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THE CONSTRUCTIVE JORDAN CURVE THEOREM

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ABSTRACT. This paper presents a constructive treatment of the Jordan curve theorem. It is shown that, given a Jordan curve, and a point whose distance to the curve is positive, then there is a finite procedure to decide whether the point is inside or outside the curve. Also, given two points that are either both inside, or both outside, the curve, then there is a finite procedure that constructs a polygonal path joining the two points, that is bounded away from the curve. Finally, a finite procedure is given for constructing a point inside the curve.

1. **Introduction.** Since the publication of Bishop's book [2] on contructive analysis, there has been a resurgence of interest in the constructive approach to mathematics. The Jordan curve theorem provides a pertinent illustration of this approach. The main concern of the Jordan curve theorem is the construction of a path joining two points and missing a curve. Indeed, the heart of the theorem may be stated as follows: *Given any three points off a Jordan curve, two of them can be connected by a path missing the curve.* The constructive approach requires finding an explicit, finite procedure for computing this path.

The standard treatments of the Jordan curve theorem (see, for example, [1], [4], [5], [6]) do not address themselves to this computation, nor can they be modified easily to supply it. The usual approach is to prove abstract existence, by reductio ad absurdum, and by appeals to nonconstructive existence theorems such as the Heine-Borel theorem. The purpose of this paper is to demonstrate that such an approach is neither necessary nor desirable. By viewing the problem constructively, we are led to a proof that is as simple as any, while considerable insight is gained into a theorem which is often considered to be a triviality. For when phrased in terms of an explicit construction of a path, the difficulty becomes apparent, even if you are sure that the curve has an inside and an outside. Brouwer gave the first constructive proof of the Jordan curve theorem in a rather formidable paper [3]. Our approach parallels Brouwer's intuitionist treatment but is in the spirit of modern constructivism.

All mathematical objects dealt with here have computational meaning. A point in the plane is given by a pair of real numbers. A real number is given by providing rational numbers which approximate

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that real number as closely as desired. One practical effect of this is the impossibility, in general, of deciding whether two given real numbers are equal, for we can only compare their rational approximations. A classical example of this is the real number r that is approximated within $1/n$ by the rational number $1/k_n$, where k_n is the least integer, not exceeding n, such that the sequence 0123456789 appears in the first k_n digits of the decimal expansion of π, or $k_n = n$ if there is no such integer. Since we possess algorithms that compute the decimal expansion of π to any number of digits, we can compute r as closely as we please. At present, however, we know of no finite computation that would settle the question of whether r is 0 or not. Thus, we must keep in mind that we may not be able to tell if a given point is on a given curve or not, or whether a curve meets some other geometric object.

Despite the fact that we may be unable to decide whether a given real number r is zero, we can handle certain situations that seem to require such a decision. If $x > y$ (i.e., $x - y$ exceeds some known positive rational number), then we can decide from appropriate rational approximations to x, y, and r which of the two inequalities, $|r| < x$ or $|r| > y$ holds. Thus, for any $\epsilon > 0$ and any real number r we can assert either $|r| > \epsilon > 0$ or $|r| < 2\epsilon$. This approximate dichotomy will suffice.

Although we cannot in general argue by reductio ad absurdum, we can use this method of proof in the following way. If A and B are the only possible outcomes of a finite computation, then we can demonstrate that A occurs by showing that the occurrence of B leads to a contradiction.

2. **What is a Jordan curve?** From a constructive point of view, the manner of presentation of a mathematical object is an essential part of its nature. A Jordan curve is usually defined to be a subset of the plane which is homeomorphic to a circle. Constructively, we must be *given* a Jordan curve. This is accomplished by supplying a particular homeomorphism f from the unit circle into the plane. The function f is to be viewed as an integral part of the Jordan curve, providing the necessary numerical data for computation. Whereas distinct functions may give rise to equivalent curves, we will not, in general, be able to decide whether two given curves are equivalent.

To locate a curve in the plane, to within $\epsilon > 0$, we need to construct a finite number of points on the curve, such that every point on the curve is within ϵ of one of these finitely many points. This may be done by selecting equally spaced points around the unit circle, such that the distance between adjacent points is sufficiently small, and looking at their images under f. However, we have to know how close points x and y on the unit circle must be, for the distance between $f(x)$ and $f(y)$ to be less than ϵ — that is, we need to know a modulus of continuity, ω, for f.

In addition, we need a modulus of continuity for f^{-1}, the inverse of f. For suppose our Jordan curve looks like Figure 1, and we are required to construct a path from a to b that misses the curve. In order to find our way through the narrow passage, with only a finite number of points on the curve to guide us, we must have information about its width. This data is supplied by the modulus of continuity for f^{-1}. Roughly speaking, the distances from a to b to the curve, together with ω, yield a lower bound for $|x - y|$. The modulus of continuity for f^{-1} converts this to a lower bound for $|f(x) - f(y)|$. Since we may not be able to compute the values of such a modulus from f and ω, we must include it as part of the data given with the curve. This is most simply done by redefining ω to be the minimum of the moduli of continuity of f and f^{-1}. Then ω serves as a common modulus of continuity for f and f^{-1}. We will also need the more general notion of a *closed path.*

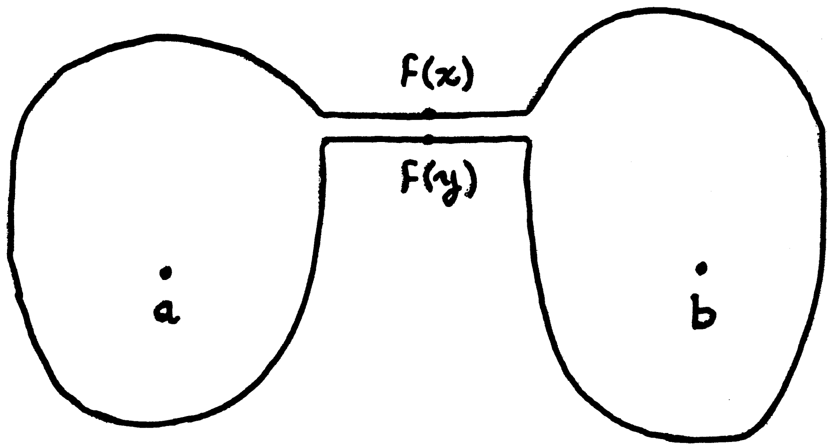

Figure 1

DEFINITION. A *closed path* is a subset J of the plane, together with a continuous function f from the unit circle C onto J, and a positive function ω that serves as a modulus of continuity for f; i.e., for any $\delta > 0$ and $z_1, z_2 \in C$, if $|z_1 - z_2| < \omega(\delta)$, then $|f(z_1) - f(z_2)| < \delta$. A closed path is a *Jordan curve* if, in addition, f is a homeomorphism and ω serves also as a modulus of continuity for f^{-1}; i.e., for any $\delta > 0$ and $z_1, z_2 \in C$, if $|f(z_1) - f(z_2)| < \omega(\delta)$, then $|z_1 - z_2| < \delta$. It will be convenient to require that $\omega(\delta) < \frac{1}{2}\min(1, \delta)$.

The map f and the modulus ω are integral parts of the closed path. Nevertheless, we shall abuse the language in the customary way, and refer to the closed path (J, f, ω) simply as J. If the words "unit circle" are replaced by the words "unit interval" in the above definition, then we have the definition of a *path* (joining $f(0)$ and $f(1)$) and a *Jordan arc*.

If J is a closed path, and a is a point in the plane, then the distance from a to J can be computed (as closely as we wish). To compute the distance within $1/n$, place points around the unit circle at intervals of length less than $\omega(0.5/n)$, and compute the distances from a to the images of these points to within $0.5/n$. Then the smallest of these numbers is the desired approximation. If the distance from a to J is (known to be) positive, that is, if we have some rational number $r > 1/n$ which approximates the distance to within $1/n$, then we say that a is *off* J. To say that a is *on* J means $a = f(z)$ for some z on C.

A path J_1 is *bounded away* from a path J_2 if, for some $\delta > 0$, $|x_1 - x_2| \geqq \delta$ for each x_1 on J_1 and x_2 on J_2. A *polygonal path* P joining the points a and b is a sequence of points $a = a_0, a_1, \cdots, a_n = b$, together with the line segments $\overline{a_{i-1}a_i}$, $1 \leqq i \leqq n$. A polygonal path P joining a point to itself may be identified with a unique, uniformly parameterized, closed path. If, in addition, P is *simple*—that is, if $n \geqq 3$ and any two nonconsecutive line segments of P are bounded away from each other (and, for $n = 3$, vertices are bounded away from opposite sides)—then this closed path becomes a Jordan curve by appropriately altering ω.

JORDAN CURVE THEOREM. *Given a Jordan curve, J, we can construct two points a and b, off J, such that:*

(1) *Given any point c off J, we can construct a polygonal path that is bounded away from J, and that joins c to one of the two points, a or b.*

(2) *Any polygonal path joining a and b comes arbitrarily close to J.*

Since J is bounded, any two points sufficiently far from J can be joined by a polygonal path that is bounded away from J. These points, and the points off J that can be joined to them by such paths, are said to be *outside* J. The other points off J are said to be *inside* J. Our first problem is to develop a procedure for deciding whether a point off J is inside J or outside J.

3. **The index.** To determine whether a point is inside or outside of a Jordan curve, we compute a number called the index of the point with respect to the curve. This number is most naturally introduced in its computational form — with respect to a sequence of points, rather than with respect to a curve. We denote the distance between two points, a and b, by $|a - b|$. Throughout this paper, if we are dealing with a sequence $x_1, \cdots, x_n$, then x_{n+1} is understood to mean x_1.

DEFINITION. A sequence of points $x_1, \cdots, x_n$ is said to be *admissible* for the point a if $|x_i - x_{i+1}| < |a - x_j|$, for $1 \leqq i \leqq n$, and $1 \leqq j \leqq n$. If $x_1, \cdots, x_n$ is admissible for a then the *index* of $x_1, \cdots, x_n$ with respect to a is defined by

$$\operatorname{ind}(a; x_1, \cdots, x_n) = \frac{1}{2\pi} \sum_{i=1}^{n} \text{angle } x_i a x_{i+1},$$

where all angles are taken (strictly) between $-\pi/3$ and $\pi/3$, and clockwise angles are taken to be negative.

The constructive theory of angles in the Euclidean plane is straightforward and will be assumed. The admissibility condition is more than is necessary to insure an unambiguous interpretation of the term "angle $x_i a x_{i+1}$" in the definition of index. However, this notion of admissibility facilitates comparison of related sequences (see Proposition 3). The index is supposed to measure how many times the sequence $x_1, \cdots, x_n$ goes around the point a.

PROPOSITION 1. *If* $x_1, \cdots, x_n$ *is admissible for* a, *then* $\operatorname{ind}(a; x_1, \cdots, x_n)$ *is an integer.*

PROOF. In this proof we shall use a capital A in the word angle when the angle under consideration is not necessarily between $-\pi/3$ and $\pi/3$. Let $A_j = \sum_{i=1}^{j}$ angle $x_i a x_{i+1}$ − Angle $x_1 a x_{j+1}$, where Angle $x_1 a x_{j+1}$ is determined up to an integral multiple of 2π. Then $A_j - A_{j-1} =$ angle $x_j a x_{j+1}$ − Angle $x_1 a x_{j+1}$ + Angle $x_1 a x_j$ is an integral multiple of 2π for $1 \leqq j \leqq n$. But $A_0 = -$Angle $x_1 a x_1$ is an integral multiple of 2π. Hence, A_n is an integral multiple of 2π. But $\operatorname{ind}(a; x_1, \cdots, x_n) = (A_n +$ Angle $x_1 a x_{n+1})/2\pi$ is then an integer, since $x_{n+1} = x_1$.

PROPOSITION 2. *If* $y_1, \cdots, y_n$ *is a circular permutation of the admissible sequence* $x_1, \cdots, x_n$, *then* $y_1, \cdots, y_n$ *is admissible, and* $\operatorname{ind}(a; y_1, \cdots, y_n) = \operatorname{ind}(a; x_1, \cdots, x_n)$.

PROOF. This follows directly from the definition.

PROPOSITION 3. *If* $x_1, \cdots, x_n$ *and* $x_1, \cdots, x_j, v, x_{j+1}, \cdots, x_n$ *are admissible for a, then* $\text{ind}(a; x_1, \cdots, x_n) = \text{ind}(a; x_1, \cdots, x_j, v, x_{j+1}, \cdots, x_n)$.

PROOF. The admissibility of the two sequences implies that the distance between any two of the points x_j, x_{j+1}, and v, is less than the distance from any one of them to a. Hence, angle $x_j a x_{j+1}$ = angle $x_j a v$ + angle $v a x_{j+1}$, and the proposition follows.

The index of a closed path with respect to a point is computed by approximating the path with a sequence of points. We say that $y_1, \cdots, y_n$ is a δ-mesh on J if $y_j = f(x_j)$, where the z_j are arranged consecutively around the unit circle C in the counterclockwise sense, starting at $z_1 = (1, 0)$, and $|z_{j+1} - z_j| < \omega(\delta)$ for $1 \leqq j \leqq n$. Observe that in a δ-mesh the points on J between y_i and y_{i+1} are within δ of y_i and y_{i+1}, not merely within δ of some y_j.

DEFINITION. Let J be a closed path, a a point whose distance from J is at least δ, and $y_1, \cdots, y_n$ a δ-mesh on J. Then the *index of J with respect to a* (or, the *index of a with respect to J*) is defined to be $\text{ind}(a; y_1, \cdots, y_n)$, and is written $\text{ind}(a; J)$.

Note that since $y_1, \cdots, y_n$ is a δ-mesh on J, and $|a - y_j| \geqq \delta$ for $1 \leqq j \leqq n$, then $y_1, \cdots, y_n$ is admissible for a. To show that the definition does not depend on the choice of δ-mesh, let $x_1, \cdots, x_m$ be another δ-mesh on J. Then we may interleave the x's and y's to form an admissible sequence $z_1, \cdots, z_{m+n}$. Repeated application of Proposition 3 then shows that $\text{ind}(a; y_1, \cdots, y_n) = \text{ind}(a; z_1, \cdots, z_{m+n}) = \text{ind}\,(a; x_1, \cdots, x_m)$.

If J is the perimeter of a square, described in the counterclockwise sense, then it is easily seen that $\text{ind}(a; J) = 0$ for points outside J, and $\text{ind}(a; J) = 1$ for points inside J. Although we do not reduce the general Jordan curve theorem to the Jordan curve theorem for polygons, but rather attack the general case directly, the next result, relating the index of a Jordan curve to the index of a nearby polygon, is central to the development. Here, for the first time, we need J to be a Jordan curve, and we make use of the fact that ω is also a modulus of continuity for f^{-1}.

LEMMA. *Let J be a Jordan curve,* δ *a positive number, and P a simple closed polygon with sides of length less than* $0.3\omega(\omega(\delta))$, *each of whose vertices is closer than* $0.3\omega(\omega(\delta))$ *to J. Then there is an integer m such that* $\text{ind}(a; P) = m\, \text{ind}(a; J)$ *for any point a whose distance from J is at least* 2δ.

Proof. Let $y_1, \cdots, y_n$ be a $0.05\omega(\omega(\delta))$-mesh on J. Let $x_1, \cdots, x_s$ be a consecutive listing of the vertices of P, and choose k_i so that $|x_i - y_{k_i}| < 0.35\omega(\omega(\delta))$. Then by repeated application of Proposition 3, $\text{ind}(a; P) = \text{ind}(a; x_1, \cdots, x_s) = \text{ind}(a; x_1, y_{k_1}, y_{k_2}, \cdots, y_{k_s}) = \text{ind}(a; y_{k_1}, \cdots, y_{k_s})$. If $y_j = f(z_j)$, for $1 \leqq j \leqq n$, then $|z_{k_i} - z_{k_{i+1}}| <, \omega(\delta) < \frac{1}{2}$, because $|y_{k_i} - y_{k_{i+1}}| \leqq |y_{k_i} - x_i| + |x_i - x_{i+1}| + |x_{i+1} - y_{k_{i+1}}| < \omega(\omega(\delta))$. Let Z_i denote the sequence of z's, starting at z_{k_i} and going to $z_{k_{i+1}}$ along the shorter arc of C, and let Y_i denote the corresponding sequence of y's. Then each term in Z_i is within $\omega(\delta)$ of z_{k_i} and $z_{k_{i+1}}$, so each term in Y_i is within δ of y_{k_i} and $y_{k_{i+1}}$. It follows, by repeated application of Proposition 3, that $\text{ind}(a; P) = \text{ind}(a; y_{k_1}, \cdots, y_{k_s}) = \text{ind}(a; Y_1, \cdots, Y_s)$, where $Y_1, \cdots, Y_s$ is to be thought of as the sequence of y's resulting from the concatenation of the Y's. Notice that any subsequence of the form $y_i y_j y_i$, or $y_i y_i$, may be replaced by y_i without changing the value of the index. Thus, we may reduce the sequence $Y_1, \cdots, Y_s$ to a sequence which either consists of the single term y_{k_1}, or consists of a circular permutation of the sequence $y_1, \cdots, y_n$, or of the sequence $y_n, \cdots, y_1$, repeated a positive integer number of times. In any event, $\text{ind}(a; P) = m\, \text{ind}(a; J)$ for some integer m. Observe that m is independent of a.

4. **The connection.** If we are given two points a and b that are off the Jordan curve J, then we may compute the integers $\text{ind}(a; J)$ and $\text{ind}(b; J)$. These integers tell the whole story regarding the possibility of joining a and b by a polygonal path that is bounded away from J. First we show that if two points can be joined by a path that is bounded away from J, then they necessarily have the same index. In fact, this much holds if J is simply a closed path.

Proposition 4. *Let J be a closed path, and S a set of points whose distance to J exceeds $\delta > 0$. Then* $\text{ind}(a; J)$ *is uniformly continuous on S as a function of a.*

Proof. Let $x_1, \cdots, x_n$ be a δ-mesh on J. Suppose a and b are in S, and $|a - b| < \epsilon\delta/4n$, where $0 < \epsilon < 1$. Then

$$|\text{angle } x_i a x_{i+1} - \text{angle } x_i b x_{i+1}| \leqq |\text{angle } a x_i b| + |\text{angle } a x_{i+1} b|$$
$$\leqq 4|b - a|/\delta < \epsilon/n.$$

Thus, $|\text{ind}(a; J) - \text{ind}(b; J)| < \epsilon$.

Corollary. *If a and b can be joined by a path that is bounded away from the closed path J, then* $\text{ind}(a; J) = \text{ind}(b; J)$.

PROOF. Suppose a and b are joined by a path P bounded away from J. By Proposition 4 we can find an $\epsilon > 0$ such that, if p and q are on P and $|p - q| < \epsilon$, then $|\text{ind}(p; J) - \text{ind}(q; J)| < 1$, so, since the index is an integer, $\text{ind}(p; J) = \text{ind}(q; J)$. Choose a sequence of points $a = a_0, a_1, \cdots, a_n = b$ on P such that $|a_i - a_{i-1}| < \epsilon$, $1 \leqq i \leqq n$. Then $\text{ind}(a; J) = \text{ind}(a_0; J) = \text{ind}(a_1; J) = \cdots = \text{ind}(a_n; J) = \text{ind}(b; J)$.

The converse of this corollary, for Jordan curves, is the core of the Jordan curve theorem. We prove a little more.

THEOREM 1. *If a and b are off the Jordan curve J, then either a and b can be connected by a polygonal path which is bounded away from J, or* $\text{ind}(a; J)$ *and* $\text{ind}(b; J)$ *differ by* 1.

PROOF. Let the distances from a and b to J be at least $2\delta > 0$. If $|a - b| < 2\delta$, then a and b can be connected by a straight line that is bounded away from J. If $|a - b| > \delta$, enclose J, a, and b in a circle that stays at least δ away from them, and tesselate the interior of this circle with regular hexagons of diameter $h < 0.1\omega(\omega(\delta))$, so that a lies at the center of a hexagon H_a, and b lies at the center of a hexagon H_b. Let L denote the class of straight line segments joining centers of adjacent hexagons. Paint these segments red or green, so that every red segment comes within $2h$ of J, and every green segment stays at least h away from J. Now color green those hexagons whose centers are connected to a by a sequence of green segments. Color H_a green. Color the rest of the hexagons red. If H_b is green, then a and b are connected by a sequence of green segments, which constitute a polygonal path that is bounded away from J. We shall show that if H_b is red, then $\text{ind}(a; J)$ and $\text{ind}(b; J)$ differ by 1.

Connect b to a by any simple sequence S of segments from L. Let H_v be the first green hexagon that this sequence enters, starting from b, and let v be the center of H_v. let H_w be the (red) hexagon from which H_v was entered, and let w be its center. Notice that the segment joining the centers of two adjacent hexagons of differing colors must be red. An edge common to hexagons of differing color is called a *separating edge*. For instance, the edge E, crossed by the segment $\overline{vw}$ is a separating edge. No separating edge can join the border of the tesselated region, since every separating edge is crossed by a red segment, which is closer to J than $2h$.

Each vertex not on the border of the tesselated region lies on exactly three edges, and exactly three hexagons. Examination of the possible colorings of these three hexagons shows that either zero or two of these edges are separating. It follows that each end of a separating edge is joined to exactly one other separating edge. Thus the edge

E lies on a unique path comprised of separating edges. This path forms a simple closed polygon P since it is finite, and each vertex lies on exactly two edges.

Since H_v is green, a is connected to v by a sequence of green segments. Since every green segment is at least $h/4$ away from any separating edge, $\text{ind}(a; P) = \text{ind}(v; P)$. Also, since the segments of S joining b to w lie in red hexagons, they are at least $h/4$ from any separating edge, so $\text{ind}(b; P) = \text{ind}(w; P)$. Construct a square Q in H_v on the edge E (see Figure 2). Replace the edge E in the polygon P by the other three sides of the square Q. Call the resulting polygon P'. Choose orientations so that $\text{ind}(x; P') = \text{ind}(x; P) + \text{ind}(x; Q)$ for $x = v$ and $x = w$. Since $\overline{vw}$ is bounded away from P', we have $\text{ind}(v; P') = \text{ind}(w; P')$. But $\text{ind}(v; Q)$ and $\text{ind}(w; Q)$ differ by 1, so $\text{ind}(v; P)$ and $\text{ind}(w; P)$ differ by 1. Hence, $\text{ind}(a; P)$ and $\text{ind}(b; P)$ differ by 1.

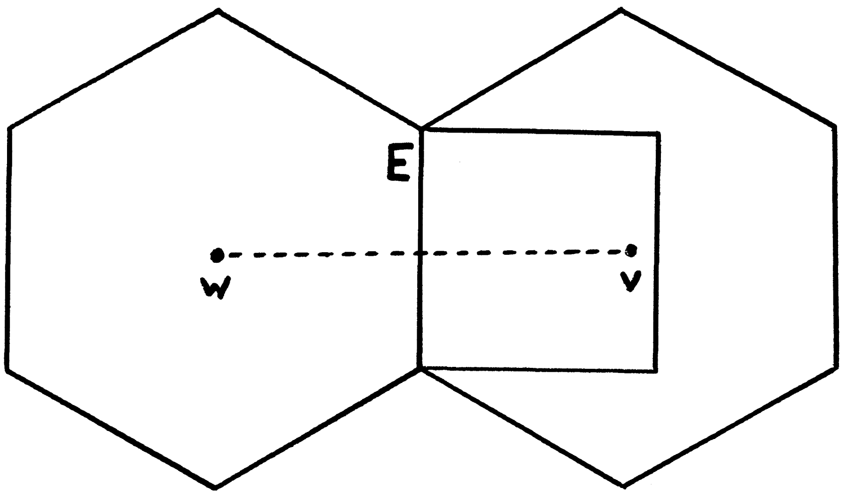

Figure 2

Since any point on a separating edge is closer to J than $2.5h$, the hypotheses of the lemma are satisfied, so $\text{ind}(a; P) = m\,\text{ind}(a; J)$, and $\text{ind}(b; P) = m\,\text{ind}(b; J)$, for some integer m. Hence $|m| = 1$, and $\text{ind}(a; J)$ and $\text{ind}(b; J)$ differ by 1. Q.E.D.

Note that $\operatorname{ind}(a; J) = 0$ if a is sufficiently far away from J. Hence, a point a off J is outside J precisely when $\operatorname{ind}(a; J) = 0$. Since indices of points off J cannot differ by more than 1, then, after possibly re-orienting J, every point off J has index either 0 or 1. Thus, any two inside points have index 1, so they can be connected by a polygonal path which is bounded away from J. There remains the problem of finding a point of index 1.

5. **Getting in.** To construct a point inside J, that is, a point with index different from 0, we construct a finite number of points, following the ideas in [5], and show that they cannot all have the same index. If A and B are polygonal paths, joining a and b and b to c respectively, we let $A + B$ denote the path joining a to c comprised of the line segments of A followed by those of B.

THEOREM 2. *Let c be a point on the Jordan curve J. If $\delta > 0$, then there exist points a and b, within δ of c, such that* $\operatorname{ind}(a; J) \neq \operatorname{ind}(b, J)$.

PROOF. Let $c = f(z)$ and $d = f(-z)$. Consider a square with boundary Q, center c, and edge of length e, where $e < \min(\delta, |c - d|)$. Place points $p_1, \cdots, p_n$ around Q so that the distance between adjacent points is less than $0.5\omega(\omega(e/4))$. Paint the segments $\overline{p_1p_2}, \cdots, \overline{p_{n-1}p_n}, \overline{p_np_1}$ red or green, so that each red segment is bounded away from $f(C_1)$, while each green segment is bounded away from $f(C_2)$, where C_1 and C_2 are the two halves of the unit circle C between z and $-z$. This is possible because $f(C_1)$ and $f(C_2)$ don't get too close together near Q. More precisely, if $x \in C_1$, $y \in C_2$, and $|f(x) - f(y)| < \omega(\omega(e/4))$, then $|x - y| < \omega(e/4)$, thus either the arc xzy or the arc $x(-z)y$ has diameter less than $\omega(e/4)$. So $|x - z| < \omega(e/4)$ or $|x + z| < \omega(e/4)$. Thus, $|f(x) - f(z)| < e/4$ or $|f(x) - f(-z)| < e/4$, so $f(x)$ (and, similarly, $f(y)$) is at least $e/4$ away from Q.

If all the segments are the same color, then c and d are joined by a Jordan arc, $f(C_1)$ or $f(C_2)$, that is bounded away from Q. Hence, by the corollary to Proposition 4, $\operatorname{ind}(c; Q) = \operatorname{ind}(d; Q)$. However, it is clear that $\operatorname{ind}(c; Q) = 1$ while $\operatorname{ind}(d; Q) = 0$. Let $w_1, \cdots, w_k$ be those points that are common endpoints of segments of different colors. We may assume that their points are arranged consecutively around Q, that $\overline{w_jw_{j+1}}$ is red if j is even, and that $\overline{w_jw_{j+1}}$ is green if j is odd (note that k must be even; and recall our convention that $w_{k+1} = w_1$). Since w_j is off J, we may compute $\operatorname{ind}(w_j; J)$. Suppose the numbers $\operatorname{ind}(w_j; J)$ are equal for $1 \leqq j \leqq k$. Then, by Theorem 1, we can find polygonal paths P_j joining w_j to w_{j+1}, for $1 \leqq j \leqq k$, that are bounded away from J.

Since $f(C_1)$ joins c to d and is bounded away from $P_j + \overline{w_{j+1}w_j}$, if j is even, we have $\text{ind}(c; P_j + \overline{w_{j+1}w_j}) = \text{ind}(d; P_j + \overline{w_{j+1}w_j})$, for $j = 2, 4, 6, \cdots, k$. Similarly, if $P = \overline{w_1w_2} + P_2 + \overline{w_3w_4} + P_4 + \cdots + \overline{w_{k-1}w_k} + P_k$, then $\text{ind}(c; P) = \text{ind}(d; P)$, since $f(C_2)$ joins c and d and is bounded away from P. Now $Q = \overline{w_1w_2} + \overline{w_2w_3} + \cdots + \overline{w_{k-1}w_k} + \overline{w_kw_1}$, so it follows directly from the definition of index that $\text{ind}(x; Q) = \text{ind}(x; P) - \sum_{j \text{ even}} \text{ind}(x; P_j + \overline{w_{j+1}w_j})$, for $x = c$ and $x = d$. Thus $\text{ind}(c; Q) = \text{ind}(d; Q)$. But $\text{ind}(c; Q) = 1$ and $\text{ind}(d; Q) = 0$. Hence, there exist i and j such that $\text{ind}(w_i; J) \neq \text{ind}(w_j; J)$.

Theorem 2 also shows that J is the common boundary of the inside points and the outside points.

COROLLARY. *Every point on J is a limit of points inside (outside) J. Conversely, if a is a limit of inside (outside) points, but a is not an inside (outside) point, then a is on J.*

PROOF. The first statement follows immediately from Theorem 2. To prove the second, we note that a cannot be a positive distance from J, for then a, being a limit of inside points, would be inside, by Proposition 4. Let M_n be a $0.3/n$-mesh on J, for $n = 1, 2, 3, \cdots$. Then there is a $y_n = f(x_n)$ in M_n, such that $|a - y_n| < 1/n$, lest $|a - y_n| > 0.5/n$ for all y_n in M_n, and hence $|a - y| > 0.2/n$ for all y on J. The y_n form a Cauchy sequence on J converging to a. Thus the x_n form a Cauchy sequence on the unit circle C. If x is the limit of the x_n, then x is on C and $f(x) = a$, so a is on J.

6. **The arc.** A slight modification allows us to show that a Jordan arc does not separate the plane. If J is a Jordan arc, and a is a point off J, then we *define* $\text{ind}(a; J)$ to be 0. A δ-mesh on J is defined to be the image of a sequence of points $0 = z_1 < z_2 < \cdots < z_n = 1$, where $z_{i+1} - z_i < \omega(\delta)$, for $1 \leqq i \leqq n - 1$. To show that the lemma then holds with J a Jordan arc and $\delta < \omega(1)$, we observe, in the penultimate sentence of the proof, that the case of a circular permutation of $y_1, \cdots, y_n$, or of $y_n, \cdots, y_1$, cannot occur because $|y_1 - y_n| \geqq \omega(1)$ is too large. Then Theorem 1 holds with J a Jordan arc, so every pair of points off J can be connected by a polygonal path that is bounded away from J.

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