$, uniformly in t and s, $0 \leqq r \leqq k$.

The boundedness statement follows from Lemma 5.6, inequality (5.13), and relation (2.9).

REMARK 5.8. The function $(\partial/\partial t)^r\, G(t, s; \alpha, T)$ is uniformly continuous in the square $a \leqq t \leqq b$, $a \leqq s \leqq b$, for $0 \leqq r \leqq k - 2$; this follows easily from the representation (5.5).

PROOF OF THEOREM 5.2. Due to uniform convergence to zero of the first factor in the integrand, by virtue of Theorem 5.1, the result is a simple consequence of elementary integral inequalities.

PROOF OF THEOREM 5.3. By (2.11) we write

$$u_p(t) - u_0(t) = \int_a^b [G(t, s; \alpha_p, T_p) - G(t, s; \alpha_0, T_0)]\, f(s)\, ds.$$

Therefore, since $U \in C^k[a, b]$, relation (2.9) gives

$$\begin{aligned} (d/dt)^r[u_p(t) - u_0(t)] = \int_a^b &(\partial/\partial t)^r[G(t, s; \alpha_p, T_p) \\ &- G(t, s; \alpha_0, T_0)]\, f(s)\, ds \end{aligned}$$

for $0 \leqq r \leqq k$. The result now follows from routine integral estimates and Theorem 5.2.

6. Special representations for Green's function. The purpose of this section is to obtain useful formulas for the computation of Green's function of § 2. A formula for the Green's function of a k-point problem for $y^{(k)} = 0$ is recorded in (6.9), and for a 2-point problem in (6.11). A practical method for computing G via the convergence principle is illustrated.

The function $h(t, s) \equiv U(t)W^{-1}(s)e$ (U any basis, W its Wronskian matrix) satisfies $h^{(i)}(s, s) = 0$, $0 \leqq i \leqq k - 2$, $h^{(k-1)}(s, s) = 1$, where $h^{(i)} \equiv (\partial/\partial t)^i h$.

Assume $\alpha = (n_0, \cdots, n_\nu)$, $T = \{s_0 < \cdots < s_\nu\}$ are given, $|\alpha| = k$, and $G(t, s; \alpha, T)$ exists for $Ku = 0$. List the standard unit row vectors in R^k as $e_{ij}(0 \leqq j \leqq n_i - 1,\ 0 \leqq i \leqq \nu) : e_{ij}$ has a one in position $k_{ij} \equiv j + 1 + \sum_{r<i} n_r$ and zeros elsewhere.

LEMMA 6.1. *Let U^*, W^* be given as in* (2.10). *Then*

$$W^{*-1}(s)e = \sum_{i=0}^{\nu} \sum_{j=0}^{n_i - 1} h^{(j)}(s_i, s)e^T_{ij}. \tag{6.1}$$

Proof: Let U be a fixed basis for $Ku = 0$. Denote by Z the $k \times k$ matrix whose rows are $U^{(j)}(s_i)$ $(0 \leqq j \leqq n_i - 1,\ 0 \leqq i \leqq \nu)$. Then $U^* = UZ^{-1}$, hence $W^{*-1}(s) = ZW^{-1}(s)$. Position k_{ij} of $W^{*-1}(s)e$ is occupied by $U^{(j)}(s_i)W^{-1}(s)e$, which is clearly equal to $h^{(j)}(s_i, s)$. The proof is complete.

PROPOSITION 6.2. *Let $u^*_{ij}(t)$ be the solution of $Ku = 0$ satisfying*

$$u^{(r)}(s_m) = 0\ (n \neq i, 0 \leqq r \leqq n_m - 1, 0 \leqq m \leqq \nu),$$
$$u^{(r)}(s_i) = 0\ (0 \leqq r \leqq n_i - 1, r \neq j), u^{(j)}(s_i) = 1.$$

Define $h(t, s)$ as above. Then (*notation as in* § 2):

$$G(t, s; \alpha, T) = \sum_{i=0}^{\nu} \sum_{j=0}^{n_i - 1} u^*{}_{ij}(t)[\epsilon(t - s) - \chi_{E_i}(s)]\ h^{(j)}(s_i, s). \tag{6.2}$$

PROOF. We have $U^* = \sum_{i=0}^{\nu} \sum_{j=0}^{n_1 - 1} u^*_{ij}e^T_{ij}$. By Lemma 6.1 and the definition of $V(s)$ [§ 2]:

$$[\epsilon(t - s)I - V(s)]\, W^{*-1}(s)e$$

$$= \sum_{i=0}^{\nu} \sum_{j=0}^{n_i - 1} [\epsilon(t - s) - \chi_{E_i}(s)]\ h^{(j)}(s_i, s)e^T_{ij}.$$

Since $G(t, s; \alpha, T) = U^*(t)[\epsilon(t - s)I - V(s)]\, W^{*-1}(s)e$, the result follows from orthogonality of the vectors $\{e_{ij}\}$.

CONSTANT COEFFICIENTS

Let us now record some simple formulas for the case of constant coefficients. We assume that $Ku \equiv u^{(k)} + \sum_{i=0}^{k-1} a_i u^{(i)}$, where $a_0, \cdots, a_{k-1}$ are real numbers. Further, $H(a, b; \alpha, T)$, and for simplicity, $\alpha = (1, 1, \cdots, 1)$, $T = \{a = s_0 < \cdots < s_{k-1} = b\}$.

Define the *shift operator* π_j as follows: for $0 \leqq j \leqq k-1$,

$$\pi_j T = \sum_{i<j} (s_i - s_j)e_{i+1} + \sum_{i>j} (s_i - s_j)e_i. \tag{6.3}$$

Here, $e_1, \cdots, e_{k-1}$ are the standard unit vectors of R^{k-1}.

For a constant coefficient equation $Ku = 0$, we can calculate in one inversion problem the solution $U(t; \tau)$ of the following problem:

$$\begin{cases} \tau = \sum_{i=1}^{k-1} \tau_i e_i, \tau_i \neq \tau_j \text{ for } i \neq j,\ \tau_i \neq 0, \\ Ku = 0, u \not\equiv 0, \\ u(0) = 1, u(\tau_i) = 0 \text{ for } 1 \leqq i \leqq k-1. \end{cases} \tag{6.4}$$

Similarly, one inversion problem will calculate the solution $u = H(t)$ of the problem

$$Ku = 0, u^{(i)}(0) = 0\ (0 \leqq i \leqq k-2), u^{(k-1)}(0) = 1. \tag{6.5}$$

PROPOSITION 6.3. *For the constant coefficient equation $Ku = 0$ with simple zero assignment at points $s_0 < s_1 < \cdots < s_{k-1}$, the Green's function is given by*

$$G(t, s; \alpha, T) = \sum_{i=0}^{k-1} U(t - s_i; \pi_i T)[\epsilon(t-s) - \chi_{[s_0, s_i]}(s)]\, H(s_i - s). \tag{6.6}$$

The functions U, H, π_j are defined by (6.3)–(6.5).

PROOF. Apply Proposition 6.2, then observe that the equation is translation invariant, hence $h(s_i, s) = H(s_i - s)$ and $u^*_{i,0}(t) \equiv U(t - s_i; \pi_i T)$, $0 \leqq i \leqq k-1$.

THE EQUATION $y^{(k)} = 0$

As an application of (6.6), we obtain the formula of Das and Vatsala [7] for the Green's function of the k-point problem

$$y^{(k)} = 0,\ y(s_i) = 0\ (0 \leqq i \leqq k-1). \tag{6.7}$$

The formula to be given here is many times more compact than that in [7], and in contrast to the work of Das and Vatsala, is obtained directly, without mathematical induction.

The function $U(t; \tau)$ of (6.4) is computed without linear algebra:

$$U(t; \tau) = \prod_{i=1}^{k-1} \left(\frac{\tau_i - t}{\tau_i} \right). \tag{6.8}$$

Likewise, the function H of (6.5) is calculated without linear algebra: $H(t) = t^{k-1}/(k-1)!$. We verify immediately from (6.8) that

$$U(t - s_j; \pi_j T) = \frac{p(t)}{p'(s_j)(t - s_j)},$$

where $p(t) = \prod_{i=0}^{k-1} (t - s_i)$. It follows from Proposition 6.3 that

$$(6.9)\quad G(t, s; \alpha, T) = \frac{p(t)}{(k-1)!} \sum_{j=0}^{k-1} \frac{(s_j - s)^{k-1}}{p'(s_j)(t - s_j)} [\epsilon(t - s) - \chi_{[s_0, s_j]}(s)].$$

In this relation, $\alpha = (1, 1, \cdots, 1)$, $T = \{s_0 < s_1 < \cdots < s_{k-1}\}$.

Two-Point Problems for $Ku = 0$

Assume that K has arbitrary continuous coefficients on $[a, b]$. If proposition 6.2 is specialized to $\alpha = (\ell, k - \ell)$, $1 \leqq \ell \leqq k - 1$, and $T = \{a < b\}$, then the Green's function for the two-point problem is given by [notation as in Proposition 6.2]

$$(6.10)\quad G(t, s; \alpha, T) = \begin{cases} \displaystyle\sum_{j=0}^{\ell-1} \overset{*}{u}_{0,j}(t)\, h^{(j)}(a, s), & s \leqq t, \\[2ex] \displaystyle -\sum_{j=0}^{k-\ell-1} \overset{*}{u}_{1,j}(t) h^{(j)}(b, s), & s \geqq t. \end{cases}$$

Two-Point Problems for $y^{(k)} = 0$

In the case of the equation $y^{(k)} = 0$ with boundary conditions $y^{(i)}(a) = 0$ $(0 \leqq i \leqq \ell - 1)$, $y^{(j)}(b) = 0$ $(0 \leqq j \leqq k - \ell - 1)$, the Green's function has the explicit representation (6.10) where the functions involved are

$$\overset{*}{u}_{0,j}(t) \equiv \frac{1}{j!} \sum_{r=j}^{\ell-1} c_{r,j} \left(\frac{t-a}{a-b}\right)^r \left(\frac{t-b}{a-b}\right)^{k-\ell} \qquad (0 \leqq j \leqq \ell - 1),$$

$$\overset{*}{u}_{1,i}(t) \equiv \frac{1}{i!} \sum_{r=i}^{k-\ell-1} d_{r,i} \left(\frac{t-b}{b-a}\right)^r \left(\frac{t-a}{b-a}\right)^{\ell} \qquad (0 \leqq i \leqq k - \ell - 1),$$

$$h(t, s) \equiv (t - s)^{k-1}/(k-1)!,$$

and the coefficients $c_{r,j}$ and $d_{r,i}$ are determined by the following recursions:

$$\sum_{r=j}^{p} \binom{k-\ell}{p-r} c_{r,j} = 0 \quad (p > j), \quad \sum_{r=i}^{q} \binom{\ell}{q-r} d_{r,i} = 0 \quad (q > i),$$

with $c_{j,j} = (a - b)^j$, $d_{i,i} = (b - a)^i$. A binomial expansion gives

$$0 = (d/dt)^{q-i-1}[(-1)^{q-i-1}(t-1)^{q-i}t^{\ell-1}]\,|_{t=1}$$
$$= -\frac{1}{\ell}\sum_{r=i}^{q}\binom{\ell}{q-r}\binom{\ell+r-i-1}{r-i}(-1)^{r-1}.$$

Therefore,

$$d_{r,i} = (b-a)^i(-1)^{r-i}\binom{\ell+r-i-1}{r-i},$$
$$c_{r,j} = (a-b)^j(-1)^{r-j}\binom{k-\ell+r-j-1}{r-j}.$$

Insertion of the values of $d_{r,i}$, $c_{r,j}$ into the identities for $u^*_{0,j}$ and $u^*_{1,i}$ gives, by virtue of formula (6.10),

$$(6.11)\quad G(t,s;\alpha,T) = \begin{cases} \displaystyle\sum_{j=0}^{\ell-1}\Big[\sum_{p=0}^{\ell-1-j}\binom{k-\ell+p-1}{p}\Big(\frac{t-a}{b-a}\Big)^p\Big] \\ \qquad\displaystyle\frac{(t-a)^j(a-s)^{k-j-1}}{j!(k-j-1)!}\Big(\frac{b-t}{b-a}\Big)^{l-\ell}, \\ \displaystyle-\sum_{i=0}^{k-\ell-1}\Big[\sum_{q=0}^{k-\ell-i-1}\binom{\ell+q-1}{q} \\ \displaystyle\Big(\frac{b-t}{b-a}\Big)^q\Big]\frac{(t-b)^i(b-s)^{k-i-1}}{i!(k-i-1)!}\Big(\frac{t-a}{b-a}\Big)^{\ell}, \end{cases}$$

for $a \leqq s \leqq t \leqq b$ and $a \leqq t \leqq s \leqq b$, respectively. Here, $\alpha = (\ell, k-\ell)$, $1 \leqq \ell \leqq k-1$, and $T = \{a < b\}$. Notice that the second line of (6.10) is obtained from the first line by replacing a, b, ℓ by $b, a, k-\ell$, respectively, except for the sign -1.

Practical Computation of G

The computation of the Green's function $G(t, s; \alpha_0, T_0)$ for an arbitrary multipoint problem can theoretically be reduced to (1) the determination of the Cauchy function $h(t, s)$, and (2) the calculation of the basis $U_0(t)$ satisfying $Z[U_0; \alpha_0, T_0] = I$. Indeed, formula (6.2) then gives a formula for G.

The Cauchy function $h(t, s)$ can be found by initial value methods, because it is the solution of $Ku = 0$ with initial conditions $u^{(i)}(s) = \delta_{i,k-1}$. In the case of constant coefficients, $h(t, s) = H(t - s)$, where H is the solution of the initial value problem $Ku = 0$, $u^{(i)}(0) = \delta_{i,k-1}$. The basic tools for determining $h(t, s)$ are the Runge-Kutta methods for numerical solution of differential equations and Laplace transform methods. If K has constant coefficients and a basis is explicitly known for $Ku = 0$, the determination of H can be reduced to a problem in linear algebra, to which the methods of numerical linear algebra are applicable.

The basis $U_0(t)$ cannot be found explicitly, unless the operator K is extremely simple. However, numerical approximations to $U_0(t)$ of high accuracy may still be obtainable, which yield an acceptable approximation to G.

The basis $U_0(t)$ can be approximated to a high degree of accuracy using the sequence of fundamental Niccoletti solutions $\{U_p\}_{p=1}^{\infty}$, the order of approximation being given in Corollary 4.15. The utility of the sequence $\{U_p\}_{p=1}^{\infty}$ depends largely upon the selection of the sequence $(\alpha_p, T_p) \to (\alpha_0, T_0)$ and the ease of calculation of U_p. The following remarks outline the advantages and difficulties of this method of approximation of $U_0(t)$.

The usual way to select $\{(\alpha_p, T_p)\}$ is to let α_p assign simple zeros, and let T_p cluster at T_0 with rate $1/p$. For example, if $\alpha_0 = (2, 2)$, $T_0 = \{a < b\}$ and $Ku \equiv u^{\text{iv}}$, then we would select $\alpha_p = (1, 1, 1, 1)$, $T_p = \{a < a + 1/p < b - 1/p < b\}$.

The next step is to determine U_p by the formula $U_p = V_p\mathcal{N}_p$, where $\mathcal{N}_p$ is the normalizing matrix of § 4 and V_p satisfies the identity $Z_p(V_p) = I$. Indeed, we then have (see Def. 4.14, Lemma 4.5) $Z_p(V_p\mathcal{N}_p) = Z_p(V_p)\mathcal{N}_p = I \cdot \mathcal{N}_p = \mathcal{N}_p$, and by uniqueness $U_p = V_p\mathcal{N}_p$. In the fourth order exsmple discussed above,

$$\mathcal{N}_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & p^{-1} & 0 & 0 \\ 0 & 0 & 1 & -p^{-1} \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad V_p = (v_{p1}, v_{p2}, v_{p3}, v_{p4}),$$

$$v_{pi} = \frac{Q_p(t)}{Q_p{}'(t_i)(t - t_i)}, \quad 1 \leqq i \leqq 4,$$

where $t_1 = a$, $t_2 = a + 1/p$, $t_3 = b - 1/p$, $t_4 = b$, $Q_p(t) = \prod_{i=1}^{4} (t - t_i)$.

Finally, the components of U_0 are found by the limit relation

$$U_0(t) = \lim_{p \to \infty} U_p(t) = \lim_{p \to \infty} V_p(t)\mathcal{N}_p.$$

The usual tool here is L'Hospital's rule, but machine computation could replace this analytical procedure. Returning to the fourth order example, the desired basis $U_0 = (u_{00}, u_{01}, u_{10}, u_{11})$ is given by $(x = (t-a)/(b-a))$

$$u_{00} = \lim_{p\to\infty} (v_{p1} + v_{p2}) = (1+2x)(1-x)^2,$$

$$u_{01} = \lim_{p\to\infty} p v_{p2} = x(1-x)^2,$$

$$u_{10} = \lim_{p\to\infty} v_{p3} + v_{p4} = x^2(3-2x),$$

$$u_{11} = \lim_{p\to\infty} -p v_{p3} = x^2(x-1).$$

The labor saved in writing down $v_{p1}, \cdots, v_{p4}$ was expended in the above limiting process. However, this limiting procedure may in fact be quite appropriate when it is possible to write down V_p with ease, but in contrast, the inversions necessary to obtain $U_0(t)$ directly are formidable. Such a situation occurs even for the operator $(d/dt)^k$ when the number of interior boundary conditions is large and these boundary conditions have high mulitplicity.

The Green's function $G(t, s; \alpha_0, T_0)$ can now be written down with the help of (6.2). Indeed, for the fourth order problem $u^{\text{iv}} = f$, $u(0) = u'(0) = u(1) = u'(1) = 0$, the preceding discussion and relation (6.10) gives (compare with (6.11).):

$$(6.12)\quad G(t, s; \alpha, T) = \begin{cases} \dfrac{1}{3!}\left[\dfrac{(a-s)^3(t-b)^2}{(a-b)^2} + \dfrac{3(a-s)^2(t-a)(t-b)^2}{(a-b)^2} \right. \\ \qquad\qquad \left. - \dfrac{2(a-s)^3(t-a)(t-b)^2}{(a-b)^3}\right], \\ \dfrac{-1}{3!}\left[\dfrac{3(b-s)^2(t-a)^2(t-b)}{(b-a)^2}\right. \\ \qquad \left. - \dfrac{2(b-a)^3(t-a)^2(t-b)}{(b-a)^3} + \dfrac{(b-s)^3(t-a)^2}{(b-a)^2}\right] \end{cases}$$

for $t \geqq s$ and $t < s$, respectively.

The work involved in writing down (6.12) (or even (6.11)) can be reduced by the observation that the first line is obtained from the second line by a suitable substitution. More precisely, the reader can easily verify the following fact.

PROPOSITION 6.3. *Let K have constant coefficients. The Green's function G for $Ku = 0$ with $\alpha = (\ell, k-\ell)$, $1 \leqq \ell \leqq k-1$, $T = \{a < b\}$ is obtained as follows: Compute G for $s \leqq t$, then for $s > t$, replace a, b, ℓ by $b, a, k-\ell$, respectively, and multiply by -1.*

If K has constant coefficients, then known formulas for G give rise to formulas for G with other boundary conditions. For example, the problem (1) $u''' = f$, $u(a) = u'(a) = 0 = u(b)$ has Green's function

$$
(6.13)\qquad G_{2,1}(t,s) = \begin{cases} \left[1 + \dfrac{t-a}{b-a}\right] \dfrac{b-t}{b-a} \cdot \dfrac{(a-s)^2}{2} \\ \quad + (t-a) \cdot \dfrac{b-t}{b-a} \cdot (a-s), s \leqq t, \\ -\left(\dfrac{t-a}{b-a}\right)^2 \cdot \dfrac{(b-s)^2}{2}, t \leqq s, \end{cases}
$$

and the problem (2) $u''' = f$, $u(a) = 0 = u(b) = u'(b)$ has Green's function

$$
(6.14)\qquad G_{1,2}(t,s) = \begin{cases} \dfrac{(a-s)^2}{2} \left(\dfrac{b-t}{b-a}\right)^2, s \leqq t, \\ \cdot \left[1 + \dfrac{b-t}{b-a}\right] \cdot \dfrac{(b-s)^2}{2} \cdot \dfrac{t-a}{b-a} \\ \quad - (t-b)(b-s) \cdot \dfrac{t-a}{b-a}, \ t \leqq s. \end{cases}
$$

The connection between formulas (6.13) and (6.14) is the following: if K has constant coefficients, $Ku = 0$, $u^{(i)}(c) = 0$, $0 \leqq i \leqq m$, then $v(t) = u(d + c - t)$ satisfies $K^*v = 0$ and $v^{(i)}(d) = 0$, $0 \leqq i \leqq m$ (see Def. 1.4). For $1 \leqq \ell \leqq k - 1$, let $U_\ell(t)$ denote the row vector basis $(u^*_{00}, \cdots, u^*_{10}, \cdots)$ appearing in (6.10), then $K = (-1)^k K^*$ implies

$$
(6.15)\qquad U_{k-\ell}(t) = U_\ell(b + a - t) \cdot \left[\begin{array}{c:c} 0 & B \\ \hdashline A & 0 \end{array}\right]
$$

where A is $(k - \ell) \times (k - \ell)$, B is $\ell \times \ell$,

$$
A = \operatorname{diag}(1, -1, \cdots, (-1)^{k-\ell-1}),
$$
$$
B = \operatorname{diag}(1, -1, \cdots, (-1)^{\ell-1}).
$$

The effect of this observation on (6.13)–(6.14) is that (6.14) is obtained from (6.13) by replacing t by $b + a - t$ and s by $b + a - s$, except for sign:

$$
(6.16)\qquad G_{1,2}(t,s) = -G_{2,1}(b + a - t, b + a - s).
$$

In general, one can prove the following, using (6.10), (6.15), and the identities $h(t, s) = H(t - s)$, $H^{(j)}(x) = (-1)^{k-1+j}H^{(j)}(-x)$.

PROPOSITION 6.4. *Let K have constant coefficients, $K^* = (-1)^k K$, and put $\alpha = (\ell, k - \ell)$, $T = \{a < b\}$, $G_{\ell,k-\ell}(t, s) = G(t, s; \alpha, T)$. If $G_{\ell,k-\ell}(t, s)$ exists, then so does $G_{k-\ell,\ell}(t, s)$ and*

$$G_{k-\ell,\ell}(t, s) = (-1)^k G_{\ell,k-\ell}(a + b - t, a + b - s). \tag{6.17}$$

7. **The sign of Green's function.** The purpose of this section is to record some technical results on the sign of Green's function and the associated integral operator. These results are to be used in §§ 8 and 9.

LEMMA 7.1. *Let K be disconjugate on $[a, b]$, $f \in C[a, b]$, $f > 0$ on $[a, b]$. Then*

$$\prod_{i=0}^{\nu} (t - s_i)^{-n_i} \int_a^b G(t, s; \alpha, T) f(s)\, ds > 0 \tag{7.1}$$

for $a \leqq t \leqq b$ (interpret (7.1) as a limit for $t \in T$).

LEMMA 7.2. *Let K be disconjugate on $[a, b]$. Then*

$$G(t, s; \alpha, T) \cdot \prod_{i=0}^{\nu} (t - s_i)^{-n_i} > 0, a < s < b, a \leqq t \leqq b. \tag{7.2}$$

(Interpret (7.2) as a limit for $t \in T$).

REMARK 7.3. If K is conjugate in $[a, b]$, and $G(t, s; \alpha, T)$ exists, then (7.2) may fail at a finite number of points.

A derivation of (7.1), (7.2) based on the seemingly extraneous notion of *zero component* may be found in Coppel [6], pages 106–109.

A weak version of (7.2) was obtained recently by Das and Vatsala [7] in the special case $\alpha = (1, 1, \cdots, 1)$ for $Ku \equiv u^{(k)}$ by direct computation. They apparently did not know this was folklore in the Russian literature.

A brief history of results concerning the sign of G can be found in Coppel [**6**; p. 138].

For even order self-adjoint equations with $\alpha = (\ell, k - \ell)$, $T = \{a < b\}$, the Green's function G is *strictly totally positive of order k*; see Karlin [**10**], Chapter 10.

Much work has been done by Russian mathematicians on the question of the sign of G in the presence of conjugacy. For the most part, very little has been settled on this question, and it deserves further study. For positive results, see Peterson [15] and the references therein to the Russian literature.

8. **Norm estimates for $y^{(k)} = 0$.** The purpose of this section is to obtain norm estimates for $G(t, s; \alpha, T)$ in the spaces $C^k[a, b]$ and $L^1[a, b]$, for the equation $y^{(k)} = 0$.

Let us begin with some elementary estimates in the space $L^1[a, b]$. It is clear that the solution $y(t)$ of $y^{(k)} = 1$, $y(s_i) = 0$ $(0 \leqq i \leqq k-1)$ is $y(t) = (1/k!) \prod_{i=0}^{k-1} (t - s_i)$. Since G is one-signed for fixed t (Lemma 7.2), it follows that

$$(8.1) \quad \int_a^b |G(t, s; \alpha, T)|\, ds = \left| \int_a^b G(t, s; \alpha, T)\, ds \right| = |y(t)|.$$

On the other hand, we can now limit via the Green's function convergence theorem in (8.1) to get

$$(8.2) \qquad \int_a^b |G(t, s; \beta, S)|\, ds = \frac{1}{k!} \prod_{i=0}^{\nu} |t - t_i|^{n_i}$$

for arbitrary $\beta = (n_0, \cdots, n_\nu)$ and $S = \{t_0 < \cdots < t_\nu\}$, $|\beta| = k$. Of course, (8.2) could also be obtained in the same way as (8.1).

A nontrivial estimate has been obtained for the uniform norm. In reference [12], Nehari gives an elementary proof of the inequality of Beesack [3] in the special case when $\gamma = (1, 1, \cdots, 1)$, $R = \{a = t_0 < \cdots < t_{k-1} = b\}$:

$$(8.3) \qquad |G(t, s; \gamma, R)| \leqq \frac{\left| \prod_{i=0}^{k-1} (t - t_i) \right|}{(b-a)(k-1)!}.$$

We now have a conceptually simple proof of the general result of Beesack [3]:

$$(8.4) \qquad |G(t, s; \beta, S)| \leqq \frac{\left| \prod_{i=0}^{\nu} (t - t_i)^{n_i} \right|}{(b-a)(k-1)!}.$$

Indeed, we can select a sequence $\gamma_p = (1, 1, \cdots, 1)$ and R_p such that $(\gamma_p, R_p) \to (\beta, S)$, $\beta = (n_0, \cdots, n_\nu)$, $S = \{t_0 < \cdots < t_\nu\}$, and apply the Green's function convergence theorem to inequality (8.3) to obtain (8.4).

Estimates for the derivatives of G can also be obtained. A class of boundary conditions can be isolated for which the estimates are extremely easy: $\alpha = (n_0, n_1, \cdots, n_\nu)$, $T = \{s_0 < \cdots < s_\nu\}$, $n_0 > 1$, $n_\nu > 1$.

In order to study this class of boundary conditions, consider first the case $n_0 = n_\nu = 2$, $n_i = 1$ for $0 < i < \nu$. If $n = k - 1$ and $s \in [a, b]$ is fixed, then $g(x) \equiv (\partial/\partial t)\ G(x, s; \alpha, T)$ has zeros at $a = a_1 < \cdots < a_n = b$, $g \in C^{n-1}[a, b]$, and $g^{(n-1)}$ has the characteristic jump discontinuity of a Green's function. Following the methods of Nehari [12], we obtain

$$|g(x)| \leqq \frac{\left|\prod_{i=1}^{n} (x - a_i)\right|}{(b - a)(n - 1)!} \quad (a \leqq x \leqq b).$$

The problem with this relation is that $a_i = a_i(s)$ for $0 < i < n$, and we really don't know the location of these points. However, we can at least claim $|a_{i+1} - a_i| \leqq 2h$ where

$$h = \max\{s_{i+1} - s_i : 0 \leqq i \leqq \nu - 1\} \equiv \text{mesh } (T).$$

Then, following Das and Vatsala [7, Lemma 4.1], $\prod_{i=1}^{n} |x - a_i| \leqq (n - 1)^{n-1}h^n$ and this gives the estimate

$$\left| \frac{\partial}{\partial t}\ G(t, s; \alpha, T) \right| \leqq \frac{(n - 1)^{n-1}h^n}{(b - a)(n - 1)!}. \tag{8.5}$$

Suppose now that $n_0 > 1$, $n_\nu > 1$. Then we can select a sequence $\alpha_p = (2, 1, 1, \cdots, 1, 2)$ and corresponding $T_p = \{a = s_{0,p} < \cdots < s_{\nu_p p} = b\}$ such that, in the limit, $n_i - 1$ points of T_p cluster at s_i $(0 \leqq i \leqq \nu)$, and $T \cap T_p = T$. Therefore, $(\alpha_p, T_p) \to (\alpha, T)$ and the Green's function convergence theorem applies. For p sufficiently large, inequality (8.5) is valid with (α, T) replaced by (α_p, T_p), h the same, $n = k - 1$. Therefore, limiting gives

$$\left| \frac{\partial}{\partial t}\ G(t, s; \alpha, T) \right| \leqq \frac{(k - 2)^{k-2}[\text{mesh}(T)]^{k-1}}{(b - a)(k - 2)!}, \tag{8.6}$$

valid whenever $|\alpha| = k$, $n_0 \geqq 2$, $n_\nu \geqq 2$.

In the same way, one can establish the inequality

$$|(\partial/\partial t)^r\ G(t, s; \alpha, T)| \leqq \frac{(k - r - 1)^{k-r-1}[\text{mesh}(T)]^{k-2-r}}{(b - a)(k - r - 1)!} \tag{8.7}$$

valid for $0 \leqq r \leqq \min(n_0, n_\nu) - 1$, $\alpha = (n_0, \cdots, n_\nu)$, $T = \{s_0 < \cdots < s_\nu\}$, where G is the Green's function for the operator $Ku \equiv u^{(k)}$. If (α, T) does not concentrate a zero in (a, b), then we may replace mesh(T) by $(b - a)/2$ in (8.7).

On the other hand, using the inequality

$$\prod_{i=1}^{n} |x - a_i| \leqq \left(\frac{n-1}{n}\right)^{n-1} \frac{(b-a)^n}{n}$$

proved by Beesack [3, p. 808] gives instead of (8.7) the inequality

$$|(\partial/\partial t)^r G(t, s; \alpha, T)| \leqq \left(\frac{k-r-1}{k-r}\right)^{k-r-1} \frac{(b-a)^{k-r-1}}{(k-r)!} \tag{8.8}$$

for $0 \leqq r \leqq \min(n_0, n_\nu) - 1$.

Relation (8.7) is good when mesh(T) is small, but (8.8) is better when $T \cap (a, b) = \emptyset$ and k is large. Neither estimate is particularly outstanding, except in degenerate cases.

In contrast, excellent estimates for the derivatives of G have been obtained by Ostroumov [14] for 2-point problems for $K = (d/dt)^k$. It is reasonable to conjecture that good estimates exist when K is disconjugate; see Bates and Gustafson [1], [2].

9. **Norm estimates for $Ku = 0$.** In this section, the problem of norm estimation of Green's functions for $Ku = 0$ is considered. The spaces of interest are $C[a, b]$ and $L^1[a, b]$.

LEMMA 9.1. *Suppose $\alpha = (1, 1, \cdots, 1)$, $T = \{a = s_0 < \cdots < s_{k-1} = b\}$. Let $u_j(t)$ be the solution of $Ku = 0$ which takes the value 1 at s_j and is zero at the other points of T. Then (notation of § 6)*

$$\int_a^b G(t, s; \alpha, T)\, ds = \sum_{j=0}^{k-1} u_j(t) \int_{s_j}^t h(s_j, s)\, ds. \tag{9.1}$$

PROOF. Integrate (6.2) in this special case. The problem is to compute $\int_a^b [\epsilon(t - s) - \chi_{E_i}(s)]\, h(s_i, s)\, ds$. Break this integral into two integrals, over $[a, t]$ and $[t, b]$. Considering cases leads to its value being $\int_{s_i}^t h(s_i, s)\, ds$, hence (9.1).

The convergence theorem can now be used to obtain from (9.1) estimates for the L^1 norm of a Green's function. The critical assumption of disconjugacy of K on $[a, b]$ is needed to obtain $|\int_a^b G| = \int_a^b |G|$. The details are left to the reader, with § 8 serving as the model.

PROPOSITION 9.2. *Let $\alpha^* = (n_0, \cdots, n_\nu)$, $T^* = \{a = s_0 < \cdots < s_\nu = b\}$, $|\alpha^*| = k$, and assume K is disconjugate.*

Let $\mathcal{G}$ be a collection of pairs (α, T) such that $\alpha = (1, 1, \cdots, 1)$, $|\alpha| = k$, $T = \{a = t_0 < \cdots < t_{k-1} = b\}$, and there exists at least one sequence in $\mathcal{G}$ which converges to (α^*, T^*).

Define $u_j(t;\, T)$ *to be the solution of* $Ku = 0$ *with value* 1 *at* t_j *and value* 0 *at the other points of* T, *for each* $(\alpha, T) \in \mathcal{I}$.

Then

$$(9.2)\quad \int_a^b |G(t, s; \alpha^*, T^*)|\, ds \leqq \sup_{(\alpha,T)\in F} \left| \sum_{j=0}^{k-1} u_j(t;\, T) \int_{t_j}^{t} h(t_j, s)\, ds \right|.$$

In a similar manner, the uniform norm of G can be estimated in terms of "nearby" Green's functions built from simple boundary conditions:

PROPOSITION 9.3. *Notation and assumptions as in* 9.2, *except delete the hypothesis of disconjugacy. Then*:

$$(9.3)\quad |G(t, s; \alpha^*, T^*)| \leqq \sup_{(\alpha,T)\in \mathcal{I}} \left| \sum_{j=0}^{k-1} u_j(t;\, T)[\epsilon(t-s) - \chi_{E_j}(s)]\, h(s_j, s) \right|.$$

REMARK 9.4. The use of relations (9.1)–(9.3) in nonlinear and linear boundary value problems has been illustrated by Wend [19], Beesack [3], Das and Vatsala [7], and others.

It would be interesting to develop some estimates for the norm of G in the space $C^r[a, b]$. In this direction the following result is recorded:

PROPOSITION 9.5. *Notation and assumptions as in* 9.3. *The following inequality is valid for* $0 \leqq r \leqq k-2$.

$$(9.4)\quad \begin{aligned} &|(\partial/\partial t)^r\, G(t, s; \alpha^*, T^*)| \\ &\qquad \leqq \sup_{(\alpha,T)\in \mathcal{I}} \left| \sum_{j=0}^{k-1} u_j^{(r)}(t;\, T)[\epsilon(t-s) - \chi_{E_j}(s)]\, h(s_j, s) \right|. \end{aligned}$$

REMARK 9.6. The question of sharpness of (9.2), (9.3) can be resolved by appeal to the forthcoming paper of Bates and Gustafson [2], wherein it is shown that for disconjugate operators K, $(\alpha_p, T_p) \rightarrow (\alpha^*, T^*)$ implies

$$\begin{aligned} \lim_{p\to\infty} |G(t, s; \alpha_p, T_p)| &= |G(t, s; \alpha^*, T^*)| \\ &= \sup_{p>1} |G(t, s; \alpha_p, T_p)|. \end{aligned}$$

ACKNOWLEDGMENT. The referee's pertinent comments and valuable suggestions have shaped this paper, and given me guidance unavailable elsewhere. I take this opportunity to thank him.

Thanks are also due to the participants of the Seminar in Differential Equations at the University of Utah, who listened patiently through preliminary versions of this paper.

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