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RIEMANN'S FUNCTIONAL EQUATION

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It is well known that $\zeta(s)$, the Riemann Zeta function, satisfies the functional equation

$$(1) \qquad \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

In 1921, it was shown by Hamburger (See [2]) that $\zeta(s)$, as a member of a wide class of ordinary Dirichlet series, could be characterized by equation (1). Hamburger considered, in fact, a more general problem, the solution of the functional equation

$$(2) \qquad \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)f(s) = \pi^{-(1-s)/2}\,\Gamma\left(\frac{1-s}{2}\right)g(1-s),$$

where "solution" here (and for the balance of the paper) refers to a pair of Dirichlet series, $(f(s), g(s))$ satisfying equation (2). Hamburger imposed conditions on $f(s)$ and $g(s)$ which, together with the fact that they satisfy equation (2), necessitated that they satisfy $f(s) = g(s) = c\zeta(s)$.

Subsequently, other researchers have found different sets of conditions, which, when imposed on the solutions of (2), have again led to the same conclusion that $f(s) = g(s) = c\zeta(s)$. (See [1] and [5].) For example, a corollary to a theorem in [5] is the

THEOREM 1. *Let* $f(s) = \sum_{j=1}^{\infty} a_j\mu_j^{-s} = \prod_{j=1}^{\infty} (1-\pi_j^{-s})^{-1}$, $1 < \pi_1 \leqq \pi_2 \leqq \cdots, \pi_j \to \infty$, *be a general Dirichlet series, with an Euler product representation, which converges for* $\mathrm{Re}(s) > 1$. *Suppose that* $f(s) = E(s)(s-1)^{-1}$, *where* $E(s)$ *is an entire function of finite order such that* $E(1) = 1$ *and* $E(0) = 1/2$. *Further, let* $g(s) = \sum_{k=1}^{\infty} b_k\nu_k^{-s}$, $1 \leqq \nu_1 < \nu_2 < \cdots, \nu_k \to \infty$, *be a general Dirichlet series which converges absolutely for* $\mathrm{Re}(s) \geqq 2$. *Then, if* $f(s)$ *and* $g(s)$ *are related by equation* (2), *it follows that* $f(s) = g(s) = \zeta(s)$.

We mention that an essential difference between the hypotheses of Hamburger's theorem and the hypothesis of Theorem 1, is that Hamburger assumed that $\mu_j \in Z^+, j = 1, 2, \cdots$, whereas in Theorem 1, $f(s)$ may be a general Dirichlet series, albeit with an Euler product representation.

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In the present paper, we shall exhibit a class of pairs of general Dirichlet series, $(f(s), g(s))$, which cannot satisfy equation (2). Thus, we prove the

THEOREM 2. *Let* $f(s) = \sum_{j=1}^{\infty} a_j \mu_j^{-s}$, $0 < \mu_1 < \mu_2 < \cdots$, $\mu_j \to \infty$, *be a general Dirichlet series, absolutely convergent for* $\mathrm{Re}(s) > 1$, *and such that* $f(s) = E(s)(s-1)^{-1}$, *where* $E(s)$ *is an entire function of finite order. Further, let* $g(s) = \sum_{k=1}^{\infty} b_k \nu_k^{-s}$, $0 < \nu_1 < \nu_2 < \cdots$, $\nu_k \to \infty$, *be another general Dirichlet series, absolutely convergent for* $\mathrm{Re}(s) \geqq 2$, *such that* $\sum_{k=1}^{\infty} |b_k| < \infty$. *Then* $(f(s), g(s))$ *cannot be a solution of* (2).

We shall need a lemma, which states that if $f(s)$ and $g(s)$, satisfying the hypotheses of Theorem 2, are related by equation (2), then an analogue of the Poisson summation formula, given in equation (3), holds. The connection between the Riemann functional equation and the Poisson summation formula has been known for some time (see [3], [4]). In the following Lemma, we present a new proof of this result in a form sufficient for the proof of Theorem 2.

LEMMA. *Suppose that* $f(s)$ *and* $g(s)$ *are Dirichlet series, satisfying the hypotheses of Theorem* 2, *and related by equation* (2). *Let* $\varphi(x) \in L^1(-\infty, \infty)$ *be thrice continuously differentiable such that* $\varphi^{(3)} \in L^1(-\infty, \infty)$, $\varphi(x) = O(|x|^{-2})$, $\varphi'(x) = O(|x|^{-2})$, *as* $|x| \to \infty$. *Then*

$$\sum_{j=-\infty}^{\infty} a_j \varphi(\mu_j) = \sum_{k=-\infty}^{\infty} b_k \varphi(2\pi\nu_k), \tag{3}$$

where we have put $\mu_{-j} = -\mu_j$, $a_{-j} = a_j$, $\mu_0 = 0$, $a_0 = 2E(0)$ *and* $\nu_{-k} = -\nu_k$, $b_{-k} = b_k$, $\nu_0 = 0$, $b_0 = E(1)$.

PROOF OF LEMMA. Suppose that $f(s)$ and $g(s)$ satisfy the hypotheses of the lemma. Then an argument of Siegel (see [6]) shows that the following equation

$$E(0) + \sum_{j=1}^{\infty} a_j \, e^{-2\pi\mu_j\tau} = \frac{E(1)}{2\pi\tau} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{b_k \tau}{\nu_k^2 + \tau^2} \tag{4}$$

holds when $\mathrm{Re}(\tau) > 0$. We now deduce (3) from (4). In equation (4), put $\tau = \delta + iu$, $\delta > 0$, multiply the result by $(1 - |u|/T) \cos 2\pi x u$, $T > 0$, and integrate the product over $-T < u < T$, obtaining

$$\frac{T}{2} \sum_{j=-\infty}^{\infty} a_j K((x - \mu_j)T) = \frac{1}{2} \sum_{|\nu_k| \leqq T} b_k \left(1 - \frac{\nu_k}{T} \right) e^{2\pi i \nu_k x}, \tag{5}$$

where $K(x) = ((\sin \pi x)/\pi x)^2$. Now choose $\varphi(x)$ satisfying the hypotheses of the lemma. Multiply equation (5) by $\varphi(x)$ and integrate over $-\infty < x < \infty$:

$$(6)\ \frac{T}{2}\int_{-\infty}^{\infty}\varphi(x)\sum_{j=-\infty}^{\infty} a_j K((x-\mu_j)T)\,dx = \frac{1}{2}\sum_{|\nu_k|\leqq T} b_k\left(1-\frac{\nu_k}{T}\right)\hat{\varphi}(2\pi\nu_k).$$

Since $f(s)$ is absolutely convergent at $s = 2$, and since $|\varphi(x)| = O(|x|^{-2})$ as $|x| \to \infty$, it follows that

$$\int_{-\infty}^{\infty}|\varphi(x)|\sum_{j=-\infty}^{\infty}|a_j|K((x-\mu_j)T)\,dx < \infty,$$

for all $T > 0$. Therefore, we may interchange the order of integration and summation on the sinister side of (6). We obtain

$$\begin{aligned} &\frac{T}{2}\int_{-\infty}^{\infty}\varphi(x)\sum a_j K((x-\mu_j)T)\,dx \\ &= \frac{T}{2}\sum a_j \int_{-\infty}^{\infty}\varphi(x)K((x-\mu_j)T)\,dx \\ (7)\quad &= \frac{T}{2}\sum a_j \int_{-\infty}^{\infty}\varphi(\mu_j)K((x-\mu_j)T)\,dx \\ &\quad + \frac{T}{2}\sum a_j \int_{-\infty}^{\infty}(\varphi(x)-\varphi(\mu_j))K((x-\mu_j)T)\,dx \\ &= \frac{1}{2}\sum a_j\varphi(\mu_j) + \frac{T}{2}\sum a_j\int_{-\infty}^{\infty}(\varphi(x)-\varphi(\mu_j))K((x-\mu_j)T)\,dx, \end{aligned}$$

since $\int_{-\infty}^{\infty} T\,K((x-\mu_j)T)\,dx = 1$. We shall now show that the second expression on the dexter side of (7) goes to zero as $T \to \infty$. Thus, choose $\delta > 0$, and consider the integral

$$\int_{-\infty}^{\infty}(\varphi(x)-\varphi(\mu_j))T\,K((x-\mu_j)T)\,dx = \int_{|x-\mu_j|<\delta} + \int_{|x-\mu_j|\geqq\delta} = I_1 + I_2.$$

Regarding I_1, we have the estimate

$$|I_1| \leqq \int_{|x-\mu_j|<\delta}\delta|\varphi'(\xi)|T\,K((x-\mu_j)T)\,dx,\ \mu_j-\delta<\xi<\mu_j+\delta,$$

$$= O(\delta|\mu_j|^{-2})\int_{-\infty}^{\infty}T\,K((x-\mu_j)T)\,dx = O(\delta|\mu_j|^{-2}),$$

where the implied constant is independent of $\delta > 0$. Regarding I_2, we have

$$|I_2| \leqq \int_{|x-\mu_j|\geqq\delta} \frac{|\varphi(x)| + |\varphi(\mu_j)|}{T(x-\mu_j)^2}\, dx$$
$$= O\left(\frac{|\mu_j|^{-2}}{T\delta}\right).$$

Combining the estimates for I_1 and I_2, we see that

$$\frac{T}{2} \sum a_j \int (\varphi(x) - \varphi(\mu_j))K((x-\mu_j)T)\, dx = O\left(\delta + \frac{1}{T\delta}\right) \sum \frac{|a_j|}{|\mu_j|^2}.$$

Taking $\delta = T^{1/2}$, and letting $T \to \infty$, we see that the sinister side of (6) tends to the sinister side of (3) as $T \to \infty$.

On the other hand, since $\varphi^{(3)} \in L^1(-\infty, \infty)$, then for $t \neq 0$,

$$\hat{\varphi}(t) = \int_{-\infty}^{\infty} e^{itx}\varphi(x)\, dx = \frac{1}{(it)^3}\int_{-\infty}^{\infty} e^{itx}\varphi^{(3)}(x)\, dx$$
$$= O(|t|^{-3}), \qquad |t| \to \infty.$$

It follows from this fact and from $\sum_{k=1}^{\infty} |b_k|\, |\nu_k|^{-2} < \infty$, that the dexter side of (6) also tends to the dexter side of (3) as $T \to \infty$. This proves the lemma.

Proof of Theorem. Define functions $\varphi_j(x)$ on $\mu_j \leqq x \leqq \mu_{j+1}$ by

$$\varphi_j(x) = c_j \exp(-(x-\mu_j)^{-2} - (x-\mu_{j+1})^{-2}),$$

where the non-zero constants c_j satisfy

$$\sum_{j=-\infty}^{\infty} |c_j|\, |\mu_{j+1} - \mu_j| < \infty, \tag{8}$$

$$|c_j| \leqq \mu_{j+1}^{-2} + \mu_j^{-2}, \quad \text{if } j(j+1) \neq 0,$$
$$\leqq 1, \quad \text{if } j \text{ or } j+1 = 0. \tag{9}$$

Then define a function $\varphi(x)$ on the real line by $\varphi(x) = \varphi_j(x)$, $\mu_j \leqq x \leqq \mu_{j+1}$. It is clear that $\varphi(x)$ satisfies the hypotheses of the lemma (i.e., the conditions $\varphi(x)$, $\varphi^{(3)}(x) \in L^1(-\infty, \infty)$ follow from (8); and the growth conditions on $\varphi(x)$ and $\varphi'(x)$ follows from (9)) and that $\varphi(x)$ vanishes only at $x = \mu_j$, $j = 0, \pm 1, \pm 2, \cdots$. Also, as we have already shown, $|\hat{\varphi}(t)| = O(|t|^{-3})$, $|t| \to \infty$, so that $\hat{\varphi} \in L^1(-\infty, \infty)$.

Now for each real y, define a function $\varphi(x, y) = e^{2\pi i y x}\varphi(x)$. Then $\varphi(x, y)$ satisfies the hypotheses of the lemma, and for all real y, $\varphi(x, y)$ vanishes *only* when $\varphi(x)$ vanishes. Therefore,

$$0 = \sum_{j=-\infty}^{\infty} a_j \varphi(\mu_j, y) = \sum_{k=-\infty}^{\infty} b_k\, \hat{\varphi}(2\pi(\nu_k + y)) = h(y).$$

Since $\hat{\varphi} \in L^1$, and since $\sum_{k=-\infty}^{\infty} |b_k| < \infty$, we know that $h(y) \in L^1$. And it follows that for any real u,

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} e^{-2\pi i y u} h(y)\, dy \\ &= \sum_{k=-\infty}^{\infty} b_k \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iyu} \varphi(y + \nu_k)\, dy \\ &= \left(\sum b_k e^{i\nu_k u} \right) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iyu} \varphi(y)\, dy \\ &= \left(\sum b_k e^{i\nu_k u} \right) \varphi(u) \\ &= \theta(u)\varphi(u). \end{aligned}$$

Therefore, $\theta(u) \equiv 0$, since $\theta(u)$ is continuous. It follows that $b_k = 0$ for all k. Hence, by equation (2), and the uniqueness of analytic continuation, $f(s) \equiv 0$. But this contradicts $f(s) = E(s)(s-1)^{-1}$, $E(1) = 1$. This proves Theorem 2.

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