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SEPARABLE ALGEBRAS OVER PRUFER DOMAINS

JOSEPH A. WEHLEN

ABSTRACT. Employing sheaf-theoretic techniques and idempotent lifting properties, the following extension of the Wedderburn Principal Theorem is shown: If A is any finitely generated algebra over an almost Dedekind domain R such that A modulo its Baer lower radical is separable over R, then A contains a separable subalgebra S which adds with the lower radical as R-modules to give A.

Certain structural results concerning separable algebras over arbitrary Prufer domains are obtained. These parallel results already known for Dedekind domains.

Finally, results relating the Hochschild and the weak global dimensions of an algebra with the weak global dimension of the ground ring are obtained. The ground rings include some Prufer domains and their generalizations.

The purpose of this article is to examine the structure of commutative and non-commutative algebras over Prufer domains, and in particular over almost Dedekind domains. We shall prove for these last rings a generalization of the Wedderburn Principal Theorem.

In § 1, it will be shown that every finitely generated, torsion-free, separable commutative algebra S over a Prufer domain R is again a Prufer domain. If R is an almost Dedekind domain, then so also is S.

In § 2, similar results will be obtained for non-commutative separable algebras over these same types of ground ring. Moreover, since a finitely generated, torsion-free algebra over a Prufer domain is projective, one may write any separable algebra over a Prufer domain as a direct sum of the torsion ideal and the projective ideal. This decomposition can be "lifted" to any finitely generated algebra A which is separable modulo its lower radical. An application of sheaf-theoretic techniques as in [3] to this decomposition obtains the Wedderburn-like structure theorem for almost Dedekind domains.

In § 3, we will give results relating the Hochschild dimension and the weak global dimension of a finitely generated, torsion-free algebra

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over an almost Dedekind domain and certain generalizations. These extend results of [17] concerning noetherian rings.

CONVENTIONS. All rings, modules, and ring homomorphisms shall be unitary. All homological dimensions shall be left dimensions unless otherwise specified.

The Hochschild dimension of an R-algebra A (R-dim A) is the left projective dimension of A considered as an $A^e = A \otimes_R A^{op}$-module. An R-algebra A is called separable if R-dim $A = 0$.

By a Prufer domain we shall mean a not necessarily noetherian domain over which every finitely generated torsion-free module is projective. A Bezout domain is a Prufer domain in which every finitely generated ideal is principal. An almost Dedekind domain is a Prufer domain in which every proper prime ideal is maximal; equivalently, it is a domain each of whose localizations is a DVR.

By a finitely generated or projective R-algebra A we shall mean that A is finitely generated or projective as an R-module. The unqualified word ideal shall mean "two-sided ideal." $L(A)$ shall denote the (Baer) lower radical of the ring A; $J(A)$, its Jacobson radical. A commutative ring is called connected if it has no idempotents but 0 and 1.

1. **Commutative Algebras.** We begin the study of commutative algebras over Prufer domains with a theorem which is the analogue of [**11**, Theorem 4.3, page 473].

THEOREM 1.1. *Let R be a Prufer domain and S be a finitely generated, commutative, separable R-algebra. Then $S = S^* \oplus t(S)$, where S^* is a direct sum of projective Prufer domains over R and $t(S)$ is a finitely generated torsion algebra, faithful over* $R/\text{annih}_R(t(S))$ (cf. [**6**, page 6]).

PROOF. Clearly $S/t(S)$ is a finitely generated, torsion-free R-module and as such must be projective over R; hence, by the separability of S, $S/t(S)$ is S-projective [**6**, Proposition 2.3, page 48]. Denote this by S^*. Therefore, the following exact sequence splits as R-algebras: $0 \to t(S) \to S \to S^* \to 0$. We may clearly assume that S^* is a connected ring for the purposes of the proof. Now $S^* \otimes Q$ is a separable commutative algebra over the field of quotients Q of R. Therefore it must be a direct sum of fields. Let e be a primitive idempotent in $S^* \otimes Q$. Define the function f by: $f(s) = (s \otimes 1)e$.

Now $(S^* \otimes 1)e$, being finitely generated and torsion-free over R, is projective and so we have also the split exact sequence:

$$0 \to \ker(f) \to S^* \to (S^* \otimes 1)e \to 0.$$

Since S^* is connected, $\ker(f)$ must be zero [6, Corollary 2.6, page 96], and hence $S^* \otimes Q$ is also connected. Thus S^* is a domain.

To see that S^* has the Prufer property, we need only take I to be a finitely generated ideal of S^*. Since I is a finitely generated torsion-free R-module, I is R-projective and hence S^*-projective by the separability of S^*.

COROLLARY 1.1.1. *A finitely generated, torsion-free, separable, connected commutative algebra S over a Prufer domain R is again a Prufer domain.*

With a slight additional effort one may obtain the following result when R is an almost Dedekind domain:

THEOREM 1.2. *If R is an almost Dedekind domain, every finitely generated, torsion-free, separable, connected commutative R-algebra is an almost Dedekind domain.*

PROOF. Since S is torsion-free over R, S_p is a finitely generated, projective, separable R_p-algebra for every prime and hence maximal ideal of R. This entails that S_p/pS_p be a direct sum of fields; since R_p is a DVR, S_p must be a semi-local Dedekind domain; therefore, pS is the intersection of maximal ideals of S. Now if P is a non-zero, non-maximal prime of S, then for some prime p of R, P_p would be a non-zero, non-maximal prime in a Dedekind domain. This is impossible.

It is known that not every finitely generated, torsion-free, separable algebra over a Bezout domain is a Bezout domain. This is shown by the following example due to Pierre Samuel [15, Corollary to Theorem 5.1, page 16]: Let $R = Q[X, Y]/(X^2 + 2Y^2 + 1)$ and $S = R \otimes_Q Q[i]$, where Q denotes the field of rational numbers. Then R is a principal ideal domain, S is a finitely generated, projective and separable R-algebra which is a Dedekind domain, but not a PID. However, by Corollary 1.1.1 and [13, Theorems 63 and 64, pages 38–9], the following is true:

THEOREM 1.3. *A finitely generated, torsion-free, separable commutative algebra S over a semi-local Bezout domain is again a semi-local Bezout domain.*

In order to obtain further information about separable algebras over Prufer rings, one must study the structure of algebras over the proper homomorphic images of a Prufer domain R. We present only one result in this direction as a complete characterization is at present impossible due to the lack of knowledge about homomorphic images of Prufer domains.

THEOREM 1.4. *If R is an almost Dedekind domain and S is a finitely generated, commutative, separable R-algebra such that $R/\text{annih}_R(S)$ is semi-prime, then S is an absolutely flat ring.*

PROOF. Since $R/\text{annih}_R(S)$ is semi-prime and every prime ideal is maximal, $R/\text{annih}_R(S)$ is absolutely flat by [13, exercise 32, page 64]. Then by [7, Theorem 1, page 4] the result follows.

2. **Non-commutative algebras.** We begin with three basic theorems about general separable algebras over the various types of Prufer domains. $Z(A)$ shall denote the center of the algebra A.

THEOREM 2.1. *If R is a Prufer domain and A is a finitely generated, torsion-free R-algebra which is R-separable, then every finitely generated left (or right) ideal of A is A-projective, A is a semi-prime ring which is a direct sum of prime rings.*

PROOF. That every finitely generated ideal I of A is A-projective follows directly from the fact that it is finitely generated and torsion-free over R and hence R-projective. By the separability of A over R, I is then projective over A [6, Proposition 2.3, page 48]. By Theorem 1.1, we may assume that $R = Z(A)$. In this case, the Morita Theory [6, pages 54–55] yields that (0) is a prime ideal.

THEOREM 2.2. *Let R be an almost Dedekind domain and A be a finitely generated, torsion-free R-algebra which is R-separable. A is a semi-prime ring in which every non-maximal prime ideal is a direct summand of A and in which every finitely generated left (or right) ideal is projective. A is a direct sum of prime rings.*

THEOREM 2.3. *A central separable algebra A over a Bezout domain R again has the property that every finitely generated ideal is principal.*

The Theorems 2.2 and 2.3 are both direct consequences of the Morita Theory [6, pages 54–55]. In 2.3, the author points out that in case the Bezout domain R is semi-local he knows of no finitely generated one-sided ideal in A which is not principal.

We now obtain the direct sum decomposition of a non-commutative algebra A in terms of idempotents lifted from $A/L(A)$.

THEOREM 2.4. *If A is a finitely generated algebra over a Prufer domain R such that $A/L(A)$ is R-separable, then there exist idempotents e_p and e_t in A such that $A = e_pAe_p \oplus e_pAe_t \oplus e_tAe_p \oplus e_tAe_t$ with the property that $e_pAe_p/e_pL(A)e_p$ is R-projective and $e_pAe_t \oplus e_tAe_p \oplus e_tAe_t \subset t(A)$, the torsion submodule of A.*

PROOF. Since $A/L(A)$ is R-separable, it follows from Theorem 1.1 that there are central idempotents e_p and e_t such that $e_p(A/L(A))$ is the projective subalgebra and $e_t(A/L(A))$ is the torsion subalgebra of $A/L(A)$. Since $L(A)$ is a nil ideal, A is an SBI-ring [cf., **10**, page 54] with respect to $L(A)$. Whence the idempotents lift to give the Pierce decomposition described.

The preceding result enables one to determine special cases in which a finitely generated algebra A over a Prufer domain R, for which $A/L(A)$ is R-separable, contains a separable subalgebra S such that $S + L(A) = A$ as R-modules. The subalgebra S is called an *inertial subalgebra* of A. If every finitely generated R-algebra A with $A/L(A)$ R-separable contains an inertial subalgebra, R is said to be a *weak inertial coefficient ring* (WIC-ring). R is said to satisfy the *WIC-uniqueness statement* if for every finitely generated algebra A with $A/L(A)$ separable over R with inertial subalgebras S and S' there is an element n in $L(A)$ such that $S = (1 - n)S'(1 - n)^{-1}$. The concept of WIC-ring was defined in [**18**]; in [**9**, Corollary 3 to Theorem 1], E. C. Ingraham has shown that every noetherian ring is a WIC-ring for which the WIC-uniqueness statement holds.

To prove that every almost Dedekind domain is a WIC-ring, we shall employ the sheaf-theoretic techniques used by W. C. Brown in [**3**] and as described by R. S. Pierce in [**14**]. All undefined terms will be as in [**14**].

Let $X(R)$ denote the decomposition space of the commutative ring R, i.e., the set of maximal ideals of the Boolean ring of idempotents of R. For each x in $X(R)$, we shall denote by R_x the ring R/xR.

LEMMA 2.5. *Let R be an almost Dedekind domain and A be a finitely generated algebra over R with $A/L(A)$ separable and torsion-free over R, then $L(A)$ is nilpotent.*

PROOF. Let $S = R/I$ denote R modulo the annihilator of $t(A)$. Then I is a product of powers of the maximal ideals of R. As noted before, $S/L(S)$ is an absolutely flat ring and since $L(S)$ is nil, the natural homomorphism $f\colon S \to S/L(S)$ when restricted to $B(S)$, the set of idempotents in S considered as a Boolean ring, induces an isomorphism of $B(S)$ onto $B(S/L(S))$. Therefore, $X(S) = X(S/L(S))$. Taking into account the structure of I, one may check that for each y in $X(S)$, S_y is a complete local ring with nilpotent maximal ideal.

Now it is simple to check that since $L(A)$ is an R-direct summand of A, that $L(A)^t \subset t(L(A))$ for some integer t. Let $B = A/IA$. By the preceding paragraph, for each y in $X(S)$, there is an $n(y)$ such that each product of $n(y)$ generators of $L(A)$ is zero in B_y. Hence there is a neighborhood $N(y)$ on which these equations hold for each z in $N(Y)$. Using the partition property, there are finitely many open and closed sets $N^*(y_i)$ covering $X(S)$ for which the equations hold. Set $v = \max n(y_i)$. Since the product of any v generators of $L(A)$ is zero for each x in $X(S)$, each product of v generators of $L(A)$ is zero in B. Hence in A the product of any tv generators of $L(A)$ is zero and so $L(A)$ is nilpotent.

Thus we are now able to prove the first part of the structure theorem.

THEOREM 2.6. *Let R be an almost Dedekind domain and A a finitely generated, R-algebra with $A/L(A)$ separable and torsion-free over R. A contains an inertial subalgebra S such that* $S \cap L(A) = (0)$.

PROOF. The proof is in two steps. Suppose first that $L(A)^2 = (0)$. Since $A/L(A)$ is also finitely generated and R-torsion-free (hence, R-projective), the existence is guaranteed by [1, Proposition 12, page 16] and [5, Lemma 1, page 79].

For the general case, one first notes that $A/L(A)^2$ contains an inertial subalgebra $\bar{S}_1$ by the above. Thus there is a subalgebra S_1 of A containing $L(A)^2$ such that $S_1 + L(A) = A$. Moreover, since $S_1/L(A)^2 = A/L(A)$, $L(A)^2 = S_1 \cap L(A) = L(S_1)$. Assume now that for $k = 1, \cdots, n$, one has constructed S_k so that: $S_k + [L(A)]^{2^k} = S_{k-1}$ and $S_k \cap [L(A)]^{2^k} = [L(A)]^{2^{k+1}}$ with $S_k/[L(A)]^{2^{k+1}}$ separable over R.

We now construct an S_{n+1} with the same properties. For convenience, set $K = [L(A)]^{2^n}$. Since S_n/K is separable and projective over R, there is a subalgebra S_{n+1} of S_n such that $S_{n+1} + K = S_n$ and S_{n+1}/K^2 is R-separable and $S_{n+1} \cap K = K^2$.

Thus we have defined a sequence S_k with the above properties. Since $L(A)$ is nilpotent, there is an r such that $[L(A)]^{2^r} = (0)$ and so the sequence becomes constant with S_r. Hence, S_r is separable over R and $S_r + L(A) = A$. Therefore, S_r is an inertial subalgebra of A.

THEOREM 2.7. *An almost Dedekind domain R is a weak inertial coefficient ring for which the WIC-uniqueness statement holds.*

PROOF. By Theorem 2.4 we may separately consider the algebras e_pAe_p and e_tAe_t. By Theorem 2.6, e_pAe_p contains an inertial subalgebra S_p.

Let S denote the proper homomorphic image of R so that e_tAe_t is a faithful S-algebra. Now again, each stalk S_y of S (y in $X(S)$) is a complete local ring with nilpotent radical and so by a slight alteration of the proof of [3, Theorem 1, page 369], we may obtain the existence of an inertial subalgebra S_t of e_tAe_t. Since $L(A) = e_pL(A)e_p \oplus e_pAe_t \oplus e_tAe_p \oplus e_tL(A)e_t$, [10, page 54], one sees that $S_p \oplus S_t$ is the desired inertial subalgebra.

The fact that R satisfies the WIC-uniqueness statement now follows from [9, Corollary 1 to Theorem 3].

The author knows of no finitely generated algebra A over a Prufer domain R with $A/L(A)$ R-separable which does not contain an inertial subalgebra. In fact, it is proved in [9, Theorem 4] that if A is commutative, then A contains an inertial subalgebra; if $L(A)$ is nilpotent it is simple to use the proof of 2.6 and 2.7 to show the existence of an inertial subalgebra for any finitely generated A.

By an unpublished result of E. C. Ingraham, it is known that any almost Dedekind domain R, which is the union of a chain of inertial coefficient rings (IC-rings) is again an IC-ring (R is an IC-ring if every finitely generated algebra which is separable modulo its Jacobson radical contains an inertial subalgebra.) It appears that an almost Dedekind domain R is an IC-ring if and only if the Jacobson radical of R is zero. Some evidence for the truth of this conjecture is supplied by [9, Theorem 4].

3. **Weak global dimension and Hochschild dimension.** In this section we give results analogous to those of [17] and which depend on those results for their proof. Moreover, we generalize these new results further to a special type of locally noetherian ring.

We say that a ring R is *weakly globally isodimensional* if w. gl dim R = w. gl dim R_m for each maximal ideal m of R. We say that an R-algebra A is *cohomologically isodimensional* if R-dim $A = R_m$-dim A_m for each maximal ideal m of R.

Theorem 3.1. *Let R be an almost Dedekind domain and A be a finitely generated, torsion-free R-algebra of finite Hochschild dimension. Then the following are true*:

(a) w. gl dim $A = R$-dim $A + 1$,
(b) R-dim $A^e = 2\,(R$-dim $A)$,
(c) w. gl dim $A^e = 2\,(R$-dim $A) + 1$,
(d) If $A/L(A)$ is R-separable, w. $\mathrm{hd}_A(A/L(A)) = R$-dim A.

Proof. We will use the well-known result that w. gl dim $A = \sup$ (w. gl dim A_m). Since A_m is noetherian for every maximal ideal of an almost Dedekind domain, we recall w. gl dim A_m = gl dim A_m. Thus [17, Theorem 8, p. 77] applies. For (a),

$$\begin{aligned}\text{w. gl dim } A &= \sup(\text{w. gl dim } A_m)\\ &= \sup(R_m\text{-dim } A_m + \text{gl dim } R_m)\\ &= \sup(R_m\text{-dim } A_m) + 1\\ &= R\text{-dim } A + 1\,,\end{aligned}$$

where the last equality follows from [16, Theorem 2.1, page 129].

(b) follows directly from [16, Theorem 2.1, page 129] and [4, IX, Proposition 2.6, page 166]. Finally, if $A/L(A)$ is R-separable, then $[A/L(A)]/m[A/L(A)] = (A/mA)/J(A/mA)$; hence, (d) follows from the arguments of [16, Theorem 2.1, page 129] whose application leads to the string of equalities:

$$\begin{aligned} \text{w. hd}_A\,(A/L(A)) &= \sup \text{w. hd}_{A/mA}([A/L(A)]/m[A/L(A)]) \\ &= \sup\,(R/m\text{-dim } A/mA) \\ &= R\text{-dim } A. \end{aligned}$$

COROLLARY 3.1.1. *If A and R are as in the theorem, A is cohomologically isodimensional if and only if A is weakly globally isodimensional.*

We may extend the results of Theorem 3.1 to a class of rings we shall call locally regular. A commutative ring R is said to be *locally regular* if for each maximal ideal m of R, R_m is a regular local ring, and if R has finite weak global dimension. If R is noetherian, then R is nothing more than a regular ring of finite global dimension. Clearly every almost Dedekind domain is locally regular.

Since the results follow easily from [17, Theorem D, page 78], Theorem 3.1, and a standard induction argument, we will omit the proof.

THEOREM 3.2. *Let R be a locally regular ring and let A be an R-algebra which is an R-progenerator of finite Hochschild dimension. Suppose further that either R is weakly globally isodimensional or that A is cohomologically isodimensional.*

(a) w. gl dim $A =$ R-dim $A +$ w. gl dim R,

(b) w. gl dim $A^e = 2(R\text{-dim } A) +$ w. gl dim R,

(c) w. gl dim $A^e =$ w. gl dim R *if and only if A is R-separable,*

(d) *If $A/L(A)$ is R-separable and torsion-free, then* w. $\text{hd}_A(A/L(A)) = R$-dim A.

THEOREM 3.3. *Let R be a locally regular domain, A a finitely generated, torsion-free, commutative, separable R-algebra. A is flat as an R-module. If, in addition, A is R-projective, then A is a locally regular ring of the same weak global dimension.*

PROOF. The last part follows from the properties of separable algebras over regular local rings [cf., 17, Theorem B, page 77; 11, Lemma 4.1, page 473]. To show the first part, we notice that for each maximal ideal m of R, A_m is a finitely generated, torsion-free, separable R_m-algebra and hence by the result cited above A_m is R_m-free. Thus we have the following chain of equalities: $\text{w. hd}_R(A) = \sup(\text{w. hd}_{R_m}(A_m)) = 0$. Thus A is R-flat.

This result may be viewed as a non-noetherian extension of [**11**, Lemma 4.1, page 473].

COROLLARY 3.3.1. *Let R be a locally regular domain, A a finitely generated, torsion-free, separable R-algebra. Then A is R-flat.*

We conclude the paper with some results which relate the Hochschild dimension and the weak global dimension of algebras over local Prufer domains with principal maximal ideal. The results follow from a non-commutative generalization of some work of Chr. U. Jensen in [**12**].

We define an ideal x of a commutative ring R to be t-faithfully flat (projective) if there is a sequence of proper ideals

$$x_1 \subsetneq x_2 \subsetneq \cdots \subsetneq x_{t-1} \subsetneq x_t = x,$$

such that x_i/x_{i-1} is a faithfully flat (projective) R/x_{t-1}-module.

THEOREM 3.4. *Let A be an R-algebra which is finitely generated and free as an R-module.*

(a) *Let x be a t-faithfully flat ideal of R. If* $B \neq 0$ *is a right* A/xA-*module of finite weak dimension, then* $\text{w. hd}_A(B) = \text{w. hd}_{A/xA}(B) + t$.

(b) *Let x be a t-faithfully projective ideal of R. If* $B \neq 0$ *is a right* A/xA-*module of finite projective dimension, then* $\text{hd}_A(B) = \text{hd}_{A/xA}(B) + t$.

As the proofs are quite similar, we will prove only part (a).

PROOF. We follow the proof of [**12**, Theorem 4, page 398], adapting it to the algebra situation.

It is well-known that $\text{w. hd}_A(B) \leqq \text{w. hd}_{A/xA}(B) + \text{w. hd}_A(A/xA)$. By an inductive argument quite similar to that of [**17**, Theorem B, page 77], we may reduce to the case where x is R-faithfully flat, so that $\text{w. hd}_A(B) \leqq \text{w. hd}_{A/xA}(B) + 1$.

Now since $\text{w. hd}_A(A/xA) = 1$, using the spectral sequence (3) of [**12**, page 398], $E^2_{p,q} = \text{Tor}_p^{A/xA}(B, \text{Tor}_q^A(A/xA, C)) = 0$ for $q > 1$. Therefore, $E^2_{p,q} = E^\infty_{d,1}$ which is a subquotient of $\text{Tor}^A_{d+1}(B, C)$ where $d = \text{w. hd}_{A/xA}(B)$.

Since $0 \to xA \to A \to A/xA \to 0$ is exact, for any left A/xA-module C we have: $0 = \text{Tor}_1{}^A(A, C) \to \text{Tor}_1{}^A(A/xA, C) \to xA \otimes_A C \to A/xA \otimes_A C \to 0$ is exact. Therefore

$$\text{Im}(\text{Tor}_1{}^A(A/xA, C) \to xA \otimes_A C) = \text{Ker}(xA \otimes_A C \to C) = xA \otimes_A C,$$

since $A \otimes_A C = A/xA \otimes_A C$. Whence $\mathrm{Tor}_1{}^A(A/xA, C) = xA \otimes_A C$.
Now

$$\begin{aligned} B \otimes_{A/xA}(xA \otimes_A C) &= (B \otimes_{R/x} R/x) \otimes_{A\otimes_R R/x} (xA \otimes_A C) \\ &= (B \otimes_A xA) \otimes_{R/x \otimes_R A^m} (R/x \otimes_{R/x} C) \\ &\quad \text{(by the middle-four interchange)} \\ &= (B \otimes_A xA) \otimes_{A/xA} C. \end{aligned}$$

Whence $\mathrm{Tor}^{A/xA}(B, xA \otimes_A C) = \mathrm{Tor}^{A/xA}(B \otimes_A xA, C)$. Therefore $E^2_{d,1} = \mathrm{Tor}_d{}^{A/xA}(xA \otimes_A B, C)$.

Now, by the usual argument based on [2, Chapter 1, Proposition 4, page 47], we have that $\mathrm{w.\,hd}_{A/xA}(B \otimes_A xA) = \mathrm{w.\,hd}_{A/xA}(B)$; therefore, there exists a left A/xA-module C such that $\mathrm{Tor}_d{}^{A/xA}(xA \otimes_A B, C) \neq 0$ since $\mathrm{w.\,hd}_{A/xA}(B) = d$ by assumption. But, then, $E^2_{d,1} \neq 0$ implies that $\mathrm{Tor}^A_{d+1}(B, C) \neq 0$. This entails the required equality.

THEOREM 3.5. *Let R be a local ring with maximal ideal m which is t-faithfully flat and with* w. gl dim $R = t$. *Let A be a finitely generated, projective, faithful R-algebra of finite Hochschild dimension.*

(a) w. gl dim $A = R$-dim $A +$ w. gl dim R,

(b) w. gl dim $A =$ w. $\mathrm{hd}_A(A/N)$,

(c) w. gl dim $A^e = 2(R\text{-dim } A) +$ w. gl dim R,

(d) *A is R-separable if and only if* w. gl dim $A^e =$ w. gl dim R.

PROOF. By 3.4(2), we have that

$$\begin{aligned} \mathrm{w.\,hd}_A(A/N) &= \mathrm{w.\,hd}_{A/mA}(A/N) + \mathrm{w.\,gl\,dim}\,R \\ &= \mathrm{w.\,gl\,dim}\,A/mA + \mathrm{w.\,gl\,dim}\,R \\ &= R\text{-dim}\,A + \mathrm{w.\,gl\,dim}\,R \\ &\geqq \mathrm{w.\,gl\,dim}\,A. \end{aligned}$$

The last two inequalities follow from [16, Theorem 2.1, page 129] and [17, Proposition 1, page 76] respectively. Thus equality must hold, which proves both (a) and (b). (c) follows as in [17, Theorem C, page 78] by means of [4, IX, Proposition 2.6, page 166]; (d) is a corollary of (c).

One should notice that in case m is t-faithfully projective, then $\mathrm{w.\,hd}_A(A/N) = \mathrm{hd}_A(A/N)$. If, however, the ring R is assumed to be non-noetherian, then gl dim A is in general strictly greater than $\mathrm{hd}_A(A/N)$. This is particularly obvious in the case of a valuation domain.

The conclusions of Theorem 3.5 remain valid if we replace the assumption that R is local by the following assumptions: (a) R is a commutative ring of finite weak global dimension; (b) that for each maximal ideal m of R, m is t-faithfully flat and w. gl dim $R_m = t$; (c) A/N is separable over R; and (d) R is weakly globally isodimensional or A is cohomologically isodimensional.

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Ohio University, Athens, Ohio 45701