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SEVEN DIFFERENT PROOFS THAT L^∞/H^∞ IS NOT SEPARABLE

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Although one would have no difficulty in conjecturing that L^∞/H^∞ is not separable, there is no proof of this fact in general circulation. The space L^∞/H^∞ is interesting because it is isometrically isomorphic to the dual of H^1. In this didactic paper, we present seven different proofs that it is not separable. Their variety affords possible lines of generalization as well as the framing of related questions, and displays the state of the art of H^p spaces today. The first two proofs are the authors', and the remaining are due to B. A. Taylor, C. L. Fefferman, Joel Shapiro, A. Pelczynski, and J. Garnett, respectively. We thank them for permission to present their proofs here. It is likely that other proofs will be found. Indeed, we have been informed that H. S. Shapiro and A. L. Shields are preparing a joint manuscript containing a general result that implies our estimate on $\|\Delta - g\|_\infty$ in the first proof. At the end of the paper, we prove a related theorem to our title theorem.

An immediate corollary of the title theorem is that H^1 is not homeomorphic, as a topological space, to its second dual. In particular, H^1 is not reflexive, as was shown in [**9**, § 7] and subsequently in [**8**, § 2.11–2.12].

The first of the seven proofs is based on the theory of cluster sets, and explicitly exhibits in L^∞/H^∞ a collection of disjoint open balls which has the cardinality, c, of the continuum. Our second proof gives no information beyond the assertion of the theorem, and relies on the known non-reflexivity of H^1 and a Banach space lemma from [**5**]. For completeness, we include a condensed proof, based entirely on [**9**, § 7] that H^1 is not reflexive. Our third proof is due to B. A. Taylor, and uses the same construction as Proof 1, but shows directly why it works by means of the conjugate Poisson integral. The fourth proof is due to C. L. Fefferman, and uses the linear homeomorphism [**3**], [**4**] of the space of functions of bounded mean oscillation (BMO) onto the normed conjugate space of H^1. The fifth proof is due to Joel Shapiro, and uses interpolation sequences for the pair (H^1, ℓ^1). This proof seems the most likely to extend the statement that H^1 is not reflexive to more general domains than the unit disc, a problem that was

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mentioned in [8]. The sixth proof is due to A. Pelczynski and exhibits a subspace of H^1 that is isomorphic to ℓ^1. The final proof is due to J. Garnett and involves function algebra techniques.

As conventions, unless specified otherwise, we mean by L^∞, H^∞, and H^1, respectively, the spaces $L^\infty(T)$, $H^\infty(T)$, and $H^1(T)$, where $T = \{z : |z| = 1\}$. We let $D = \{z : |z| < 1\}$ and freely identify functions in $L^\infty(T)$ with their Poisson integrals, which are, of course, bounded harmonic functions in D.

PROOF 1. For $f \in L^\infty$, denote by (f) the coset $f + H^\infty$. For $0 \leqq \alpha \leqq 2\pi$, let

$$f_\alpha(e^{i\theta}) = \begin{cases} 1 & 0 \leqq \theta \leqq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that, in the norm of L^∞/H^∞, $\|(f_\alpha) - (f_\beta)\| \geqq 1/2$ when $\alpha \neq \beta$. Now for $\alpha < \beta$,

$$\Delta(e^{i\theta}) = f_\beta(e^{i\theta}) - f_\alpha(e^{i\theta}) = \begin{cases} 1 & \alpha < \theta \leqq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that for *any* function $\Delta \in L^\infty$ that is zero everywhere in an interval to the left of $e^{i\theta_0}$ (as viewed from inside the disc), and 1 everywhere in an interval to the right of $e^{i\theta_0}$, where these intervals have $e^{i\theta_0}$ as common endpoint, we have $\|\Delta - g\|_\infty \geqq 1/2$ for any $g \in H^\infty$. Without loss of generality, we take $\theta_0 = 0$. Suppose, by way of contradiction, that $\|\Delta - g\|_\infty \leqq \epsilon < 1/2$. We require some results from the theory of cluster sets, adapted to the case at hand, and we use [1] as reference. For the definition of the cluster set, let f be a function defined in D, with values in the complex plane $\mathcal{C}$, and let $z_0 = e^{i\theta_0} \in T$. The cluster set $C(f, z_0)$ of f at z_0 is defined as the set of points α of the Riemann sphere $\hat{\mathcal{C}}$ such that there exists a sequence $\{z_n\}$ in D with $\lim z_n = z_0$ and $\lim f(z_n) = \alpha$. It follows ([1, p. 3]) that $C = C(f, z_0)$ is a non-empty closed set, and that if f is continuous in D, then C must be connected. We now consider the boundary cluster set $C_B(f, z_0) = C_{B\ell}(f, z_0) \cup C_{Br}(f, z_0)$. Here, $C_{B\ell}$ is the left boundary cluster set defined as follows:

$$C(f, 0 < \theta - \theta_0 < \eta) = \bigcup_{0<\theta-\theta_0<\eta} C(f, e^{i\theta}),$$

$$C_{B\ell}(f, e^{i\theta_0}) = \bigcap_{\eta>0} C^-(f, 0 < \theta - \theta_0 < \eta),$$

while the right boundary cluster set C_{Br} is analogously defined. In [1, p. 82] it is shown that $C_{Br} \subset C$, and of course $C_{B\ell} \subset C$. If f is bounded, then C_{B_r} and C_{B_ℓ} are non-empty compact sets in $\mathcal{C}$. The one non-trivial result we use from cluster set theory is a theorem of Iversen [**1**, p. 91] that if h is analytic in D, then for every $e^{i\theta} \in T$, $\partial C(h, e^{i\theta}) \subset C_B(h, e^{i\theta})$. To continue with our proof, it is easy to see that $C_{B_r}(\Delta, 1) = \{1\}$ and $C_{B_\ell}(\Delta, 1) = \{0\}$. It follows from a simple argument that $C_{B_r}(g, 1) \subset B_r$ and $C_{B_\ell}(g, 1) \subset B_\ell$, where $B_r = \{z : |z-1| \leqq \epsilon\}$ and $B_\ell = \{z : |z| \leqq \epsilon\}$. But B_r and B_ℓ are a positive distance apart, and this contradicts $\partial C(g, 1) \subset B_r \cup B_\ell$ since $C(g, 1)$ is connected, and the first proof is concluded. We have implicitly used the following topological lemma, whose simple proof we omit.

LEMMA. *Let C be a compact connected set in $\mathcal{C}$, and let B_1 and B_2 be two closed convex sets such that $\partial C \subset B_1 \cup B_2$. If $C \cap B_1 \neq \emptyset$ and $C \cap B_2 \neq \emptyset$ then $B_1 \cap B_2 \neq \emptyset$.*

We say that a Banach space B is pseudo-reflexive if it is homeomorphic as a topological space to its second dual B^{**}. It was shown in [9, § 7] that H^1 is not reflexive. The next result is a corollary of our main theorem.

PROPOSITION. *H^1 is not pseudo-reflexive.*

PROOF. As is well-known, L^∞/H^∞ is isometrically isomorphic to $(H^1)^*$. Since H^1 is separable, it follows that if H^1 were pseudo-reflexive, then $(H^1)^{**}$ would be separable. But by a well-known result ([**6**, p. 34]), if the dual B^* of a Banach space B is separable, then B must be separable. Thus, we would have $(H^1)^*$ separable, and hence L^∞/H^∞ separable, which we have proved impossible.

We now give the second proof of our main theorem.

PROOF 2. We need first the following fact, which is a special case of [**5**, p. 178, part e].

LEMMA. *If B is a weakly sequentially complete Banach space, and if B^* is separable, then B is reflexive.*

Now it is well-known that H^1 is weakly sequentially complete — to see this, just observe that H^1 may be regarded as a closed subspace of the space of all bounded complex Borel measures on T. Hence if L^∞/H^∞ were separable, then H^1 would be reflexive, which it is not, by [**9**, § 7].

Incidentally, we have proved the following result.

PROPOSITION. *If the Banach space B is separable and weakly sequentially complete, then B is reflexive if it is pseudo-reflexive.*

PROOF THAT H^1 IS NOT REFLEXIVE. (From [9, § 7].) Let $F_n(z) = (1 + c_n) \cdot (z + 1)/(c_n z + 1)$, where $-1 < c_n < 1$ and $c_n \to -1$. We remark first that F_n is analytic in $\bar{D} = \{z : |z| \leqq 1\}$. Let $f_n(z) = F_n'(z)$. Both $\{f_n\}$ and $\{F_n\}$ converge to 0 uniformly on compact subsets of D, because $|F_n(z)| \leqq 2|1 + c_n|/(1 - |z|)$, and it is an elementary fact that if $\{F_n\}$ so converges to 0, then the same is true of $\{F_n'\}$. Now $w = F_n(z)$ maps $\{z : |z| \leqq 1\}$ onto $\{w : |w - 1| \leqq 1\}$. Computing the length of the image of $\{z : |z| = 1\}$, we have

$$2\pi = \int_0^{2\pi} \left| \frac{d}{d\theta} F_n(e^{i\theta}) \right| d\theta = \int_0^{2\pi} |f_n(e^{i\theta})| \, d\theta$$

so that f_n lies in the unit ball of H^1. We shall prove that $\{f_n\}$ does not converge weakly to 0; the same proof works for any subnet of $\{f_n\}$, and since the only possible weak limit of any subnet of $\{f_n\}$ is 0, we will have proved that the unit ball in H^1 is not weakly compact, so that H^1 cannot be reflexive. Note now that $F_n(1) = 2$ for all n, but that if $0 < \theta < 2\pi$, then $F_n(e^{i\theta}) \to 0$ as $n \to \infty$. Fixing such a θ, we have

$$F_n(e^{i\theta}) = F_n(1) + i \int_0^{\theta} f_n(e^{it}) e^{it} \, dt.$$

Since $\int_0^\theta (\,\cdot\,) e^{it} \, dt \in (H^1)^*$, if $\{f_n\}$ converged weakly (to 0), the last equation would yield the absurd conclusion $0 = 2 + 0$.

PROOF 3. (B. A. Taylor). Analogously to the first proof, we show that $\|\Delta - g\|_\infty \geqq 1$ for all $g \in H^\infty$, where we now suppose only that

$$\operatorname{ess\,lim\,inf} \{\operatorname{Re} \Delta(e^{i\theta}) : \theta \to 0+\} \geqq 1$$

and

$$\operatorname{ess\,lim\,sup} \{\operatorname{Re} \Delta(e^{i\theta}) : \theta \to 0-\} \leqq -1,$$

and the conclusion follows much as before.

Let $g = u + iv$ and suppose that $\|\Delta - g\|_\infty \leqq 1 - 2\epsilon < 1$. Then

$$\operatorname{ess\,lim\,inf} \{u(e^{i\theta}) : \theta \to 0+\} > \epsilon > 0,$$

$$\operatorname{ess\,lim\,sup} \{u(e^{i\theta}) : \theta \to 0-\} < -\epsilon < 0.$$

Let $v^*(z) = v(z) - v(0)$. Then [7, p. 78]

$$v^*(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\theta - t) Q_r(t) \, dt,$$

where $Q_r(t) = 2r \sin t/(1 - 2r\cos t + r^2)$. Hence

$$v^*(r) \leqq \frac{-\epsilon r}{\pi} \int_{-\pi}^{\pi} \frac{|\sin t|}{1 - 2r\cos t + r^2}\, dt + \text{const.}$$

But

$$r \int_0^{\pi} \frac{\sin t}{1 - 2r\cos t + r^2}\, dt = r \int_{-1}^{1} \frac{ds}{1 - 2rs + r^2} = \log\left(\frac{1+r}{1-r}\right),$$

which approaches $+\infty$ as $r \to 1-$, so that v^* and hence g cannot be bounded.

Proof 4. (C. L. Fefferman) We use the result ([3], [4]) that $(H^1)^*$ is linearly homeomorphic as a Banach space to BMO. This result was proved for the Hardy class H^1 in a half-space, but the proof is even a bit simpler for the disc. (Alternatively, one knows [7, p. 130] that H^1 of the half-plane and H^1 of the disc are isometrically isomorphic.) We identify T with $[0, 2\pi]$ and note that a function $f \in L^1(T)$ belongs to BMO when

$$\|f\|_{BMO} = \sup \frac{1}{|I|} \int_I |f - \mathrm{av}_I f| < \infty,$$

where I runs over intervals in T and $\mathrm{av}_I(f) = (1/|I|)\int_I f$. For convenience, we identify two functions in BMO whose difference is a constant a.e. Now consider again the function $\Delta = f_\beta - f_\alpha$ of Proof 1. A simple estimate, choosing I as a small interval centered at a jump of Δ, shows that $\|\Delta\|_{BMO} \geqq 1/2$. The consequent non-separability of BMO implies that of L^∞/H^∞.

Proof 5. (J. Shapiro) For $f \in H^1$, and $\{z_k\}$ a sequence in D, define $T(f) = \{(1 - |z_k|^2)f(z_k)\}$.

We use the result (see [2, Theorems 9.1, 9.2]) that there is a sequence $\{z_k\}$ in D such that T is a bounded linear transformation of H^1 onto ℓ^1. Such a sequence is called an (H^1, ℓ^1) interpolation sequence. Using the canonical homomorphisms, we see that $(H^1/\ker T)^*$ is linearly homeomorphic to $(\ell^1)^*$, which is linearly isometric ($\cong$) to ℓ^∞. But $(H^1/\ker T)^* \cong (\ker T)^\perp \subset (H^1)^* \cong L^\infty/H^\infty$, where $\perp$ denotes the annihilator in $(H^1)^*$. Since ℓ^∞ is not separable, neither is L^∞/H^∞.

It is clear from the above considerations that if G is a Riemann surface, then $H^1(G)$ will not be reflexive as soon as there exists an $(H^1(G), \ell^1)$ interpolation sequence for a suitable operator T. It seems likely that this will be the case as soon as $H^1(G)$ is not trivial, but we have no proof.

PROOF 6. (Pelczynski). This proof establishes that H^1 contains a subspace isomorphic to ℓ^1, which implies that $L^\infty/H^\infty = (H^1)^*$ has a quotient space isomorphic to ℓ^∞ and is consequently not separable. To this end, we produce a sequence $\{g_{n_j}\}$ of functions in H^1 such that for every finite complex sequence $a_1, a_2, \cdots, a_r$ we have

$$\frac{1}{2}\sum_{j=1}^{r} |a_j| \leqq \int_0^{2\pi} \left| \sum_{j=1}^{r} a_j g_{n_j}(t) \right| d\lambda(t) \leqq \sum_{j=1}^{r} |a_j|,$$

where $d\lambda(t) = (2\pi)^{-1}\,dt$. This is enough for our purpose. Let $f_n(t) = [2^{-1}(1 + e^{it})]^n$ for $t \in [0, 2\pi]$ and $n = 1, 2, 3, \cdots$. A direct computation shows that for some positive constant c

$$(1) \qquad \|f_n\| = \int_0^{2\pi} |f_n(t)|\,d\lambda(t) = \int_0^{2\pi} \left|\cos\frac{t}{2}\right|^n d\lambda(t) \geqq cn^{-1/2}.$$

Let us put $g_n = f_n\|f_n\|^{-1}$. Clearly, (1) implies that

$$(2) \qquad \lim g_n(t) = 0$$

uniformly for $\alpha \leqq t \leqq \beta$, for any pair (α, β) with $0 < \alpha < \beta < 2\pi$. Furthermore, for any fixed n we have

$$(3) \qquad \lim_{\alpha\to 0}\int_0^{\alpha} |g_n(t)|\,d\lambda(t) = \lim_{\beta\to 2\pi}\int_\beta^{2\pi} |g_n(t)|\,d\lambda(t) = 0.$$

Using (2) and (3), we inductively define an increasing sequence $\{n_k\}$ of positive integers, and sequences $\{\alpha_k\}$ and $\{\beta_k\}$ of real numbers for which

$$(4) \qquad 0 < \alpha_{k+1} < \alpha_k < \beta_k < \beta_{k+1} < 2\pi,\ \lim_{k\to\infty}\alpha_k = 0,\ \lim_{k\to\infty}\beta_k = 2\pi,$$

and

$$(5) \qquad \int_{\alpha_k}^{\beta_k} |g_{n_j}(t)|\,d\lambda(t) < 4^{-j-1} \quad \text{for } j > k,$$

$$(6) \qquad \int_{\alpha_k}^{\beta_k} |g_{n_j}(t)|\,d\lambda(t) > \|g_{n_j}\| - 4^{-k-1} = 1 - 4^{-k-1}$$

$$\text{for } j = 1, 2, \cdots, k.$$

Let us set $B_1 = [\alpha_1, \beta_1]$, $B_s = [\alpha_s, \beta_s]\setminus[\alpha_{s-1}, \beta_{s-1}]$ for $s = 2, 3, \cdots$. Next fix a positive integer r and any sequence $\{a_1, a_2, \cdots, a_r\}$ of complex numbers. By (5) and (6), we see that if $j > k$, then

$$\int_{B_k} |g_{n_j}(t)|\,d\lambda(t) \leqq \int_{\alpha_k}^{\beta_k} |g_{n_j}'(t)|\,d\lambda(t) < 4^{-j-1} < 4^{-k}.$$

If $j < k$ then

$$\int_{B_k} |g_{n_j}(t)|\, d\lambda(t) \leqq \int_0^{2\pi} |g_{n_j}(t)|\, d\lambda(t) - \int_{\alpha_{k-1}}^{\beta_{k-1}} |g_{n_j}(t)|\, d\lambda(t) < 4^{-k}.$$

But if $j = k$, then

$$\int_{B_k} |g_{n_k}(t)|\, d\lambda(t) = \int_{\alpha_k}^{\beta_k} |g_{n_k}(t)|\, d\lambda(t) - \int_{\alpha_{k-1}}^{\beta_{k-1}} |g_{n_k}(t)|\, d\lambda(t)$$

$$\geqq 1 - 4^{-k-1} - 4^{-k-1} \geqq \frac{7}{8}.$$

(If $k = 1$ we could take $\alpha_0 = \beta_0 = \pi$, say.) Thus for $k \leqq r$

$$\int_{B_k} \Big| \sum_{j=1}^{r} a_j g_{n_j}(t) \Big|\, d\lambda(t) \geqq \frac{7}{8} |a_k| - \sum_{j \neq k} |a_j| 4^{-k}$$

$$\geqq \frac{7}{8} |a_k| - \sum_{j=1}^{r} |a_j| 4^{-k}.$$

Hence

$$\int_0^{2\pi} \Big| \sum_{j=1}^{r} a_j g_{n_j}(t) \Big|\, d\lambda(t) \geqq \sum_{k=1}^{r} \int_{B_k} \Big| \sum_{j=1}^{r} a_j g_{n_j}(t) \Big|\, d\lambda(t)$$

$$\geqq \frac{7}{8} \sum_{k=1}^{r} |a_k| - \sum_{k=1}^{\infty} \sum_{j=1}^{r} |a_j| 4^{-k} \geqq \frac{1}{2} \sum_{j=1}^{r} |a_j|.$$

In the other direction since $\|g_{n_j}\| = 1$ for $j = 1, 2, 3, \cdots$, we get

$$\int_0^{2\pi} \Big| \sum_{j=1}^{r} a_j g_{n_j}(t) \Big|\, d\lambda(t) \leqq \sum_{j=1}^{r} |a_j|.$$

This completes the proof of the desired inequality, and the rest follows.

Proof 7. (Garnett). This proof is based on some material in Chapter 10 of [7]. We identify L^{∞} with $C(X)$, where X is the spectrum of L^{∞}. Then $(L^{\infty}/H^{\infty})^* = (H^{\infty})^{\perp}$ where $(H^{\infty})^{\perp}$ is the space of Borel measures μ on X that annihilate H^{∞}. Now suppose that L^{∞}/H^{∞} is separable. By Theorem 2.10.1 of [6] it follows that the unit ball B of $(H^{\infty})^{\perp}$ is sequentially compact in the weak star topology of $(L^{\infty})^* = M(X)$. Take a sequence $\{p_j\}$ of distinct complex numbers with $|p_j| = 1$, $p_j \to 1$.

Let φ_j be a complex homomorphism of H^{∞} with $\varphi_j(z) = p_j$, but $\varphi_j \notin X$. For example, if $\{z_{n,j}\}$ is a Blaschke sequence, $z_{n,j} \to p_j$

as $n \to \infty$, then let φ_j be any cluster point of $\{z_{n,j}\}$ in the maximal ideal space of H^∞. Now choose $f_j \in H^\infty$ such that $\varphi_j(f_j) = 0$ and $|f_j| = 1$ on X. For example, we may take f_j to be the Blaschke product over the above sequence $\{z_{n,j}\}$. Now let $\mu_j = f_j\, dm_j$, where m_j is the representing measure for φ_j. Passing to a subsequence if necessary, suppose μ_j converges weak star to μ_0. Choose $h \in L^\infty$ so that $h(z) = (-1)^j \bar{f}_j(z)$ almost everywhere on an arc that contains p_j. Then $\int h\, d\mu_j = (-1)^j$, which is a contradiction. The last equality uses the fact that $\int F\, dm_j$ depends only on the values of F near p_j, which we now prove. Let $G(z) = (1 + \bar{p}_j z)/2$, so that $G(p_j) = 1$, and $|G(z)| < 1$ for $z \neq p_j$. Now $1 = \int G^n\, dm_j$, but as $n \to \infty$, $\int G^n\, dm_j \to m_j(\tilde{p}_j)$ (where $\tilde{p}_j$ is the fiber at p_j) by the dominated convergence theorem. This proof is now complete.

Finally, we prove a rather easy result. Here, $L^\infty(D)$ is the space of essentially bounded measurable functions in D and $H^\infty(D)$ is the space of bounded analytic functions in D, both with the essential supremum norm.

PROPOSITION. *$L^\infty(D)/H^\infty(D)$ is not separable.*

PROOF. For $0 < r < 1$, let

$$f_r(z) = \begin{cases} \bar{z} \text{ for } |z| \leqq r, \\ 0 \text{ otherwise.} \end{cases}$$

Then for $0 < r < s < 1$,

$$f_s(z) - f_r(z) = \begin{cases} \bar{z} \text{ for } r < |z| \leqq s, \\ 0 \text{ otherwise.} \end{cases}$$

Let g belong to $H^\infty(D)$, and let $\Delta = f_s - f_r + g$. Then by the Cauchy integral theorem,

$$I = \frac{1}{2\pi} \int_{\rho=r}^{\rho=s} \int_{\theta=-\pi}^{\theta=\pi} \Delta(\rho e^{i\theta}) e^{i\theta}\, d\theta\, d\rho = \int_{\rho=r}^{\rho=s} \rho\, d\rho = \tfrac{1}{2}(s+r)(s-r).$$

So if $\epsilon > 0$ and if $\|\ \|_Q$ denotes the norm in $L^\infty(D)/H^\infty(D)$, then we have, for suitable $g \in H^\infty(D)$,

$$|I| \leqq (\|(f_s) - (f_r)\|_Q + \epsilon) \left[\frac{1}{2\pi} \int_{\rho=r}^{\rho=s} \int_{\theta=-\pi}^{\theta=\pi} d\theta\, d\rho \right].$$

Since the expression in square brackets equals $(s - r)$, we see that $\|(f_s) - (f_r)\|_Q \geqq (s + r)/2$ and restricting our attention, say, to $r, s \geqq 1/2$, we conclude that $L^\infty(D)/H^\infty(D)$ is not separable.

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