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ON THE EXISTENCE OF SOLUTIONS TO MULTIPOINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. We are concerned with the nth order differential equation $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ where it is assumed throughout that f is continuous on $[\alpha, \beta) \times R^n$, $\alpha < \beta \leqq \infty$, and that solutions of initial value problems are unique and exist on $[\alpha, \beta)$. Our main concern will be to show that under certain conditions the uniqueness of solutions of multipoint boundary value problems implies the existence of solutions of such problems.

1. **Introduction.** We will consider the following nonlinear differential equation in this work:

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}), \tag{1.1}$$

where $x \in I = [\alpha, \beta), \alpha < \beta \leqq +\infty$.

The following assumptions are assumed to be satisfied throughout this work:

(A) f is continuous on $[\alpha, \beta) \times R^n$, and

(B) solutions of initial value problems (IVP's) are unique and extend throughout $[\alpha, \beta)$.

DEFINITION 1.1. We say that $y \in C^n[\alpha, \beta)$ has an $(i_1, \cdots, i_m)$-distribution of zeros, $0 \leqq i_k \leqq n$, $\sum_{k=1}^m i_k = n$, on $[c, d] \subset [\alpha, \beta)$ provided there are points $c \leqq x_1 < \cdots < x_m \leqq d$ such that $y(x)$ has a zero of order at least i_k at x_k, $k = 1, \cdots, m$.

DEFINITION 1.2. Let $R = \{r > t$: there exist distinct solutions $u(x)$ and $v(x)$ of equation (1.1) such that $u(x) - v(x)$ has an $(i_1, \cdots, i_m)$-distribution of zeros, $0 \leqq i_k \leqq n$, $\sum_{k=1}^m i_k = n$, on $[t, r]\}$. If $R \neq \emptyset$, set $r_{i_1 \cdots i_m}(t) = \inf R$. If $R = \emptyset$, set $r_{i_1 \cdots i_m}(t) = +\infty$.

REMARK 1.3. If $t \leqq x_1 < \cdots < x_m < r_{i_1 \cdots i_m}(t) \leqq +\infty$, then solutions of equation (1.1) satisfying the boundary conditions

$$y^{(l_j-1)}(x_j) = C_{j,l_j};\ C_{j,l_j} \in R^1 = (-\infty, \infty); \tag{1.2}$$
$$l_j = 1, \cdots, i_j; j = 1, \cdots, m,$$

when they exist are unique.

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Equation (1.1) along with boundary conditions (1.2) is called an $(i_1, \cdots, i_m)$-boundary value problem (BVP) and will be referred to in this paper as the $(i_1, \cdots, i_m)$-BVP (1.1), (1.2). In the linear case it is well known that if $t \leqq x_1 < \cdots < x_m < r_{i_1 \cdots i_m}(t) \leqq +\infty$, then for the given linear equation there always exists a unique solution satisfying the boundary conditions (1.2). Uniqueness implies existence results of this type for the nonlinear case is what this paper is primarily concerned with.

Definition 1.4. The first conjugate point $\eta_1(t)$, for the nonlinear equation (1.1) is defined by

$$\eta_1(t) = \min \left\{ r_{i_1 \cdots i_m}(t) : \sum_{k=1}^{m} i_k = n \right\}.$$

Definition 1.5. If I is an interval and $I \subset [\alpha, \beta)$, then we say that I is an interval of disconjugacy for equation (1.1) provided there do not exist distinct solutions $y(x)$, $z(x)$ of equation (1.1) such that $y(x) - z(x)$ has at least n zeros, counting multiplicities, on I. It is clear from Definition 1.4 that if $I \subset [t, \eta_1(t))$, then I is an interval of disconjugacy and that $[t, \eta_1(t))$ is a maximal half open interval of disconjugacy.

Remark 1.6. It is possible that $\eta_1(t) = t$ for the nonlinear equation (1.1). For example, if $n = 2$ and we consider the equation

$$y'' = -y^3,$$

it was shown in [9] that for this equation $\eta_1(t) = t$, for all t. However, if f satisfies a uniform Lipschitz condition with respect to $y, y', \cdots, y^{(n-1)}$ on compact subintervals of $[\alpha, \beta)$, then , using estimates for the Green's function ([3]) and the fact that $\eta_1(t) = r_{1\cdots 1}(t)$, ([2]), one can use standard fixed point arguments to prove that $\eta_1(t) > t$ for all $t \in [\alpha, \beta)$. In particular, if equation (1.1) is a linear differential equation, then, as is well known, $\eta_1(t) > t$, for all t. Some of the results of this paper require the hypothesis that $\eta_1(t) > t$ for some $t \in [\alpha, \beta)$. In [11] estimates for bounds on the Green's function for the $(n-1, 1)$-BVP (1.1), (1.2) and standard fixed point arguments were used to prove that $r_{n-1,1}(t) > t$ for all $t \in [\alpha, \beta)$ in a special case. In particular it was assumed that f satisfies a uniform Lipschitz condition with respect to $y, y', \cdots, y^{(n-1)}$ on each compact subinterval of $[\alpha, \beta)$ to establish lower bounds for $r_{n-1,1}(t)$.

Definition 1.7. Let $y(x)$ be a solution of equation (1.1), and let $t \leqq x_1 < \cdots < x_m < r_{i_1 \cdots i_m}(t)$. Define

$$S(y(x); x_1^{i_1}, \cdots, \hat{x}_k^{i_k}, \cdots, x_m^{i_m}) \equiv$$

$$\{u^{(i_k-1)}(x_k) : u(x) \text{ is a solution of equation (1.1) such that}$$

$$u^{(l_j)}(x_j) = y^{(l_j)}(x_j), l_j = 0, \cdots, i_j - 1; j = 1, \cdots, m; j \neq k,$$

$$u^{(l_k)}(x_k) = y^{(l_k)}(x_k), l_k = 0, \cdots, i_k - 2$$

$$(\textit{If } i_k = 1, \text{ there is no boundary condition at } x_k)\}.$$

If the superscript i_j is 1, then the superscript will be omitted and understood to be 1.

Lasota and Opial ([6]) showed that if $t \leqq x_1 < x_2 < \eta_1(t)$, then for any solution $y(x)$ of equation (1.1) $S(y(x); x_1, \hat{x}_2) = R^1$. For $n = 3$, Jackson and Schrader ([4]) showed that if $t \leqq x_1 < x_2 < x_3 < \eta_1(t)$, then for any solution $y(x)$ of equation (1.1), $S(y(x); x_1, x_2, \hat{x}_3) = S(y(x); x_1, \hat{x}_2, x_3) = R^1$ which is equivalent to the existence of all (1, 1, 1)-BVP's (1.1), (1.2) in this case. They also showed that $S(y(x); x_1^2, \hat{x}_2) = S(y(x); \hat{x}_1, x_2^2) = R^1$ for all solutions $y(x)$ of equation (1.1) which is equivalent to the existence of all (2, 1)- and (1, 2)-BVP's (1.1), (1.2) in the case where $t \leqq x_1 < x_2 < \eta_1(t)$. Klaasen ([5]) see also Hartman [1]) showed that if in addition to (A) and (B) the compactness condition defined in Chapter 2 of this paper is assumed to be satisfied, then, if $t \leqq x_1 < \cdots < x_m < \eta_1(t)$, then for any solution $y(x)$ of equation (1.1) $S(y(x); x_1^{i_1}, \cdots, \hat{x}_k^{i_k}, \cdots, x_m^{i_m}) = R^1$, $k = 1, \cdots, m$. Peterson ([7]) has obtained similar results for the fourth order equation (1.1) not assuming uniqueness of solutions of initial value problems.

Spencer ([10]) has obtained several ordering relations between the boundary value functions, $r_{i_1 \cdots i_m}(t)$. This type of result is useful in proving uniqueness of solutions of BVP's implies existence of solutions.

2. **Main Results.** In addition to (A) and (B) we will assume the compactness condition

(C) If $\{y_j(x)\}$ is a sequence of solutions of equation (1.1) which is uniformly bounded on a nondegenerate compact subinterval $[c, d]$ contained in $[\alpha, \beta)$, then there is a subsequence $\{y_{j_k}(x)\}$ of $\{y_j(x)\}$ such that $\{y^{(i)}(x)\}$ converges uniformly on each compact subinterval of $[\alpha, \beta)$, for $i = 0, \cdots, n - 1$.

The following lemma is Corollary 3.22 in [8].

Lemma 2.1. (Brouwer's Invariance of Domain Theorem). *Let U be an open set in R^n and $\phi : U \to R^n$ a continuous and one-to-one mapping. Then ϕ is an open mapping.*

DEFINITION 2.2. Let $t \geqq \alpha$. If $r_{i_1 \cdots i_m}(s) = +\infty$ for all $s > t$, then we set $r_{i_1 \cdots i_m}(t+) = +\infty$, otherwise $r_{i_1 \cdots i_m}(t+)$ is defined by

$$r_{i_1 \cdots i_m}(t+) = \inf \{r_{i_1 \cdots i_m}(s) : s > t\}.$$

The following lemma is Lemma 2.3 in [9].

LEMMA 2.3. *Assume* $\alpha \leqq t < r_{i_1 \cdots i_m}(t+) \leqq +\infty$, *and let* $\Delta = \{(x_1, \cdots, x_m) \in R^m : t < x_1 < \cdots < x_m < r_{i_1 \cdots i_m}(t+)\}$. *Define* $\phi : \Delta \times R^n \to R^{m+n}$ *by*

$$\phi(x_1, \cdots, x_m, y_1, \cdots, y_n) =$$
$$(x_1, \cdots, x_m, y(x_1), y'(x_1), \cdots, y^{(i_1-1)}(x_1),$$
$$y(x_2), \cdots, y(x_m), \cdots, y^{(i_m-1)}(x_m)),$$

where $y(x)$ *is the solution of* (1.1) *satisfying* $y^{(l)}(x_0) = y_{l+1}$, $l = 0, \cdots, n-1$, *where* x_0 *is a fixed point in* $(t, r_{i_1 \cdots i_m}(t+))$. *Then* $\phi : \Delta \times R^n \to \phi(\Delta \times R^n)$ *is a homeomorphism, and* $\phi(\Delta \times R^n)$ *is open.*

Motivated by Lemma 2.3 we have the following important modification of that result.

LEMMA 2.4. *Assume* $\alpha \leqq t < r_{i_1 \cdots i_m}(t) \leqq +\infty$, *and let* $\Delta = \{(x_1, \cdots, x_m) \in R^m : t \leqq x_1 < \cdots < x_m < r_{i_1 \cdots i_m}(t)\}$. *Define* $\phi : \Delta \times R^n \to R^{m+n}$ *by*

$$\phi(x_1, \cdots, x_m, y_1, \cdots, y_n) =$$
$$(x_1, \cdots, x_m, y(x_1), y'(x_1), \cdots, y^{(i_1-1)}(x_1),$$
$$y(x_2), \cdots, y(x_m), \cdots, y^{(i_m-1)}(x_m)),$$

where $y(x)$ *is the solution of* (1.1) *satisfying* $y^{(l)}(t) = y_{l+1}$, $l = 0, \cdots, n-1$. *Then* $\phi : \Delta \times R^n \to \phi(\Delta \times R^n)$ *is a homeomorphism, and* $\phi(\Delta \times R^n)$ *is a relatively open subset of* $\Delta \times R^n$.

PROOF. Let $\mu = t$ and $v = r_{i_1 \cdots i_m}(t)$. Set $\bar{\Delta} = \{(x_1, \cdots, x_m) \in R^m : 2\mu - x_2 < x_1 < v$ and $\mu < x_2 < \cdots < x_m < v\}$. Define $\bar{\phi} : \bar{\Delta} \times R^n \to R^{m+n}$ by

$$\bar{\phi}(x_1, \cdots, x_m, y_1, \cdots, y_n) =$$
$$(x_1, \cdots, x_m, y(x_1), y'(x_1), \cdots, y^{(i_1-1)}(x_1),$$
$$y(x_2), \cdots, y(x_m), \cdots, y^{(i_m-1)}(x_m)),$$

if $(x_1, \cdots, x_m, y_1, \cdots, y_n) \in \Delta \times R^n$.

$$\bar{\phi}(x_1, \cdots, x_m, y_1, \cdots, y_n) =$$
$$(x_1, \cdots, x_m, y(2\mu - x_1), y'(2\mu - x_1), \cdots, y^{(i_1-1)}(2\mu - x_1),$$
$$y(x_2), y'(x_2), \cdots, y(x_m), \cdots, y^{(i_m-1)}(x_m)),$$

if $(x_1, \cdots, x_m, y_1, \cdots, y_n) \in \bar{\Delta} \times R^n \sim \Delta \times R^n$, where $y(x)$ is the solution of (1.1) satisfying $y^{(l)}(t) = y_{l+1}, l = 0, \cdots, n-1$.

Note that if $(x_1, \cdots, x_m, y_1, \cdots, y_n) \in \bar{\Delta} \times R^n \sim \Delta \times R^n$ then $2\mu - x_2 < x_1 < \mu < x_2 < \cdots < x_m < v$, and hence $\mu < 2\mu - x_1 < x_2 < \cdots < x_m < v$. The function $\bar{\phi}$ is one-to-one, since, if two points of $\bar{\Delta} \times R^n$ have the same image under ϕ, then the first m coordinates of the two points must agree, and hence, by the uniqueness of solutions of the $(i_1, \cdots, i_m)$-BVP, the remaining n coordinates of these two points must agree. It follows from continuity of solutions of IVP's with respect to initial conditions that $\bar{\phi}$ is continuous. It is clear that $\bar{\Delta} \times R^n$ is an open subset of R^{m+n}, hence by Lemma 2.1, $\phi : \bar{\Delta} \times R^n \to R^{m+n}$ is an open mapping. By the way we have defined $\bar{\phi}$, it follows that $\phi(\Delta \times R^n) = \bar{\phi}(\bar{\Delta} \times R^n) \cap [\mu, v) \times R^{m+n-1}$, hence $\phi(\Delta \times R^n)$ is a relatively open subset of $[\mu, v) \times R^{m+n-1}$, and $\phi : \Delta \times R^n$ onto $\phi(\Delta \times R^n)$ is a homeomorphism.

The following corollary to Lemma 2.4 shows that uniqueness of the $(i_1, \cdots, i_m)$-BVP (1.1), (1.2) on an interval $I = [\mu, v)$ implies "local" existence of solutions of the $(i_1, \cdots, i_m)$-BVP (1.1), (1.2).

Corollary 2.5 indeed provides a powerful tool with which to attack the uniqueness implies existence problem that concerns us in this chapter. Frequent reference will be made to this important corollary.

Corollary 2.5. *Let $t \leqq x_1 < \cdots < x_m < r_{i_1 \cdots i_m}(t)$, and let $y(x)$ be a solution of the $(i_1, \cdots, i_m)$-BVP (1.1), (1.2). Then for $\delta > 0$ sufficiently small, $t \leqq s_1 < \cdots < s_m < r_{i_1 \cdots i_m}(t)$, $|\delta_{j,l_j}| < \delta$ and $|s_j - x_j| < \delta$, $l_j = 1, \cdots, i_j; j = 1, \cdots, m$, there exists a (unique) solution $z(x)$ of (1.1) satisfying*

$$z^{(l_j-1)}(s_j) = C_{j,l_j} + \delta_{j,l_j}, l_j = 1, \cdots, i_j; j = 1, \cdots, m.$$

Furthermore, given any compact subset $K \subset [\alpha, \beta)$ and $\epsilon > 0$, δ may be picked small enough that $|z^{(i)}(x) - y^{(i)}(x)| < \epsilon, i = 0, \cdots, n-1$, for all $x \in K$.

Proof. The function ϕ defined in Lemma 2.4 is an open map, and $\phi(\Delta \times R^n)$ is a relatively open subset of $\Delta \times R^n$. Since the point $(x_1, \cdots, x_m, C_{1,1}, \cdots, C_{1,i_1}, C_{2,1}, \cdots, C_{m,i_m})$ is in the open set $\phi(\Delta \times R^n)$, there exists an open neighborhood, relative to $\Delta \times R^n$, contained in

$\phi(\Delta \times R^n)$. Hence for δ sufficiently small, the set

$$S \equiv \{(s_1, \cdots, s_m, C_{1,1} + \delta_{1,1}, \cdots, C_{m,i_m} + \delta_{m,i_m}) :$$
$$t \leqq s_1 < \cdots < s_m < r_{i_1 \cdots i_m}(t), |s_j - x_j| < \delta,$$
$$|\delta_{j,l_j}| < \delta, l_j = 1, \cdots, i_j; j = 1, \cdots, m\}$$

is contained in $\phi(\Delta \times R^n)$. By the way ϕ is defined, for each point of S there exists a corresponding solution of equation (1.1) which determines that point. The last statement of the conclusion of this corollary is a consequence of the continuity of ϕ^{-1} and the continuous dependence of solutions on initial conditions.

By slightly modifying the proof of Lemma 2.4 we can show that Corollary 2.5 holds for any closed, open or half open interval I on which we have uniqueness of solutions of the $(i_1, \cdots, i_m)$-BVP (1.1), (1.2).

REMARK 2.6. *It follows from Corollary* 2.5, *that, if for every* $[c, d] \subset [\alpha, \beta)$ *there exists a subinterval* $[c_1, d_1] \subset [c, d]$, $d_1 > c_1$, *such that there exists a unique solution to every* (1, 1, $\cdots$, 1)-BVP (1.1), (1.2), *where* $c_1 \leqq x_1 < \cdots < x_n \leqq d_1$, *then the compactness condition* (C) *holds.*

THEOREM 2.7. *Let* $1 \leqq k \leqq m$ *and* $t \leqq x_1 < \cdots < x_m < r_{i_1 \cdots i_m}(t)$ *where* $i_k = 1$. *If* (1.1) *satisfies* (A), (B) *and* (C), *then for any solution* $y(x)$ *of* (1.1), $S(y; x_1^{i_1}, \cdots, \hat{x}_k, \cdots, x_m^{i_m})$ *is an open interval.*

PROOF. It follows from Corollary 2.5 that the set $S = S(y; x_1^{i_1}, \cdots, \hat{x}_k, \cdots, x_m^{i_m})$ is an open subset of R^1. It suffices to establish that if $\sigma, \tau \in S$ with $\sigma < \tau$, then $[\sigma, \tau] \subset S$. Assume that $\sigma, \tau \in S$, and let $\gamma_0 = \sup\{\gamma \leqq \tau : [\sigma, \gamma] \subset S\}$. By Corollary 2.5, if $\gamma_0 \in S$, then $[\sigma, \gamma_0 + \epsilon) \subset S$ for $\epsilon > 0$ sufficiently small, contrary to the definition of γ_0. Hence $\gamma_0 \notin S$. If $\gamma_0 < \tau$, let $\{\gamma_n\}$ be a sequence of real numbers such that for each n, $\sigma < \gamma_n < \gamma_{n+1} < \gamma_0$, and $\{\gamma_n\}$ converges to γ_0. Let $\{u_n(x)\}$ be the corresponding sequence of solutions of equation (1.1) satisfying

$$u_n^{(l_j)}(x_j) = y^{(l_j)}(x_j), l_j = 0, \cdots, i_j - 1; j = 1, \cdots, m; j \neq k,$$
$$u_n(x_k) = \gamma_n.$$

Let $v(x)$ be the solution of (1.1) satisfying

$$v^{(l_j)}(x_j) = y^{(l_j)}(x_j), l_j = 0, \cdots, i_j - 1; j = 1, \cdots, m; j \neq k,$$
$$v(x_k) = \tau.$$

Note that $u_n(x) < u_{n+1}(x)$ for $x \in (x_k, x_{k+1})$ for each n (where we let $x_{m+1} = r_{i_1 \cdots i_m}(t)$ in case $k = m$). Similarly, $u_n(x) < v(x)$ for $x \in (x_k, x_{k+1})$. Since $u_1(x) < u_n(x) < v(x)$ for $x \in (x_k, x_{k+1})$, it follows from (C) that there exists a subsequence $\{u_{n_j}(x)\}$ such that $\{u_{n_j}{}^{(i)}(x)\}$ converges uniformly to a solution $u(x)$ of equation (1.1) on compact subsets of $[\alpha, \beta)$, where $u(x)$ satisfies

$$u^{(l_j)}(x_j) = y^{(l_j)}(x_j),\ l_j = 0, \cdots, i_j - 1;\ j = 1, \cdots, m;\ j \neq k,$$

$$u(x_k) = \gamma_0.$$

But then $\gamma_0 \in S$ which is a contradiction.

To prove Theorem 2.7 when i_k is an arbitrary positive integer we will use the following elementary lemma.

LEMMA 2.8. *Let $u(x)$ and $v(x)$ be in $C^p[x_0, x_1]$, $p > 0$. If $u^{(l)}(x_0) = v^{(l)}(x_0)$, $l = 0, \cdots, p-1$, $u^{(p)}(x_1) \leqq v^{(p)}(x_1)$, and $u^{(p)}(s) < v^{(p)}(s)$ for $s \in [x_0, x_1)$, then $u^{(l)}(s) < v^{(l)}(s)$ for $s \in (x_0, x_1)$, $l = 0, \cdots, p-1$.*

THEOREM 2.9. *Let $1 \leqq k \leqq m$ and $t \leqq x_1 < \cdots < x_m < r_{i_1 \cdots i_m}(t)$, where $i_k \geqq 2$. If equation (1.1) satisfies (A), (B) and (C), then for any solution $y(x)$ of (1.1), $S(y;\ x_1{}^{i_1}, \cdots, \hat{x}_k{}^{i_k}, \cdots, x_m{}^{i_m})$ is an open interval.*

PROOF. It follows from Corollary 2.5 that the set $S \equiv S(y;\ x_1{}^{i_1}, \cdots, \hat{x}_k{}^{i_k}, \cdots, x_m{}^{i_m})$ is an open subset of R^1. It suffices to establish that if $\gamma_0 = \sup\{\gamma : [y^{(i_k-1)}(x_k), \gamma] \subset S\}$ and $\gamma' > \gamma_0$, then $\gamma' \notin S$, and that, if $\lambda_0 = \inf\{\lambda : [\lambda, y^{(i_k-1)}(x_k)] \subset S\}$ and $\lambda' < \lambda_0$, then $\lambda' \notin S$. We will consider here only the first of these two cases since the argument for the second case is similar to that of the first case.

Suppose $\gamma' > \gamma_0$ and $\gamma' \in S$. There then must exist a solution $u(x)$ of (1.1) such that

$$u^{(l_j)}(x_j) = y^{(l_j)}(x_j),\ l_j = 0, \cdots, i_j - 1;\ j = 1, \cdots, m;\ j \neq k,$$

$$u^{(l_k)}(x_k) = y^{(l_k)}(x_k),\ l_k = 0, \cdots, i_k - 2,$$

$$u^{(i_k-1)}(x_k) = \gamma'.$$

Let $\{\gamma_n\}$ be a sequence of real numbers such that $y^{(i_k-1)}(x_k) < \gamma_n < \gamma_{n+1} < \gamma_0$ for each $n \geqq 1$ and $\lim_{n\to\infty}\gamma_n = \gamma_0$. Let $\{u_n(x)\}$ be the corresponding sequence of solutions of (1.1) such that

$$u_n{}^{(l_j)}(x_j) = y^{(l_j)}(x_j),\ l_j = 0, \cdots, i_j - 1;\ j = 1, \cdots, m;\ j \neq k,$$

$$u_n{}^{(l_k)}(x_k) = y^{(l_k)}(x_k),\ l_k = 0, \cdots, i_k - 2,$$

$$u_n{}^{(i_k-1)}(x_k) = \gamma_n.$$

Assume the sequence of functions $\{u_n{}^{(i_k-1)}(x)\}$ is uniformly bounded on $[x_k, x_k+\epsilon]$ for some $\epsilon > 0$. It then follows that $\{u_n(x)\}$ is uniformly bounded on $[x_k, x_k+\epsilon]$. But then since equation (1.1) satisfies (C) there must exist a subsequence $\{u_{n_j}(x)\}$ such that $\{u_{n_j}(x)\}$ converges uniformly to a solution $z(x)$ of (1.1) on compact subsets of $[\alpha, \beta)$, and consequently $\gamma_0 \in S$. But this leads to a contradiction. Hence $\{u_n{}^{(i_k-1)}(x)\}$ cannot be uniformly bounded on $[x_k, x_k+\epsilon]$ for any $\epsilon > 0$.

Since $\{u_n{}^{(i_k-1)}(x)\}$ cannot be uniformly bounded on $[x_k, x_k+\epsilon]$, for any $\epsilon > 0$, and $y^{(i_k-1)}(x_k) < u_n{}^{(i_k-1)}(x_k) < u^{(i_k-1)}(x_k)$, it follows by continuity that there exists a decreasing sequence $\{\delta_j\}$ such that $\delta_j > 0, \lim_{j\to\infty}\delta_j = 0$,

$$u_{n_j}{}^{(i_k-1)}(x_k+\delta_j) = y^{(i_k-1)}(x_k+\delta_j), \text{ or}$$
$$u_{n_j}{}^{(i_k-1)}(x_k+\delta_j) = u^{(i_k-1)}(x_k+\delta_j), \text{ and}$$
$$y^{(i_k-1)}(s) < u_{n_j}{}^{(i_k-1)}(s) < u^{(i_k-1)}(s),$$

for $s \in (x_k, x_k+\delta_j)$. We will only consider the case where, for $j \geqq 1$,

$$u_{n_j}{}^{(i_k-1)}(x_k+\delta_j) = y^{(i_k-1)}(x_k+\delta_j).$$

By renumbering we can assume without loss of generality that

$$u_n{}^{(i_k-1)}(x_k + \delta_n)\,_n) = y^{(i_k-1)}(x_k+\delta_n) \text{ and}$$
$$y^{(i_k-1)}(s) < u_n{}^{(i_k-1)}(s) < u^{(i_k-1)}(s),$$

for $s \in (x_k, x_k+\delta_n)$. By Lemma 2.8, $y^{(i)}(s) < u_n{}^{(i)}(s) < u^{(i)}(s)$, for $s \in (x_k, x_k+\delta_n)$ and $i = 0, \cdots, i_k - 1$. By the continuity of $y^{(i)}(x)$ and $u^{(i)}(x)$, we have $\lim_{n\to\infty}u_n{}^{(i)}(x_k+\delta_n) = y^{(i)}(x_k)$, for $i = 0, \cdots, i_k - 2$. But then, by Corollary 2.5, it follows that $\lim_{n\to\infty}u_n(x) = y(x)$ uniformly on compact subintervals of $[\alpha, \beta)$ which leads to a contradiction.

Remark 2.10. In [10] Spencer showed that if equation (1.1) satisfies (A) and (B) then $r_{1\cdots1}(t) \leqq r_{21\cdots1}(t) \leqq \cdots \leqq r_{n-1,1}(t)$. This result will be used frequently in the remainder of this chapter.

Theorem 2.11. *Let* $3 \leqq k \leqq p+1 \leqq n$ *and* $t \leqq x_1 < \cdots < x_{p+1} < r_{n-1,1\cdots1}(t)$. *If equation* (1.1) *satisfies* (A), (B) *and* (C), *then for any solution* $y(x)$ *of* (1.1)

$$S(y; x_1{}^{n-p}, x_2, \cdots, \hat{x}_k, \cdots, x_{p+1}) = S(y; x_1{}^{n-p+1}, x_3, \cdots, \hat{x}_k, \cdots, x_{p+1}).$$

PROOF. By Theorem 2.8 and the fact that $r_{n-p,1\cdots1}(t) \leqq r_{n-p+1,1\cdots1}(t)$, the sets $S_1 \equiv S(y; x_1^{n-p}, x_2, \cdots, \hat{x}_k, \cdots, x_{p+1})$ and $S_2 \equiv S(y; x_1^{n-p+1}, x_3, \cdots, \hat{x}_k, \cdots, x_{p+1})$ are open intervals about $y(x_k)$. Let $S_1 = (\lambda_1, \gamma_1)$ and $S_2 = (\lambda_2, \gamma_2)$. We will show that $\gamma_1 = \gamma_2$. The case $\lambda_1 = \lambda_2$ follows from a similar argument.

Suppose $\gamma_1 < \gamma_2$. Let $v(x)$ be the solution of (1.1) such that

$$v^{(l)}(x_1) = y^{(l)}(x_1), l = 0, \cdots, n-p,$$
$$v(x_j) = y(x_j), \quad j = 3, 4, \cdots, p+1; j \neq k,$$
$$v(x_k) = (\gamma_1 + \gamma_2)/2.$$

Let $0 < \delta < \min\{x_{k+1} - x_k, x_k - x_{k-1}\}$ (where $x_{p+2} = r_{n-p,1\cdots1}(t)$ in case $k = p+1$). For n an integer and $n > 1/(\gamma_1 - \lambda_1)$, there exists a unique solution $u_n(x)$ of (1.1) such that

$$u_n^{(l)}(x_1) = y^{(l)}(x_1), l = 0, \cdots, n-p-1,$$
$$u_n(x_j) = y(x_j), \quad j = 2, 3, \cdots, p+1; j \neq k,$$
$$u_n(x_k) = \gamma_1 - 1/n.$$

Note that $\{u_n(x)\}$ is a strictly increasing sequence of functions on (x_{k-1}, x_{k+1}). As in the proof of Theorem 2.7, $\{u_n(x)\}$ cannot be uniformly bounded on any compact subinterval of $[\alpha, \beta)$, hence for n sufficiently large there must exist $s_1 \in (x_{k-1}, x_k)$ and $s_2 \in (x_k, x_{k+1})$ such that $u_n(s_1) = v(s_1)$ and $u_n(s_2) = v(s_2)$. This is not possible, hence $\gamma_1 \geqq \gamma_2$. A similar argument shows that $\gamma_1 \leqq \gamma_2$.

REMARK 2.12. Let $2 \leqq k \leqq p \leqq n-1$ and $t \leqq x_1 < \cdots < x_{p+1} < r_{n-p,1\cdots1}(t)$. If equation (1.1) satisfies (A), (B) and (C), then for any solution $y(x)$ of (1.1)

$$S(y; x_1^{n-p}, x_2, \cdots, \hat{x}_k, \cdots, x_{p+1}) =$$
$$S(y; x_1^{n-p+1}, x_2, \cdots, \hat{x}_k, x_{k+2}, \cdots, x_{p+1}),$$

for $k = 2, \cdots, p-1$, and

$$S(y; x_1^{n-p}, x_2, \cdots, \hat{x}_p, x_{p+1}) =$$
$$S(y; x_1^{n-p+1}, x_2, \cdots, \hat{x}_p)$$

When $k = p$. The proof is similar to that of Theorem 2.11 and will be omitted.

THEOREM 2.13. *Let* $1 \leqq p \leqq n$ *and* $t \leqq x_1 < \cdots < x_{p+1} < r_{n-p,1,\cdots,1}(t)$. *Assume that equation* (1.1) *satisfies* (A), (B) *and* (C), *and let* $C_{1,l} \in R^1$, $l = 1, \cdots, n-p$ *and* $C_{j,1} \in R^1$, $j = 2, \cdots, p+1$. *If*

$S(z;\, x_1^{n-1}, \hat{x}_k) = R^1$ *for* $k = 2, \cdots, p+1$ *for every solution* $z(x)$ *of* (1.1) *such that* $z^{(l-1)}(x_1) = C_{1,l}$, $l = 1, \cdots, n-p$, *then there exists a unique solution* $y(x)$ *of* (1.1) *such that*

$$y^{(1)}(x_1) = C_{1,l},\ l = 1, \cdots, n-p,$$

$$y(x_j) = C_{j,1}, j = 2, \cdots, p+1.$$

PROOF. By Remark 2.10 we have $r_{n-p,1\cdots 1}(t) \leqq r_{n-p+1,1\cdots 1}(t) \leqq \cdots \leqq r_{n-1,1}(t)$, hence, by Theorem 2.11 and Remark 2.12,

$$\begin{aligned} &S(z;\, x_1^{n-p}, x_2, \cdots, \hat{x}_k, \cdots, x_{p+1}) \\ &\quad = S(z;\, x_1^{n-p+1}, x_3, \cdots, \hat{x}_k, \cdots, x_{p+1}) \\ &\quad\ \ \vdots \\ &\quad = S(z;\, x_1^{n-p+k-2}, \hat{x}_k, \cdots, x_{p+1}) \\ &\quad = S(z;\, x_1^{n-p+k-1}, \hat{x}_k, x_{k+2}, \cdots, x_{p+1}) \\ &\quad\ \ \vdots \\ &\quad = S(z;\, x_1^{n-1}, \hat{x}_k) = R^1, \end{aligned}$$

for $2 \leqq k \leqq p+1$ and for every solution $z(x)$ of (1.1) such that $z^{(l-1)}(x_1) = C_{1,l}$, $l = 1, \cdots, n-p$.

Let $P(q)$, $2 \leqq q \leqq p$, be the proposition:

There exists a solution $y_q(x)$ of (1.1) such that

$$\begin{aligned} &y_q^{(l-1)}(x_1) = C_{1,l}, && l = 1, \cdots, n-p, \\ &y_q(x_j) = C_{j,1}, && j = 2, \cdots, q, \\ &y_q(x_j) = y_{q-1}(x_j), && j = q+1, \cdots, p+1, \end{aligned}$$

$$\text{and } S(y_q;\, x_1^{n-p}, x_2, \cdots, \hat{x}_{q+1}, \cdots, x_{p+1}) = R^1.$$

We will show by induction on q that $P(q)$ is true for $q = 2, \cdots, p$.

Assume $q = 2$. Let $y_1(x)$ be a solution of (1.1) which satisfies $y_1^{(l-1)}(x_1) = C_{1,l}$, $l = 1, \cdots, n-p$. Since $S(y_1;\, x_1^{n-p}, \hat{x}_2, \cdots, x_{p+1}) = R^1$, there exists a solution $y_2(x)$ of (1.1) such that

$$\begin{aligned} &y_2^{(l-1)}(x_1) = y_1^{(l-1)}(x_1) = C_{1,l}, \ l = 1, \cdots, n-p, \\ &y_2(x_2) = C_{2,1}, \\ &y_2(x_j) = y_1(x_j), && j = 3, \cdots, p+1. \end{aligned}$$

Since $y_2^{(l-1)}(x_1) = C_{1,l}$, $l = 1, \cdots, n-p$, $S(y_2;\, x_1^{n-p}, x_2, \hat{x}_3, \cdots, x_{p+1}) = R^1$.

Assume that $2 < q \leqq p$ and that $P(k)$ is true for $2 \leqq k < q$. Since $S(y_{q-1};\, x_1^{n-p}, \cdots, \hat{x}_q, \cdots, x_{p+1}) = R^1$, there exists a solution $y_q(x)$ of (1.1) which satisfies $P(q)$. This completes the induction. Finally, since $S(y_p;\, x_1^{n-p}, \cdots, \hat{x}_{p+1}) = R^1$, there exists a solution $y_{p+1}(x)$ of (1.1) such that

$$y_{p+1}^{(l-1)}(x_1) = C_{1,l}, \quad l = 1, \cdots, n-p,$$
$$y_{p+1}(x_j) = C_{j,1}, \qquad j = 2, \cdots, p+1.$$

EXAMPLE 2.14. If f satisfies the hypotheses of Theorem 1.9 of [11], then it follows by Theorem 2.13 that, in any subinterval of $[a, b]$, the uniqueness of solutions of the $(n-p, 1, \cdots, 1)$-BVP (1.1), (1.2) on that subinterval implies the existence of such solutions.

THEOREM 2.15. *Let* $2 \leqq k \leqq m$ *and* $t \leqq x_1 < \cdots < x_m < r_{i_1 \cdots i_m}(t)$, *where* $i_k = 1$. *Assume that equation* (1.1) *satisfies* (A), (B), (C), *and that every* $(j_1, \cdots, j_m)$-BVP (1.1), (1.2), *where* $j_q = i_q$ *for* $q \neq k-1$ *and* $j_{k-1} = i_{k-1} - 1$ (*note that* $\sum_{k=1}^{m} j_k = n-1$), *has a solution. If* $r_{i_1 \cdots i_{k-1}-1,1,i_k, \cdots, i_m}(t) \geqq r_{i_1 \cdots i_m}(t)$, *then* $S(y;\, x_1^{i_1}, \cdots, \hat{x}_k, \cdots, x_m^{i_m}) = R^1$ for every solution $y(x)$ of (1.1).

PROOF. It follows from Theorem 2.7 that $S \equiv S(y;\, x_1^{i_1}, \cdots, \hat{x}_k, \cdots, x_m^{i_m})$ is an open interval. Let $S = (\lambda, \gamma)$. We show that $\gamma = +\infty$. The proof that $\lambda = -\infty$ is similar. Assume that $\gamma < +\infty$. Let $u(x)$ be a solution of (1.1) such that

$$u^{(l_j)}(x_j) = y^{(l_j)}(x_j),\ l_j = 0, \cdots, i_j - 1;\ j = 1, \cdots, m;\ j \neq k-1, k,$$
$$u^{(l_{k-1})}(x_{k-1}) = y^{(l_{k-1})}(x_{k-1}),\ l_{k-1} = 0, \cdots, i_{k-1} - 2,$$
$$u(x_k) = \mu, \text{ where } \mu \geqq \gamma.$$

Let $\gamma_n \in [y(x_k), \gamma)$ be such that $y(x_k) < \gamma_n < \gamma_{n+1}$ for each n, and $\lim_{n\to\infty} \gamma_n = \gamma$. Let $\{u_n(x)\}$ be the corresponding sequence of solutions of (1.1) such that

$$u_n^{(l_j)}(x_j) = y^{(l_j)}(x_j),\ l_j = 0, \cdots, i_j - 1;\ j = 1, \cdots, m;\ k \neq j,$$
$$u_n(x_k) = \gamma_n.$$

It follows that $y(x) < u_n(x) < u_{n+1}(x)$, for all $x \in (x_{k-1}, x_{k+1})$ (where $x_{m+1} = r_{i_1 \cdots i_m}(t)$ in case $k = m$), and that $\{u_n(x)\}$ cannot be uniformly bounded on $[x_{k-1}, x_{k+1}]$. For n sufficiently large there must be an $s_1 \in (x_{k-1}, x_k)$ and an $s_2 \in (x_k, x_{k+1})$ such that $u_n(s_1) = u(s_1)$ and $u_n(s_2) = u(s_2)$, which is a contradiction.

THEOREM 2.16. *Let* $t \leqq x_1 < x_2 < r_{pq}(t)$. *Assume that equation* (1.1) *satisfies* (A), (B), (C), *and that every* $(p-1, q)$-BVP (1.1), (1.2) *has a solution. If* $r_{pq}(t) \leqq \min\{r_{p,1,q-1}(t), r_{p,q-1,1}(t), r_{p-1,1,q-1,1}(t)\}$, then $S(y; x_1{}^p, \hat{x}_2{}^q) = R^1$ *for every solution* $y(x)$ *of* (1.1).

PROOF. Let $y(x)$ be a solution of (1.1), and set $S(y; x_1{}^p, \hat{x}_2{}^q) = (\lambda, \gamma)$. We will only show that $\gamma = +\infty$. *Let* γ_n be such that $y^{(q-1)}(x_2) < \gamma_n < \gamma_{n+1}$ for each n and $\lim_{n\to\infty}\gamma_n = \gamma$. Let $\{u_n(x)\}$ be the corresponding sequence of solutions of (1.1) such that for each n

$$u_n{}^{(l)}(x_1) = y^{(l)}(x_1), l = 0, \cdots, p-1,$$

$$u_n{}^{(l)}(x_2) = y^{(l)}(x_2), l = 0, \cdots, q-2,$$

$$u_n{}^{(q-1)}(x_2) = \gamma_n.$$

Let $u(x)$ be a solution of (1.1) such that

$$u^{(l)}(x_1) = y^{(l)}(x_1), l = 0, \cdots, p-2,$$

$$u^{(l)}(x_2) = y^{(l)}(x_2), l = 0, \cdots, q-2,$$

$$u^{(q-1)}(x_2) = \mu, \text{ where } \mu > \gamma.$$

We will assume that q is odd. The case where q is even is similar. Note that $u_{n+1}(x) - u_n(x)$ and $u_n(x) - y(x)$ have zeros of exact order $q-1$ at x_2 and $y^{(q-1)}(x_2) < u_n{}^{(q-1)}(x_2) < u_{n+1}{}^{(q-1)}(x_2)$. Since $r_{pq}(t) \leqq r_{p,1,q-1}(t)$, we have $y(x) < u_n(x) < u_{n+1}(x)$ for $x \in (x_1, x_2)$, and since $r_{pq}(t) \leqq r_{p,q-1,1}(t)$, we have $y(x) < u_n(x) < u_{n+1}(x)$ for $x \in (x_2, r_{pq}(t))$. Also since $u(x) - u_n(x)$ and $u(x) - y(x)$ have zeros of exact order $q-1$ at x_2 and $y^{(q-1)}(x_2) < u_n{}^{(q-1)}(x_2) < u^{(q-1)}(x_2)$, we have that $y(x) < u_n(x) < u(x)$ in a deleted neighborhood of x_2. The sequence of solutions $\{u_n(x)\}$ cannot be uniformly bounded on compact subsets of $[x_1, r_{pq}(t))$, hence, for n sufficiently large, there exist $s_1 \in (x_1, x_2)$ and $s_2 \in (x_2, r_{pq}(t))$ such that $u(s_1) = u_n(s_1)$ and $u(s_2) = u_n(s_2)$, which is a contradiction.

The following remark is a generalization of Theorem 2.16 and the proof is essentially the same.

REMARK 2.17. *Let* $t \leqq x_1 < \cdots < x_m < r_{i_1 \cdots i_m}(t)$, where $i_k \geqq 2$. Assume that equation (1.1) satisfies (A), (B) and (C), that $y(x)$ is a solution of (1.1), and that there exists a solution to every $(j_1, \cdots, j_m)$-BVP (1.1), (1.2), where $j_q = i_q$, $q \neq k$ and $j_{k-1} = i_{k-1} - 1$. If $r_{i_1 \cdots i_m}(t) \leqq \min\{r_{i_1 \cdots i_{k-1},1,i_k-1,\cdots,i_m}(t), r_{i_1 \cdots i_{k-1}-1,1,i_k-1,1,\cdots i_m}(t), r_{i_1 \cdots i_{k-1},i_k-1,1 \cdots i_m}(t)\}$, then $S(y; x_1{}^{i_1}, \cdots, \hat{x}_k{}^{i_k}, \cdots, x_m{}^{i_m}) = R^1$.

Corollary 2.18. *If in addition to the hypotheses of Theorem* 2.15, *where* $i_k = 1$, *or Remark* 2.17, *where* $i_k \geqq 2$, *we assume that there exists a solution to every* $(j_1, \cdots, j_m)$-BVP (1.1), (1.2), *where* $j_q = i_q$, $q \neq k$ *and* $j_k = i_k - 1$, *then every* $(i_1, \cdots, i_m)$*-BVP* (1.1), (1.2) *on the interval* $[t, r_{i_1 \cdots i_m}(t))$ *has a unique solution.*

Proof. We want to show that there is a solution $z(x)$ of the $(i_1, \cdots, i_m)$-BVP (1.1), (1.2). Let $y(x)$ be a solution of (1.1) such that

$$y^{(l_j-1)}(x_j) = C_{j,l_j}, l_j = 1, \cdots, i_j; j = 1, \cdots, m; j \neq k,$$

$$y^{(l-1)}(x_k) = C_{k,l_k}, l_k = 1, \cdots, i_k - 1.$$

By Theorem 2.15, if $i_k = 1$ or, by Remark 2.17, if $i_k \geqq 2$, we have $S(y; x_1^{i_1}, \cdots, \hat{x}_k^{i_k}, \cdots, x_m^{i_m}) = R^1$, hence there exists a solution $z(x)$ of (1.1) satisfying (1.2).

Example 2.19. Let $t \leqq x_1 < \cdots < x_m < r_{i_1 \cdots i_m}(t)$. Assume that equation (1.1) satisfies (A), (B), (C) and that $f(x, y, y', \cdots, y^{(n-2)}, 0) = 0$. If for some $2 \leqq k \leqq m$ we have that $r_{i_1 \cdots i_m}(t) \leqq r_{i_1 \cdots i_{k-1}-1, 1, i_k, \cdots i_m}(t)$, where $i_k = 1$ or $r_{i_1 \cdots i_m}(t) \leqq \min\{r_{i_1 \cdots i_{k-1}, 1, i_k - 1, \cdots, i_m}(t), r_{i_1 \cdots i_{k-1}, i_k-1, 1 \cdots i_m}(t), r_{i_1 \cdots i_{k-1}-1, 1, i_k-1, 1, \cdots, i_m}(t)\}$ where $i_k \geqq 2$, then there exists a (unique) solution to the $(i_1, \cdots, i_m)$-BVP (1.1), (1.2).

Proof. It is easy to see that every solution of the linear equation $y^{(n-1)} = 0$ is a solution of (1.1). It follows that there exists a solution to every $(j_1, \cdots, j_m)$-BVP (1.1), (1.2), where $j_q = i_q$, $q \neq k$ and $j_k = i_k - 1$. It then follows from Corollary 2.18 that every $(i_1, \cdots, i_m)$-BVP (1.1), (1.2) has a (unique) solution.

Theorem 2.20. *Let* $1 \leqq k \leqq m$ *and* $t \leqq x_1 < \cdots < x_m < r_{i_1 \cdots i_m}(t)$, *where* $i_k = 1$. *Assume there is a* $\delta > 0$ *such that* $\eta_1(\tau) > \tau + \delta$ *for* $\tau \in [t, x_{k+1}]$ (*where* $x_m < x_{m+1} < r_{i_1 \cdots i_m}(t)$ *in case* $k = m$). *If equation* (1.1) *satisfies* (A), (B), *and* (C), *then for any solution* $y(x)$ *of* (1.1), *we have that* $S(y; x_1^{i_1}, \cdots, x_{k-1}^{i_{k-1}}, \hat{s}, x_{k+1}^{i_{k+1}}, \cdots, x_m^{i_m}) = R^1$ *for all but finitely many* $s \in (x_{k-1}, x_{k+1})$.

Proof. Let $y(x)$ be a solution of (1.1). Suppose that there exist an infinite number of points $\{s_p\}$ such that $x_{k-1} < s_p < x_{k+1}$ and $S(y; x_1^{i_1}, \cdots, x_{k-1}, \hat{s}_p, x_{k+1}^{i_{k+1}}, \cdots, x_m^{i_m}) = (\lambda_p, \gamma_p) \neq R^1$. Then either $\lambda_p > -\infty$ for infinitely many p or $\gamma_p < +\infty$ for infinitely many p. We will consider only the case where $\gamma_p < +\infty$ for infinitely many p. Since $[x_{k-1}, x_{k+1}]$ is compact there exists a subsequence $\{s_{p_j}\}$ converging to a point $s_0 \in [x_{k-1}, x_{k+1}]$. Without loss of generality we will assume that $\lim_{p\to\infty} s_p = s_0$. If $s_0 = x_{k-1}$, then $s_0 \in [x_{k-1}, \eta_1(x_{k-1}))$. If $s_0 = x_{k+1}$, then $s_0 \in [x_{k+1} - \delta/2, \eta_1(x_{k+1} - \delta/2))$. If $s_0 \in (x_{k-1}, x_{k+1})$, then since the collection of open intervals $\{(\tau, \eta_1(\tau)) : \tau \in [x_{k-1}, x_{k+1}]\}$ is an open cover of the compact set $[s_0, x_{k+1}]$, it follows that $s_0 \in (\tau, \eta_1(\tau)) \subset [\tau, \eta_1(\tau))$ for some $\tau \in [x_{k-1}, x_{k+1}]$. In any of these cases there exists $N > 0$ and $\tau \in [x_{k-1}, x_{k+1}]$ such that $s_p \in [\tau, \eta_1(\tau))$ for $p \geqq N$. Let $\sigma_q = s_{p_q}$, $q = 1, \cdots, n$, be n of these points such that $\tau \leqq \sigma_1 < \cdots < \sigma_n < \eta_1(\tau)$. Since we are in an interval of disconjugacy, there exists ([1], [5]) a solution $v(x)$ of (1.1) such that $v(\sigma_q) = \gamma_{p_q}$, $q = 1, \cdots, n$. Since $x_1 < \cdots < x_m < r_{i_1 \cdots i_m}(t)$, it follows that $S(y; x_1^{i_1}, \cdots, \hat{x}_k^{i_k}, \cdots, x_m^{i_m}) = (\lambda, \gamma)$, $-\infty \leqq \lambda < \gamma \leqq \infty$, is an open interval. Let $u_p(x)$ be the solution of (1.1) such that

$$u_p^{(l_j)}(x_j) = y^{(l_j)}(x_j),\ l_j = 0, \cdots, i_j - 1; j = 1, \cdots, m; j \neq k,$$

$$u_p(x_k) = \mu_p,$$

where $y(x_k) < \mu_p < \mu_{p+1} < \gamma$, for $p = 1, 2, \cdots$. It follows that $u_p(x) < u_{p+1}(x)$, for $x \in (x_{k-1}, x_{k+1})$, for $p = 1, 2, \cdots$, and $u_p(\sigma_q) < \gamma_{p_q}$, for $q = 1, \cdots, n$. Furthermore $u_p(x)$ is bounded below by $y(x)$ on $[x_{k-1}, x_{k+1}]$, for each p. But $\{u_p(x)\}$ is not uniformly bounded on $[\sigma_q, \sigma_{q+1}]$, $q = 1, \cdots, n-1$. It follows that, for p sufficiently large, $u_p(x) - v(x)$ has at least n distinct zeros on $[\tau, \eta_1(\tau))$, which is a contradiction.

We will use the following lemmas in the proof of Theorem 2.23.

Lemma 2.21. *Assume that* $t < x_1 < \cdots < x_m < r_{n-m,1,\cdots,1}(t+)$. *If equation* (1.1) *satisfies* (A) *and* (B), *and* $u(x)$, $v(x)$ *are distinct solutions of* (1.1) *such that*

$$u^{(l)}(x_1) = v^{(l)}(x_1), \quad l = 0, \cdots, n - m - 1,$$

$$u(x_j) = v(x_j), \qquad j = 2, \cdots, m,$$

then $u(x) - v(x)$ *has a zero of exact order* $n - m$ *at* x_1, *a zero of exact odd order at* x_j, $j = 2, \cdots, m$, *and no other zeros in* $(x_1, r_{n-m,1\cdots1}(t+))$.

Proof. It is clear that $u(x) \neq v(x)$ for $x \neq x_j$, $j = 2, \cdots, m$, and $x \in (x_1, r_{n-m,1\cdots1}(t+))$. Since $r_{n-m+1,1\cdots1}(t+) \geqq r_{n-m,1\cdots1}(t+)$, the zero of $u(x) - v(x)$ at x_1 is of exact order $n - m$. Suppose that $u(x) - v(x)$ has an exact even ordered zero at x_k for some $k = 2, \cdots, m$. Without loss of generality we will assume that $u(x) - v(x) > 0$ for $x \in (x_{k-1}, x_{k+1})$, $x \neq x_k$, (where $x_{k+1} = r_{n-m,1\cdots1}(t+)$ in case $k = m$). By Corollary 2.5, given $\epsilon > 0$, we can pick $\delta > 0$ sufficiently small such that there exists a solution $w(x)$ of (1.1) satisfying

$$w^{(l)}(x_1) = u^{(l)}(x_1), \quad l = 0, \cdots, n-m-1,$$
$$w(x_j) = u(x_j), \qquad j = 2, \cdots, m; j \neq k,$$
$$w(x_k) = u(x_k) - \delta,$$

with $|w(x) - u(x)| < \epsilon$ for $x \in [x_1, b]$, where $x_m < b < r_{n-m,1\cdots 1}(t+)$. However, if ϵ is sufficiently small, then $v(x) - w(x)$ has a zero of order $n - m$ at x_1, zeros at x_j, $j = 2, \cdots, m$; $j \neq k$, and two odd ordered zeros in a deleted neighborhood of x_k. From this contradiction we conclude that $u(x) - v(x)$ can have no exact even ordered zero at x_j, $j = 2, \cdots, m$.

Lemma 2.22. *Let $t < x_1 < \cdots < x_{n-1} < r_{21\cdots 1}(t+)$, where $n \geqq 4$. If* (1.1) *satisfies* (A), (B), *and* (C), *and there is a $\tau \geqq t$ such that $x_j \in (\tau, \eta_1(\tau))$, $j = 1, \cdots, n-2$, then, for every solution $y(x)$ of* (1.1), *we have that* $S(y; x_1{}^2, x_2, \cdots, x_{n-2}, \hat{x}_{n-1}) = R^1$.

Proof. Let $y(x)$ be a solution of (1.1). The set $S \equiv S(y; x_1{}^2, x_2, \cdots, x_{n-2}, \hat{x}_{n-1})$ is an open interval by Theorem 2.7. Let $S = (\lambda, \gamma)$. We will show that $\gamma = +\infty$. The proof that $\lambda = -\infty$ is similar and will be omitted. Assume that $\gamma < +\infty$.

We first establish the existence of a solution $z(x)$ of (1.1) such that $z(x_1) = y(x_1)$ *and* $z(x_{n-1}) \geqq \gamma$. Let $u(x)$ be the solution of (1.1) such that

$$u(x_{n-1}) = \gamma,$$
$$u^{(l)}(x_{n-1}) = 0, l = 1, \cdots, n-1.$$

If $u(x_1) = y(x_1)$, we let $z(x) = u(x)$. If $u(x_1) < y(x_1)$, let $\tau < s_1 < \cdots < s_{n-2} < x_1 < \eta_1(\tau)$, or, if $u(x_1) > y(x_1)$, let $\tau < s_1 < \cdots < s_{n-3} < x_1 < s_{n-2} < \eta_1(\tau)$. There exists ([1], [5]) a solution $z(x)$ of (1.1) such that

$$z^{(l)}(s_1) = u^{(l)}(s_1), \qquad l = 0, 1,$$
$$z(s_j) = u(s_j), \qquad j = 2, \cdots, n-2,$$
$$z(x_1) = y(x_1).$$

Since the points $s_1, \cdots, s_{n-2}$ are within an interval of disconjugacy, it follows that $z(x) - u(x)$ must have a zero of exact order 2 at s_1 and simple zeros at s_j, $j = 2, \cdots, n-2$. Furthermore, $z(x) - u(x)$ has no other zeros in the interval $(s_1, r_{21\cdots 1}(t+))$. In either case, $u(x_1) > y(x_1)$ or $u(x_1) < y(x_1)$; it follows that $z(x) > u(x)$ for all $x \in (s_{n-2}, r_{21\cdots 1}(t+))$. This establishes the existence of the solution $z(x)$ of (1.1) such that $z(x_1) = y(x_1)$ and $z(x_{n-1}) \geqq \gamma$.

Let $\{\gamma_p\}$ be a sequence of real numbers such that $y(x_{n-1}) < \gamma_p < \gamma_{p+1} < \gamma$ for each $p > 0$, and $\lim_{p\to\infty}\gamma_o = \gamma$. Let $\{u_p(x)\}$ be the corresponding sequence of solutions of (1.1) such that

$$u_p^{(l)}(x_1) = y^{(l)}(x_1), \qquad l = 0, 1,$$
$$u_p(x_j) = y(x_j), \qquad j = 2, \cdots, n-2,$$
$$u_p(x_{n-1}) = \gamma_p.$$

Since x_j, $j = 1, \cdots, n-2$, are points in $(\tau, \eta_1(\tau))$, $u_p(x) - y(x)$ has a zero of exact order 2 at x_1 and simple zeros at x_j, $j = 2, \cdots, n-2$. We have that $y(x) \leqq u_p(x) \leqq u_{p+1}(x)$ for $x \in [x_{n-j}, x_{n-j+1}]$, $j = 2, 4, \cdots, n-1$ or $n-2$, whichever is even. Also $u_{p+1}(x) \leqq u_p(x) \leqq y(x)$, for $x \in [x_{n-j}, x_{n-j+1}]$, for $j = 3, 5, \cdots, n-1$ or $n-2$, whichever is odd. Also note that, for n odd, we have $y(x) \leqq u_p(x) \leqq u_{p+1}(x)$, for $x \in (\tau, x_1)$, and, for n even, we have $u_{p+1}(x) \leqq u_p(x) \leqq y(x)$, for $x \in (\tau, x_1)$. Furthermore, $\{u_p(x)\}$ cannot be uniformly bounded on any of the intervals (τ, x_1), (x_j, x_{j+1}), $j = 1, \cdots, n-2$. It follows that for p sufficiently large there exist ξ_p^j, $j = 2, \cdots, n-1$, such that $u_p(\xi_p^j) = z(\xi_p^j)$, $j = 2, \cdots, n-1$, and $\lim_p \xi_p^j = x_j$, $j = 2, \cdots, n-1$. It also follows that $z'(x_1) \neq y'(x_1)$. But then, for p sufficiently large, there exists $\xi_p^1 \in (\tau, x_2)$, $\xi_p^1 \neq x_1$, such that $u_p(\xi_p^1) = z(\xi_p^1)$ and $\lim_p \xi_p^1 = x_1$. By renumbering the sequences $\{u_p(x)\}$, if necessary, we will assume without loss of generality that the ξ_p^j exist for each $j = 1, \cdots, n-1$ and each $p > 0$.

We will consider here only the case $z'(x_1) > y'(x_1)$ and assume that n is even. The other three cases are handled similarly. For n even we have $u_{p+1}(x) < u_p(x) < y(x)$ for $x \in (\tau, x_1)$ and $u_p''(x_1) < y''(x_1)$. We may assume that $\xi_p^1 \in (\tau, x_1)$ for each p and that $y(x) > u_p(x) > z(x)$ for $x \in (\xi_p^1, x_1)$. By Rolle's Theorem there exists $\pi_p \in (\xi_p^1, x_1)$ for each p such that $z'(\pi_p) = u_p'(\pi_p)$. Since $\lim_{p\to\infty}\pi_p = x_1$ and $|z(\pi_p) - u_p(\pi_p)| < |z(\pi_p) - y(\pi_p)|$ for each p, we have by continuity that $\lim_{p\to\infty}|z(\pi_p) - u_p(\pi_p)| = |z(x_1) - y(x_1)| = 0$. But then by Corollary 2.5, it follows that $u_p(x)$ converges uniformly to $z(x)$ on compact subintervals of $[t, r_{21\cdots 1}(t+))$, which is a contradiction.

THEOREM 2.23. *Let $t < x_1 < x_{n-1} < r_{21\cdots 1}(t+)$, where $n \geqq 4$. If equation* (1.1) *satisfies* (A), (B), *and* (C), *and if there is a $\tau \geqq t$ such that $x_1 \in (\tau, \eta_1(\tau))$, then for every solution $y(x)$ of* (1.1), *we have* $S(y; x_1^2, x_2, \cdots, \hat{x}_k, \cdots, x_{n-1}) = R^1$, for $k = 2, \cdots, n-1$.

PROOF. Let $y(x)$ be a solution of (1.1). The set $S \equiv S(y; x_1^2, x_2, \cdots, \hat{x}_k, \cdots, x_{n-1})$ is an open interval by Theorem 2.7. Let $S = (\lambda, \gamma)$. We will show only that $\gamma = +\infty$. Assume that $\gamma < +\infty$.

Let $\{\gamma_p\}$ be a sequence of real numbers such that $y(x_k) < \gamma_p < \gamma_{p+1} < \gamma$, for each $p > 0$, and $\lim_{p\to\infty}\gamma_p = \gamma$. Let $\{u_p(x)\}$ be the corresponding sequence of solutions of (1.1) such that

$$u_p^{(l)}(x_1) = y^{(l)}(x_1), \qquad l = 0, 1,$$
$$u_p(x_j) = y(x_j), \qquad j = 2, \cdots, n-1; j \neq k,$$
$$u_p(x_k) = \gamma_p.$$

By Lemma 2.21, $u_p(x) - y(x)$ has a zero of exact order 2 at x_1, exact odd ordered zeros at x_j, $j = 2, \cdots, n-1$; $j \neq k$, and no other zeros in $(x_1, r_{21\cdots 1}(t+))$. We now consider only the case where k is even. Then $y(x) \leqq u_p(x) \leqq u_{p+1}(x)$ for $x \in [x_{k-1}, x_{k+1}]$ (where $x_n \in (x_{n-1}, r_{21\cdots 1}(t+))$ in case $k = n-1$) and also for $x \in [x_j, x_{j+1}]$, $j = 1, 3, \cdots, k-3, k+2, k+4, \cdots, n-1$ or $n-2$, whichever is even. Also $u_{p+1}(x) \leqq u_p(x) \leqq y(x)$ for $x \in [x_j, x_{j+1}]$, $j = 2, 4, \cdots, k-2, k+1, k+3, \cdots, n-1$ or $n-2$, whichever is odd.

By Lemma 2.22 there exists a solution $v(x)$ of (1.1) such that

$$v^{(l)}(x_1) = y^{(l)}(x_1), l = 0, 1,$$
$$v(x_k) = \gamma.$$

But then for p sufficiently large it follows that $u_p(x) - v(x)$ has a $(2, 1, \cdots, 1)$-distribution of zeros on $[t, r_{21\cdots 1}(t+))$, which is a contradiction.

The following remarks can be proved similar to Lemma 2.22.

REMARK 2.24. Let $n = 3$ and $t < x_1 < x_2 < r_{21}(t+)$. Assume that equation (1.1) satisfies (A), (B) and (C), and that there is a $\tau \geqq t$ such that $x_1 \in (\tau, \eta_1(\tau))$. Further assume that there exists a solution $z(x)$ to every $(1, 1)$-BVP (1.1), (1.2), for $t < x_1 < x_2 < r_{21}(t+)$. Then for every solution $y(x)$ of (1.1) we have $S(y; x_1{}^2, \hat{x}_2) = R^1$.

REMARK 2.25. If $n \geqq 4$ and $t < x_1 < \cdots < x_{n-1} < r_{21\cdots 1}(t+)$, then the phrase "there is a $\tau \geqq t$ such that $x_1 \in (\tau, \eta_1(\tau))$" can be replaced by "there exists a solution $v(x)$ to every $(2, 1)$-BVP (1.1), (1.2), for $t < x_1 < x_2 < r_{21\cdots 1}(t+)$."

Theorems similar to Theorem 2.23 and Remark 2.24 can be proved to get the existence of (unique) solutions to the $(1, \cdots, 1, 2)$-BVP (1.1), (1.2), when $t < x_1 < \cdots < x_{n-1} < r_{1\cdots 12}(t+)$, or, in the case $n = 3$, when $t \leqq x_1 < x_2 < r_{12}(t)$.

Thus far we have established existence theorems for the $(i_1, \cdots, i_m)$-BVP (1.1), (1.2) for the third order case, much of the fourth order case and certain n-th order cases. The remainder of this chapter will concern other fourth order cases.

THEOREM 2.26. *Let $t < x_1 < x_2 < x_3 < r_{121}(t+)$, and assume there is a $\tau \geqq t$ such that $\tau < x_1 < x_2 < \eta_1(\tau)$. If equation* (1.1) *satisfies* (A), (B) *and* (C), *then for every solution $y(x)$ of* (1.1) *we have that* $S(y; \hat{x}_1, x_2{}^2, x_3) = S(y; x_1, x_2{}^2, \hat{x}_3) = R^1$.

PROOF. Let $y(x)$ be a solution of (1.1). We first show that $S(y; x_1, x_2{}^2, \hat{x}_3) = R^1$. Let $S(y; x_1, x_2{}^2, \hat{x}_3) = (\lambda, \gamma)$. We will only show that $\gamma = +\infty$. Suppose $\gamma < +\infty$. Let γ_n be a sequence of real numbers such that $y(x_3) < \gamma_n < \gamma_{n+1} < \gamma$ and $\lim_{n\to\infty}\gamma_n = \gamma$. Let $\{u_n(x)\}$ be the corresponding sequence of solutions of (1.1) such that

$$u_n(x_1) = y(x_1)$$
$$u_n{}^{(l)}(x_2) = y^{(l)}(x_2), l = 0, 1$$
$$u_n(x_3) = \gamma_n.$$

It is clear that $y(x) < u_n(x) < u_{n+1}(x)$ for $x \in (x_1, x_2) \cup (x_2, r_{121}(t+))$. Since $\tau < x_1 < x_2 < \eta_1(\tau)$, it follows that $u_n(x) - y(x)$ has a zero of exact order 2 at x_2, a simple zero at x_1, and no other zeros in $(\tau, \eta_1(\tau))$, for each $n > 0$. By a method similar to the method used in the proof of Lemma 2.22, there exists a solution $u(x)$ of (1.1) such that $u(x_2) = y(x_2)$ and $u(x_3) \geqq \gamma$. The sequence of solutions $\{u_n(x)\}$ is bounded above by $y(x)$ on $[\tau, x_1]$ and below by $y(x)$ on $[x_1, x_3]$. Furthermore, $\{u_n(x)\}$ is not uniformly bounded on $[\tau, x_1]$, $[x_1, x_2]$ or $[x_2, x_3]$. It follows that $u'(x_2) \neq y'(x_2)$. Using arguments as in the proof of Lemma 2.22, it can be shown that $u_n(x)$ converges uniformly to $u(x)$ on compact subintervals of $(t, r_{121}(t+))$, which is a contradiction. Hence we conclude that $S(y; x_1, x_2{}^2, \hat{x}_3) = R^1$.

To see that $S(y; \hat{x}_1, x_2{}^2, x_3) = R^1$, we let $\{\gamma_n\}$ be a sequence of real numbers such that $y(x_1) < \gamma_n < \gamma_{n+1} < \gamma$ and $\lim_{n\to\infty}\gamma_n = \gamma$. Let $\{u_n(x)\}$ be the corresponding sequence of solutions of (1.1) such that

$$u_n(x_1) = \gamma_n$$
$$u_n{}^{(l)}(x_2) = y^{(l)}(x_2), l = 0, 1$$
$$u_n(x_3) = y(x_3).$$

Let $\mu \in (\tau, x_1)$. Since $\tau < \mu < x_1 < x_2 < \eta_1(\tau)$, there exists ([7]) a solution $u(x)$ of (1.1) such that

$$u(\mu) = y(\mu)$$
$$u(x_1) = \gamma$$
$$u^{(l)}(x_2) = y^{(l)}(x_2), l = 0, 1.$$

It follows that $u(\mu) < u_n(\mu)$ and $u(x_1) > u_n(x_1)$, hence there exists $s_n{}^1 \in (\mu, x_1)$ such that $u_n(s_n{}^1) = u(s_n{}^1)$, for each n. Furthermore, $u(x) - u_n(x)$ has a zero of exact order 2 at x_2. If $u''(x_2) < u_n''(x_2)$, then $u(x) < u_n(x)$ in a deleted right neighborhood of x_2. Hence, since $u(x) > y(x)$ for $x \in (x_2, r_{121}(t+))$ and $u_n(x_3) = y(x_3)$, it follows by continuity that there exists $s_n{}^3 \in (x_2, x_3)$ such that $u(s_n{}^3) = u_n(s_n{}^3)$, which is a contradiction. Hence we may assume $u''(x_2) > u_n''(x_2)$. But, then, for n sufficiently large, it follows that $u(x) - u_n(x)$ has a $(1, 2, 1)$-distribution of zeros on $(t, r_{121}(t+))$, which is a contradiction.

The proof of the following theorem parallels that of Theorem 2.26 and will be omitted.

THEOREM 2.27. *Let $t \leqq x_1 < x_2 < x_3 < r_{121}(t)$, and assume that there is a $\tau \geqq t$ such that $\tau < x_2 < x_3 < \eta_1(\tau)$. If equation* (1.1) *satisfies* (A), (B), *and* (C), *then, for every solution $y(x)$ of* (1.1), *we have that* $S(y; \hat{x}_1, x_2{}^2, x_3) = S(y; x_1, x_2{}^2, \hat{x}_3) = R^1$.

BIBLIOGRAPHY

1. P. Hartman, *On n-parameter families and interpolation problems for nonlinear differential equations,* Trans. Amer. Math. Soc. **154** (1971), 201–226.

2. L. K. Jackson, *Uniqueness of solutions of boundary value problems for ordinary differential equations,* SIAM J. on Applied Math. **24** (1973), 535–538.

3. ———, Lecture notes given at the Univ. of Utah 1971–72.

4. L. K. Jackson and K. Schrader, *Existence and uniqueness of solutions of boundary value problems for third order differential equations,* Journal of Differential Equations 9 (1971), 46–54.

5. G. Klaasen, *Existence theorems for boundary value problems of nth order differential equations,* The Rocky Mountain J. of Math. 3 (1973), 457–472.

6. A. Lasoto and Z. Opial, *On the existence and uniqueness of solutions of a boundary value problem for an ordinary second-order differential equation,* Colloquium Math. **18** (1967), 1–5.

7. D. Peterson, *Uniqueness , existence, and comparison theorems for ordinary differential equations,* Ph.D. thesis, Univ. of Nebraska, 1973.

8. J. T. Schwartz, *Nonlinear Functional Analysis,* Gordon and Breach Science Publishers, New York, 1969.

9. J. Spencer, *Boundary value functions for nonlinear differential equations,* Ph.D. thesis, Univ. of Nebraska, 1973.

10. ———, *Boundary value functions for nonlinear differential equations,* Journal of Differential Equations (submitted).

11. D. Sukup, *Boundary value problems for nonlinear differential equations,* Ph.D. thesis, Univ. of Nebraska, 1974.

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