ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 5, Number 4, Fall 1975

ON A CRANK-NICOLSON SCHEME FOR NONLINEAR PARABOLIC EQUATIONS

A. REYNOLDS

1. **Introduction.** In this paper we consider a Crank-Nicolson type scheme for the problem:

$$u_t = f(t, x, u, u_x, u_{xx}) \text{ in } (0, T] \times (a, b) \tag{1.1}$$

$$u(0, x) = \varphi(x), u(t, a) = \varphi_0(t), \text{ and } u(t, b) = \varphi_1(t) \tag{1.2}$$

where $\varphi(a) = \varphi_0(0)$ and $\varphi(b) = \varphi_1(0)$.

In [11], the author was able to obtain a convergence theorem for a set of finite difference analogues of (1.1), (1.2) with $(0, T] \times (a, b)$ replaced by $[0, T] \times (a, b)$. For the Crank-Nicolson type scheme included among the methods in [11] a $O(\Delta t + h^2)$ convergence result was obtained. No method was given for solving the nonlinear system of difference equations.

For the Crank-Nicolson type scheme presented here, three improvements are possible. We obtain $O((\Delta t)^2 + h^2)$ convergence, we give a convergent iterative scheme for solving the nonlinear system of difference equations, and we obtain our results without assuming that the solution of (1.1) has continuous derivatives at $t = 0$.

Consideration of this iterative procedure yields an existence and uniqueness theorem for the solution of the nonlinear system of difference equations. This existence and uniqueness theorem is a slight improvement over the analogous result in [11], in that we obtain it by requiring that $f(t, x, z, p, r)$ satisfies certain Lipschitz conditions with respect to z, p, and r whereas in [11], we assumed f had continuous partial derivatives with respect to z, p, and r.

2. **Notation and Preliminary Results.** Let

$$h = \frac{b - a}{n + 1} \text{ and } \Delta t = T/m \tag{2.1}$$

where n and m are positive integers. Also let $x_i = a + ih$ for $i = 0, 1, \cdots, n + 1$ and $t_j = j\,\Delta t$ for $j = 0, 1, \cdots, m$.

For the remainder of the paper, we will suppose 2.1 defines a mesh on $[0, T] \times [a, b]$, and if $v(t, x)$ is any function defined on this mesh we denote $v(t_j, x_i)$ by $v_{i,j}$. For any such mesh function, we let

Received by the Editors November 23, 1973 and in revised form March 19, 1974.

$$D_{-t}v_{i,j} = \frac{v_{i,j} - v_{i,j-1}}{\Delta t}, \qquad D_0 v_{i,j} = \frac{v_{i+1,j} - v_{i-1,j}}{2h},$$

$$D_+ v_{i,j} = \frac{v_{i+1,j} - v_{i,j}}{h} \text{ and } D_- v_{i,j} = \frac{v_{i,j} - v_{i-1,j}}{h}.$$

With this notation note that

$$D_+D_-v_{i,j} = \frac{v_{i+1,j} + v_{i-1,j} - 2v_{i,j}}{h^2}.$$

Throughout this paper, we will denote

$$f\left(t_{j-1/2}; x_i; \frac{v_{i,j} + v_{i,j-1}}{2}; \frac{D_0 v_{i,j} + D_0 v_{i,j-1}}{2}, \frac{D_+D_-v_{i,j} + D_+D_-v_{i,j-1}}{2}\right)$$

by $f(v_{i,j})$ where v is any mesh function.

In problem 1.1, 1.2, we replace u_t, u_x and u_{xx} by the appropriate difference quotients and consider the discrete problem

(2.2) $\quad D_{-t}v_{i,j} = f(v_{i,j})$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$

(2.3) $\quad v_{0,j} = \varphi_0(t_j)$; $v_{n+1,j} = \varphi_1(t_j)$; and $v_{i,0} = \varphi(x_i)$

for $0 \leqq i \leqq n + 1$ and $0 \leqq j \leqq m$.

We assume there exist constants $\alpha \geqq 0$, $A \geqq 0$, $B \geqq 0$, C' and C such that

(2.4a) $\quad \alpha(\bar{r} - r) \leqq f(t, x, z, p, \bar{r}) - f(t, x, z, p, r) \leqq A(\bar{r} - r)$ for $\bar{r} \geqq r$,

(2.4b) $\quad |f(t, x, z, \bar{p}, r) - f(t, x, z, p, r)| \leqq B|\bar{p} - p|$, and

(2.4c) $\quad -C(\bar{z} - z) \leqq f(t, x, \bar{z}, p, r) - f(t, x, z, p, r) \leqq C'(\bar{z} - z)$ for $\bar{z} \geqq z$.

Note no restrictions are placed on the sign of C or C'.

We now make the following assumptions on h and Δt.

(2.5a) $$\frac{\Delta t C'}{2} \leqq 1,$$

(2.5b) $$\alpha - \frac{hB}{2} \geqq 0,$$

$$\text{(2.5c)} \qquad \frac{A\Delta t}{h^2} \leqq 1 - \frac{C\Delta t}{2}.$$

The proofs of the following theorem and lemma are omitted since the techniques of proof for both are the same as the analogous theorem and lemma in [**11**].

THEOREM 2.6. *Assume* 2.4a–c *and* 2.5a–c *holds. Let v and w be any two mesh functions. If $D_{-t}v_{i,j} - f(v_{i,j}) < D_{-t}w_{i,j} - f(w_{i,j})$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$, and if $v_{0,j} < w_{0,j}$, $v_{i,0} < w_{i,0}$ and $v_{n+1,j} < w_{n+1,j}$ for $0 \leqq i \leqq n+1$ and $0 \leqq j \leqq m$, then $v_{i,j} < w_{i,j}$ for $0 \leqq n+1$ and $0 \leqq j \leqq m$.*

2.7. For the remainder of the paper we assume that problem 1.1, 1.2 has a unique solution u such that u_{xtt}, u_{ttt}, u_{xxtt} and u_{xxxx} exist and are continuous and bounded in $(0, T] \times (a, b)$.

LEMMA 2.8. *Assume f satisfies* 2.4a,b *and* 2.5b, *and let $u(t, x)$ be the solution of* 1.1, 1.2. *Let ρ be a non-negative function defined on $[0, T]$, and define $z_{i,j} = u_{i,j} + \rho_j$ for $1 \leqq i \leqq n$, $0 \leqq j \leqq m$ and $z_{n+1,j} = u_{n+1,j}$, $z_{0,j} = u_{0,j}$ for $0 \leqq j \leqq m$ where $u_{i,j} = u(t_j, x_i)$. Then*

$$f\left(t_{j-1/2}, x_i, \frac{z_{i,j} + z_{i,j-1}}{2}, \frac{D_0 z_{i,j} + D_0 z_{i,j-1}}{2}, \frac{D_+D_- z_{i,j} + D_+D_- z_{i,j-1}}{2}\right.$$

$$\leqq f\left(t_{j-1/2}, x_i, \frac{u_{i,j} + \rho_j + u_{i,j-1} + \rho_{j-1}}{2}, \frac{D_0 u_{i,j} + D_0 u_{i,j-1}}{2},\right.$$

$$\left.\frac{D_+D_- u_{i,j} + D_+D_- u_{i,j-1}}{2}\right).$$

3. **A Convergence Theorem.** For $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$ let $\alpha_i(t_j)$ and $\beta_i(t_j)$ be defined by $u_x(t_j, x_i) = D_0 u_{i,j} + \alpha_i(t_j)$ and $u_{xx}(t_j, x_i) = D_+D_- u_{i,j} + \beta_i(t_j)$. By the differentiability assumption 2.7, α_i and β_i are $0\ (h^2)$. We now assume $\rho > 0$, and ω are functions defined on $[0, T]$ and $[0, T] \times R^3$ respectively such that

$$\text{(3.1a)} \qquad \frac{\rho_j - \rho_{j-1}}{\Delta t} > \frac{(\Delta t)^2}{12} \sup_{\substack{0 < t \leqq T \\ 1 \leqq i \leqq n}} |u_{ttt}(\bar{t}, x_i)|,$$

where

$$\rho_j = \rho(t_j),$$

$$\frac{\rho_j + \rho_{j-1}}{2\Delta t} \geqq \omega \Bigg(t_{j-1/2}, \quad \left| \frac{\rho_j + \rho_{j-1}}{2} + \frac{(\Delta t)^2}{8} u_{tt}(t^{(1)}, x_i) \right|,$$

$$\text{(3.1b)} \qquad \left| \frac{\alpha_i(t_j) + \alpha_i(t_{j-1})}{2} - \frac{(\Delta t)^2}{8} u_{xtt}(t^{(2)}, x_i) \right|,$$

$$\left| \frac{\beta_i(t_j) + \beta_i(t_{j-1})}{2} - \frac{(\Delta t)^2}{8} u_{xxtt}(t^{(3)}, x_i) \right| \Bigg)$$

for any $t^{(1)}$, $t^{(2)}$ and $t^{(3)}$ in (t_{j-1}, t_j), and

$$\text{(3.1c)} \qquad f(t, x, \bar{z}, \bar{p}, \bar{r}) - f(t, x, z, p, r) \leqq \omega(t, |\bar{z} - z|, |\bar{p} - p|, |\bar{r} - r|).$$

THEOREM 3.2. *Suppose* 2.4a–c, 2.5a–c, 2.7 *and* 3.1a–c *hold. If u is the solution of* 1.1, 1.2 *and $w_{i,j}$ is the solution of* 2.2, 2.3, *then*

$$\sup_{0 \leqq i \leqq n+1} |u(t_j, x_i) - w_{i,j}| \leqq \rho(t_j) \text{ for } 0 \leqq j \leqq m.$$

PROOF. With $u_{i,j} = u(t_j, x_i)$ we have

$$D_{-t}(u_{i,j} + \rho_j) = \frac{u_{i,j} - u_{i,j-1}}{\Delta t} + \frac{\rho_j - \rho_{j-1}}{\Delta t}$$

for $1 \leqq i \leqq n$ and $1 \leqq j \leqq m$.

Introducing the notation $z = (t_{j-1/2}, x_i)$ and using Taylor's Theorem and the Intermediate Value Theorem for continuous functions, we obtain:

$$\text{(3.2.1)} \qquad D_{-t}u_{i,j} = u_t(t_{j-1/2}, x_i) + \frac{(\Delta t)^2}{24} u_{ttt}(\bar{t}, x_i)$$

where $t_{j-1} < \bar{t} < t_j$. Now since u is the solution of problem 1.1, 1.2, by 3.1a and 3.2.1 we have:

$$\text{(3.2.2)} \qquad D_{-t}(u_{i,j} + \rho_j) > f(z, \ u(z), \ u_x(z), \ u_{xx}(z)) + \frac{\rho_j - \rho_{j-1}}{2\Delta t}.$$

Expanding $u(z)$, $u_x(z)$, and $u_{xx}(z)$ in appropriate Taylor Series, and letting $y = (u_{i,j} + u_{i,j-1})/2$, we obtain from 3.2.2 that:

$$D_{-t}(u_{i,j} + \rho_j) > f\Bigg(z, y - \frac{(\Delta t)^2}{8} u_{tt}(t^{(1)}, x_i), D_0 y + \frac{\alpha_i(t_j) + \alpha_i(t_{j-1})}{2}$$

$$\text{(3.2.3)} \qquad - \frac{(\Delta t)^2}{8} u_{xtt}(t^{(2)}, x_i), D_+D_- y + \frac{\beta_i(t_j) + \beta_i(t_{j-1})}{2}$$

$$- \frac{\Delta t^2}{8} u_{xxtt}(t^{(3)}, x_i) \Bigg) + \frac{\rho_j - \rho_{j-1}}{2\Delta t}.$$

Applying 3.1b, 3.1c and lemma 2.8 in order we obtain from 3.2.3 that

$$D_{-t}(u_{i,j} + \rho_j) > f\left(z, y + \frac{\rho_j + \rho_{j-1}}{2}\right.$$

(3.2.4)

$$\left. D_0\left(y + \frac{\rho_j + \rho_{j-1}}{2}\right), D_+D_-\left(y + \frac{\rho_j + \rho_{j-1}}{2}\right)\right).$$

Letting $v_{i,j} = u_{i,j} + \rho_j$, 3.2.4. becomes

$$D_{-t}v_{i,j} - f(v_{i,j}) > 0 \text{ for } 1 \leqq i \leqq n, 1 \leqq j \leqq n. \tag{3.2.5}$$

Let $B = \{(t, x) \mid x = a, x = b, \text{ or } t = 0\}$. Since $\rho > 0$ on $[0, T]$ and $u_{i,j} = w_{i,j}$ for (t_i, x_i) in B we have that

$$v_{i,j} = u_{i,j} + \rho_j > w_{i,j} \text{ on } B. \tag{3.2.6}$$

Now from 3.2.5, 3.2.6, and theorem 2.6, we have that $w_{i,j} < u_{i,j} + \rho_j$ for $0 \leqq i \leqq n + 1$ and $0 \leqq j \leqq m$. The same argument with $-\rho$ in place of ρ yields $u_{i,j} - \rho_j < w_{i,j}$ for $0 \leqq i \leqq n + 1$ and $0 \leqq j \leqq m$ and the theorem follows.

We now wish to construct functions ρ and ω which satisfy 3.1a–c such that $\rho > 0$ and ρ is $O(\Delta t^2 + h^2)$. Let $\bar{C} = \max\{C, C'\}$ and let

$$\omega(t, z, \rho, r) = \bar{C}z + Bp + Ar \tag{3.3}$$

where C, C', A and B are the Lipschitz constants appearing in 2.4a–c. We now show that if ω is defined by 3.3, then 3.1c is satisfied. We first write

$$f(t, x, \bar{z}, \bar{p}, \bar{r}) - f(t, x, z, p, r) = E_1 + E_2 + E_3 \tag{3.4}$$

where $E_1 = f(t, x, \bar{z}, \bar{p}, \bar{r}) - f(t, x, z, \bar{p}, \bar{r})$, $E_2 = f(t, x, z, \bar{p}, \bar{r}) - f(t, x, z, p, \bar{r})$, and $E_3 = f(t, x, z, p, \bar{r}) - f(t, x, z, p, r)$.

From 2.4c it follows easily that

$$E_1 \leqq \begin{cases} C'(\bar{z} - z) = C'|\bar{z} - z| & \text{for } \bar{z} \geqq z \\ -C(\bar{z} - z) = C|\bar{z} - z| & \text{for } \bar{z} < z, \end{cases}$$

hence we have that

$$E_1 \leqq \bar{C}|\bar{z} - z|. \tag{3.5a}$$

Similarly, using 2.4a and 2.4b respectively we conclude that

$$E_3 \leqq A|\bar{r} - r| \text{ and } E_2 \leqq B|\bar{p} - p|. \tag{3.5b}$$

Now from 3.4 and 3.5a–b, we have that 3.1c is satisfied.

We now show that 3.1a is satisfied for an appropriate ρ. Let $D = (0, T] \times (a, b)$ and define

$$K = \max\{\sup_D |u_{tt}(t,x)|, \sup_D |u_{xtt}(t,x)|, \sup_D |u_{xxtt}(t,x)|\},$$

$$\bar{\alpha} = \max_{1 \leqq i \leqq n} |\alpha_i|, \text{ and } \bar{\beta} = \max_{i \leqq i \leqq n} |\beta_i|.$$

From assumption 2.7 and the definition of α_i and β_i, we have that $\bar{\alpha}$ and $\bar{\beta}$ are $O(h^2)$ and hence there exists a constant $K' > 0$ such that $\bar{\alpha} \leqq K'h^2$ and $\bar{\beta} \leqq K'h^2$. Let $M = \max\{1, K, K', A, B, \bar{C}\}$ and define

$$\rho(t) = (\Delta t)^2 \exp(8M^2 t) + K'(A+B)h^2 \exp(8Mt). \tag{3.6}$$

Then

$$\begin{aligned} \frac{\rho_j - \rho_{j-1}}{\Delta t} &\geqq (\Delta t)^2 \exp(8M^2 t_{j-1}) \left(\frac{\exp(8M^2 \Delta t) - 1}{\Delta t} \right) \\ &\geqq (\Delta t)^2 \exp(8M^2 t_{j-1})(8M^2) \geqq (\Delta t)^2 8M^2 \\ &\geqq \frac{\Delta t^2}{12} \sup_D |u_{ttt}(t,x)|, \end{aligned}$$

so 3.1a is satisfied when ρ is defined by 3.6.

Letting

$$\rho^{(j)} = \frac{\rho_j + \rho_{j-1}}{2} + \frac{\Delta t^2}{4} u_{tt}(t^{(1)}, x_i),$$

$$\alpha^{(j)} = \frac{\alpha_i(t_j) + \alpha_i(t_{j-1})}{2} - \frac{\Delta t^2}{4} u_{xtt}(t^{(2)}, x_i), \text{ and}$$

$$\beta^{(j)} = \frac{\beta_i(t_j) + \beta_i(t_{j-1})}{2} - \frac{(\Delta t)^2}{4} u_{xxtt}(t^{(3)}, x_i),$$

we have

$$\begin{aligned} \omega(t_{j-1/2}, |\rho^{(j)}|, |\alpha^{(j)}|, |\beta^{(j)}|) &= \bar{C}|\rho^{(j)}| + B|\alpha^{(j)}| + A|\beta^{(j)}| \\ &\leqq \bar{C}\left(\frac{\rho_j + \rho_{j-1}}{2}\right) + B\bar{\alpha} + A\bar{\beta} + (\bar{C} + B + A)K \frac{(\Delta t)^2}{8} \\ &\leqq \frac{M}{2}(\rho_j + \rho_{j-1}) + M(A+B)K'h^2 \exp(8Mt_{j-1}) \\ &\quad + 3M^2 \frac{\Delta t^2}{8} \exp(8\,Mt_{j-1}). \end{aligned} \tag{3.7}$$

Now choose Δt sufficiently small such that $\exp(8M^2\,\Delta t) + 1 \leqq 6$, then we have that

(3.8)

$$(\rho_j + \rho_{j-1}) \leqq (\Delta t^2) \exp(8M^2 t_{j-1})\,(6) + K'(A + B)6h^2 \exp(8\,Mt_{j-1}).$$

Using (3.8) in (3.7) yields

$$\omega(t_{j-\frac{1}{2}}, |\rho^{(j)}|, |\alpha^{(j)}|, |\beta^{(j)}|) \leqq 4M^2(\Delta t)^2 \exp(8M^2 t_{j-1}). + 4M(A + B)K'h^2 \exp(8Mt_{j-1})$$

$$\leqq \Delta t \exp(8M^2 t_{j-1}) \left(\frac{\exp(8M^2\Delta t) - 1}{2\Delta t}\right) + (A + B)K'h^2 \exp(8Mt_{j-1}) \left(\frac{\exp(8M\Delta t) - 1}{2\Delta t}\right)$$

$$= \frac{\rho_j - \rho_{j-1}}{2\Delta t}.$$

Hence 3.1b is satisfied.

Since ρ is $O(\Delta t^2 + h^2)$, an application of theorem 3.2 yields

THEOREM 3.9. *Suppose* 2.4a–c *and* 2.5a–c *hold. Let* $u(t, x)$ *be the solution of* 1.1, 1.2 *and* $w_{i,j}$ *be the solution of* 2.1, 2.2. *Then*

$$\sup_{\substack{0 \leqq i \leqq n+1 \\ 0 \leqq j \leqq m}} |u(t_j, x_i) - w_{i,j}| = O(\Delta t^2 + h^2).$$

4. **Iterative Solution of the Discrete Problem.** In this section we show problem 2.2, 2.3 has a unique solution by developing a convergent iterative procedure for solving the system of difference equations. To show that 2.2, 2.3 has a unique solution, we need only show that if $v_{i,j-1}$ for $1 \leqq i \leqq n$ is known then the problem

(4.1) $$v_{i,j} = v_{i,j-1} + \Delta t f(v_{i,j}) \text{ for } i = 1, 2, \cdots, n$$

(4.2) $$v_{0,j} = \varphi_0(t_j), v_{n+1,j} = \varphi_1(t_j)$$

has a unique solution.

We note that $v_{i,j}$ is a solution of 4.1, 4.2 if and only if $v_{i,j}$ is a solution of

(4.1′) $$2v_{i,j} = v_{i,j} + v_{i,j-1} + \Delta t f(v_{i,j}) \text{ for } 1 \leqq i \leqq n$$

(4.2′) $$v_{0,j} = \varphi_0(t_j), v_{n+1,j} = \varphi_1(t_j).$$

Hence 4.1, 4.2 has a unique solution if and only if 4.1′, 4.2′ has a unique solution. We now define an iterative procedure for solving 4.1′, 4.2′. Let ξ_i^0 be arbitrary for $i = 1, 2, \cdots, n$. For each i, $1 \leqq i \leqq n$, let

$$2\xi_i^{\beta+1} = \xi_i^\beta + v_{i,j} + \Delta t f\left(t_{j-1/2}, x_i, \frac{\xi_i^\beta + v_{i,j-1}}{2},\right. \tag{4.3}$$
$$\left.\frac{D_0\xi_i^\beta + D_0 v_{i,j-1}}{2}, \frac{D_+D_-\xi_i^\beta + D_+D_-v_{i,j-1}}{2}\right)$$

for $\beta = 0, 1, 2, \cdots$ where

$$\xi_0^\beta = \varphi_0(t_j) \text{ and } \xi_{n+1}^\beta = \varphi_1(t_j) \text{ for all } \beta.$$

In 4.3 we have used the notation

$$D_0\xi_i^\beta = \frac{\xi_{i+1}^\beta - \xi_{i-1}^\beta}{2h} \text{ and } D_+D_-\xi_i^\beta = \frac{\xi_{i+1}^\beta + \xi_{i-1}^\beta - 2\xi_i^\beta}{h^2}.$$

We let $\xi^\beta = (\xi_1^\beta, \xi_2^\beta, \cdots \xi_n^\beta)^T$ for $\beta = 0, 1, 2, \cdots$, introduce simplifying notation, and use 2.4a–c to write algorithm 4.3 in a form from which we can deduce that $\|\xi^{\beta+1} - \xi^\beta\|_\infty < \|\xi^\beta - \xi^\beta\|_\infty$ and hence conclude that the iterative scheme converges. Let $z = (t_{j-1/2}, x_i)$ and $\sigma(\beta) = (\xi_i^\beta + v_{i,j-1})/2$. With this notation we can derive from 4.3 that

$$2(\xi_i^{\beta+1} - \xi_i^\beta) = \xi_i^\beta - \xi_i^{\beta-1} + \Delta t[f(z, \sigma(\beta), D_0\sigma(\beta),$$
$$D_+D_-\sigma(\beta)) - f(z, \sigma(\beta-1), D_0\sigma(\beta-1), D_+D_-\sigma(\beta-1))]$$

or

$$2(\xi_i^{\beta+1} - \xi_i^\beta) = \xi_i^\beta - \xi_i^{\beta-1} + \Delta t(E_1 + E_2 + E_3) \tag{4.4}$$

where

$$E_1 = f(z, \sigma(\beta), D_0\sigma(\beta), D_+D_-\sigma(\beta))$$
$$- f(z, \sigma(\beta-1), D_0\sigma(\beta), D_+D_-\sigma(\beta)),$$
$$E_2 = f(z, \sigma(\beta-1), D_0\sigma(\beta), D_+D_-\sigma(\beta))$$
$$- f(z, \sigma(\beta-1), D_0\sigma(\beta-1), D_+D_-\sigma(\beta)), \text{ and}$$
$$E_3 = f(z, \sigma(\beta-1), D_0\sigma(\beta-1), D_+D_-\sigma(\beta))$$
$$- f(z, \sigma(\beta-1), D_0\sigma(\beta-1), D_+D_-\sigma(\beta-1)).$$

If $\xi_i^\beta \geqq \xi_i^{\beta-1}$, then by 2.4c, we have

$$-C\frac{(\xi_i^\beta - \xi_i^{\beta-1})}{2} \leqq E_1 \leqq C'\frac{(\xi_i^\beta - \xi_i^{\beta-1})}{2}.$$

If $\xi_i^\beta < \xi_i^{\beta-1}$, then 2.4c implies that

$$C'\frac{(\xi_i^\beta - \xi_i^{\beta-1})}{2} \leqq E_1 \leqq -C\frac{(\xi_i^\beta - \xi_i^{\beta-1})}{2},$$

hence we have

$$E_1 = \begin{cases} (-C + \gamma_0(C' + C))(\xi_i^\beta - \xi_i^{\beta-1})/2 \text{ for } \xi_i^\beta \geqq \xi_i^{\beta-1} & (4.5a) \\ (C' + \gamma_0'(-C - C'))(\xi_i^\beta - \xi_i^{\beta-1})/2 \text{ for } \xi_i^\beta < \xi_i^{\beta-1} & (4.5b) \end{cases}$$

where γ_0 and γ_0' are in $[0, 1]$. Similarly using 2.4b and 2.4a respectively yields

$$E_2 = \begin{cases} (-B + 2\gamma_1 B)D_0 \left(\frac{\xi_i^\beta - \xi_i^{\beta-1}}{2}\right) \text{ for } \xi_i^\beta \geqq \xi_i^{\beta-1} & (4.6a) \\ (B - 2\gamma_1' B)D_0 \left(\frac{\xi_i^\beta - \xi_i^{\beta-1}}{2}\right) \text{ for } \xi_i^\beta < \xi_i^{\beta-1} & (4.6b) \end{cases}$$

and

$$E_3 = \begin{cases} (\alpha + \gamma_2(A - \alpha))D_+D_- \left(\frac{\xi_i^\beta - \xi_i^{\beta-1}}{2}\right) \text{ for } \xi_i^\beta \geqq \xi_i^{\beta-1} & (4.7a) \\ (A + \gamma_2'(\alpha - A))D_+D_- \left(\frac{\xi_i^\beta - \xi_i^{\beta-1}}{2}\right) \text{ for } \xi_i^\beta < \xi_i^{\beta-1} & (4.7b) \end{cases}$$

where $\gamma_1, \gamma_1', \gamma_2$, and γ_2' are in $[0, 1]$.

We now have eight possibilities for the sum $E_1 + E_2 + E_3$. For all eight we can show the iterative scheme 4.3 converges if we replace the assumption 2.5a by $\Delta t C'/2 < 1$. That is, we have the following result.

THEOREM 4.8. *Suppose* 2.4a–c, 2.4a–b *hold and let* $\Delta t C'/2 < 1$. *Then the iterative scheme given by* 4.3 *converges to a vector* ξ *which is the unique solution of* 4.1, 4.2.

PROOF. We prove the theorem only in the case where E_1, E_2, and E_3 are given by 4.5b, 4.6b, and 4.7b respectively. Then from 4.4 we have

$$2(\xi_i^{\beta+1} - \xi_i^\beta) = \Big(1 + \frac{\Delta t}{2}\Big[C' - \gamma_0'(C' + C) - \frac{2}{h^2}(A + \gamma_2'(\alpha - A))\Big]\Big)(\xi_i^\beta - \xi_i^{\beta-1}) +$$

$$+ \frac{\Delta t}{2}\left(\frac{A + \gamma_2'(\alpha - A)}{h^2} + \frac{B - 2\gamma_1' B}{2h}\right)(\xi_{i+1}^\beta - \xi_{i+1}^{\beta-1}) +$$

$$+ \frac{\Delta t}{2}\left(\frac{A + \gamma_2'(\alpha - A)}{h^2} - \frac{B + 2\gamma_1' B}{2h}\right)(\xi_{i-1}^\beta - \xi_{i-1}^{\beta-1}).$$

Hence we have that

$$
\begin{aligned}
2|\xi_i^{\beta+1} - \xi_i^{\beta}| \leqq \; & \left| 1 + \frac{\Delta t}{2}\Big(C' - \gamma_0'(C' + C) - \frac{2}{h^2}(A + \gamma_1'(\alpha - A))\Big)\right| |\xi_{i-1}^{\beta} - \xi_i^{\beta-1}| \\
& + \frac{\Delta t}{2} \left| \frac{A + \gamma_2'(\alpha - A)}{h^2} + \frac{B - 2\gamma_1' B}{2h}\right| |\xi_{i+1}^{\beta} - \xi_{i+1}^{\beta-1}| + \\
& + \frac{\Delta t}{2} \left| \frac{A + \gamma_2'(\alpha - A)}{h^2} - \frac{(B - 2\gamma_1' B)}{2h}\right| |\xi_{i-1}^{\beta} - \xi_{i-1}^{\beta-1}|.
\end{aligned} \tag{4.8.1}
$$

From 2.4a and 2.4c we see that $\alpha - A \leqq 0$ and $C' + C \geqq 0$ so using 2.5c we obtain:

$$
\begin{aligned}
& 1 + \frac{\Delta t}{2}(C' - \gamma_0'(C' + C)\, \frac{2}{h^2}\,(A + \gamma_1'(\alpha - A)) \geqq \\
& 1 + \frac{\Delta t}{2}(-C) - \frac{2}{h^2}A \geqq 0.
\end{aligned} \tag{4.9a}
$$

Similarly using 2.5b, it is easy to show that

$$
\frac{A + \gamma_2'(\alpha - A)}{h^2} + \frac{B - 2\gamma_1' B}{2h} \geqq 0 \tag{4.9b}
$$

and

$$
\frac{A + \gamma_2'(\alpha - A)}{h^2} - \frac{(B - 2\gamma_1' B)}{2h} \geqq 0. \tag{4.9c}
$$

Now 4.9a–c and 4.8.1 imply that:

$$
\begin{aligned}
2|\xi_i^{\beta+1} - \xi_i^{\beta}| & \leqq \left(1 + \frac{\Delta t}{2}(C' - \gamma_0'(C' + C))\right) \|\xi^{\beta} - \xi^{\beta-1}\|_\infty \\
& \leqq \left(1 + \frac{\Delta t}{2} C'\right) \|\xi^{\beta} - \xi^{\beta-1}\|_\infty
\end{aligned}
$$

or

$$
|\xi_i^{\beta+1} - \xi_i^{\beta}| \leqq \frac{1 + (\Delta t/2)C'}{2} \|\xi^{\beta} - \xi^{\beta-1}\|_\infty
$$

for $1 \leqq i \leqq n$ so that

$$
\|\xi^{\beta+1} - \xi^{\beta}\|_\infty \leqq \frac{1 + (\Delta t/2)C'}{2} \|\xi^{\beta} - \xi^{\beta-1}\|_\infty .
$$

As

$$\frac{1 + (\Delta t/2)C'}{2} < 1,$$

it follows in the standard way that the sequence ξ^β converges. Noting that the Lipschitz conditions 2.4a–c imply that $f(t, x, z, p, r)$ is a continuous function of (z, p, r) completes the proof.

REMARKS. If the parabolic equation we are solving is quasilinear, there exist finite difference methods which give rise to a linear system of difference equations at each time step. In particular the reader is referred to [4], [8] and [9].

CONCLUDING REMARKS. To obtain the results of this paper it is only necessary that 2.4a–c are satisfied for (z, p, r) in certain bounded subsets of R^3 rather than in all of R^3. A detailed explanation of this is given in [11].

ACKNOWLEDGMENTS. The author wishes to express his gratitude to his former advisor, Gerald Hedstrom, for his continued encouragement.

REFERENCES

1. L. Bers, *On mildly non-linear partial differential equations of elliptic type,* J. Research Natl. Bur. Standards **51** (1953), 229–236.

2. L. Collatz, *Numerical treatment of differential equations,* Springer, 1960.

3. J. Douglas, Jr., *A survey of numerical methods for parabolic differential equations, advances in computers,* Vol. 2, pp. 1–52, Academic Press, 1961.

4. J. Douglas, Jr. and B. F. Jones, *On predictor-corrector methods for non-linear parabolic equations,* J. Soc. Indust. Appl. Math., **11** (1963), 195–204.

5. R. Gorenflo, *Diskrete Diffusionsmodelle und monotone differenzenschemata fur parabolische differentialgleichungen,* Methoden und Verfahren der Mathematischen Physik Band 1, Bericht uber eine Tagung, Oberwolfach, 1969, 143–162.

6. ———, *Monotonic difference schemes for weakly coupled systems of parabolic differential equations,* Conf. on Numerical Solution of Differential Equations (Dundee 1969), Springer, 1969, pp. 160–167.

7. R. Krawczyk, *Uber Differenzenverfahren bei parabolischen Differentialgleichungen,* Arch. Rational Mech. Anal. **13** (1963), 81–121.

8. M. Lees, *Approximate solutions of parabolic equations,* J. Soc. Ind. Appl. Math 7 (1959), 167–183.

9. M. Lees, *A linear three-level difference scheme for quasilinear parabolic equations,* Math. Comp. **20** (1967), 516–522.

10. R. M. Redheffer, *Bemerkungenüber Differentialungleichungen bei abzählbaren Ausnahmemengen,* Numer. Math **9** (1966/67), 437–445.

11. A. Reynolds, *Convergent finite difference schemes for nonlinear parabolic equations,* Siam J. Num. Anal. **9** (1972), 523–533.

12. M. E. Rose, *On the integration of nonlinear parabolic equations by implicit difference methods,* Quart. Appl. Math. **14** (1956), 237–248.

13. W. Walter, *Wärmeleitung in Systemen mit mehreren Komponenten,* Numerische Mathematik. Differentialgleichungen. Approximation-theorie (Tagungen, Oberwolfach, 1966), 177-185 Birkhauser, Basel, 1968.
14. W. Walter, *Die Linienmethode bei nichtlinearen parabolischen Differentialgleichungen,* Numer. Math. 12 (1968), 307-321.

UNIVERSITY OF TULSA, TULSA, OKLAHOMA 74104