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ω-SEMIGROUPS

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§1. **Introduction.** Let ϵ stand for the set of non-negative integers (*numbers*), V for the class of all subcollections of ϵ (*sets*), Λ for the set of isols, and Ω for the class of all recursive equivalence types (RET). The relation of inclusion is denoted $\subset$, α recursively equivalent to β by $\alpha \simeq \beta$, for sets α and β, and the RET of α by Req (α). For the purpose of this paper we say a *semigroup* is an ordered pair (α, p), where (i) $\alpha \subset \epsilon$ and (ii) p is a semigroup operation (i.e., an associative binary multiplication) on $\alpha \times \alpha$. An *ω-semigroup* is a semigroup (α, p), where p can be extended to a partial recursive function of two variables. The concept of an ω-semigroup is a recursive analogue of a semigroup and is a generalization of an ω-group. In this paper, the author shows (T2), that there are ω-semigroups which are groups but not ω-groups; but that all periodic ω-semigroups which are groups, are ω-groups (T1). Theorems T3, T5, T6, T7, and T11 give conditions for an ω-semigroup to be an ω-group. The recursive analogues of regular semigroup, inverse semigroup, and right group [ω-regular ω-semigroup, inverse ω-semigroup, and ω-right group] are studied in sections §5, §6, §8 respectively, with particular attention paid to $T(\alpha)$, the analogue of the regular semigroup of all mappings from α into α, and $I(\alpha)$, the analogue of the symmetric inverse semigroup on α. Theorems T17 and T22 relate ω-regular ω-semigroups to ω-groups and T28 relates ω-regular ω-semigroups to inverse ω-semigroups. In T42 and T43 we have two nice characterizations of an ω-right group and T45 shows that a periodic ω-semigroup that is a right group is an ω-right group. Finally section §7 gives a brief introduction to the ω-homomorphism theory of ω-semigroups. The author wishes to thank the referee for his helpful suggestions.

§2. **Basic concepts and notations.** The reader of this paper is assumed to be familiar with the notation and basic results of [2], [6], and [7].

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REMARK. We recall from [7] that an *ω-group* is an ordered pair (α, p) where (i) $\alpha \subset \epsilon$, (ii) p is a group operation on $\alpha \times \alpha$ which can be extended to a partial recursive function of two variables, (iii) the function which takes x to its inverse under p has a partial recursive extension.

REMARK. We recall from [6] that $R = \mathrm{Req}\,(\epsilon)$ and Ω_R is the set of all regressive RET's.

REMARK. As for ω-groups, for $S = (\alpha, p)$, an ω-semigroup, we usually denote $p(x, y)$ by $x \cdot y$ or just xy, for $x, y \in \alpha$. Also x^n is defined by $x^1 = x$ and $x^{n+1} = x^n \cdot x$, for $x \in \alpha$.

DEFINITION. If $S = (\alpha, p)$ is an ω-semigroup, then the *order* of S, [written: $o(S)$] is $\mathrm{Req}\,(\alpha)$.

DEFINITION. Let $S = (\alpha, p)$ be an ω-semigroup. Then
(i) S is an *r.e. semigroup* if $o(S) = R$,
(ii) S is an *isolic semigroup* if $o(S) \in \Lambda$,

DEFINITION. A function ϕ from an ω-semigroup S_1 onto an ω-semigroup S_2 is an *ω-isomorphism* if ϕ is a semigroup isomorphism which has a one-to-one partial recursive extension.

DEFINITION. An ω-semigroup S_1 is *ω-isomorphic* to an ω-semigroup S_2 [written: $S_1 \cong_\omega S_2$] if there exists an ω-isomorphism mapping S_1 onto S_2.

REMARK. If $S_1 \cong_\omega S_2$, then $o(S_1) = o(S_2)$.

REMARK. Any subsemigroup of an ω-semigroup is clearly an ω-semigroup.

NOTATION. For an ω-semigroup S, we write $H \leqq S$ to denote that H is a subsemigroup of S.

DEFINITION. A subsemigroup $H = (\beta, q)$ of an ω-semigroup $S = (\alpha, p)$ is a *recursive subsemigroup* of S [written: $H \leqq_{\mathrm{rec}} S$] if $\beta \mid \alpha - \beta$, i.e., if β is separable from $\alpha - \beta$.

REMARK. We recall from semigroup theory that a semigroup S is *periodic* if every element has finite order, i.e., for every $x \in S$, the cyclic semigroup generated by x is finite. We note that every isolic semigroup is periodic.

§3. ω-Semigroups and ω-groups.

THEOREM T1. *If S is a periodic ω-semigroup and is also a group then S is a periodic ω-group.*

PROOF. Since $S = (\alpha, p)$ is an ω-semigroup and a group then $\alpha \subset \epsilon$ and p is a group operation on $\alpha \times \alpha$ which can be extended to a partial recursive function of two variables. Thus it suffices to show that the function mapping x to its inverse under p, x^{-1}, for $x \in \alpha$ has a partial recursive extension. But, S is a periodic group. Therefore given $x \in \alpha$, $x \neq e$, there exists $n \in \epsilon - \{0\}$ such that $x^n = e$, where e is the identity of S. Hence $x^{-1} = x^{n-1}$. We can now easily see that x^{-1} has a partial recursive extension.

COROLLARY. *If $S = (\alpha, p)$ is an isolic semigroup which is a group then S is an isolic group.*

PROOF. This is immediate from the fact that every isolic semigroup is periodic.

The statement of Theorem 1 leads one to ask if every ω-semigroup which is a group is also an ω-group. The answer to this question is no as is shown by Theorem 2 which is due to Hassett.

REMARK. We need the recursive mappings j, k, ℓ defined by: $j(x, y) = x + (x + y)(x + y + 1)/2$, $j[k(n), \ell(n)] = n$. We recall that j maps ϵ^2 one-to-one, onto ϵ.

THEOREM T2. *There exists an ω-semigroup that is a group but not an ω-group.*

PROOF. Let α be an r.e. set which is not recursive and $a(x)$ be a one-to-one recursive function ranging over α. Assume without loss of generality that $a(0) = 0$ and $a(1) = 1$. For each number $n > 1$, we define a cyclic group C_n as follows:

(a) If $n \in \alpha$, $(\exists k)[a(k) = n]$. Let C_n be a cyclic group of order k, $\{x, \cdots, x^{k-1}, x^k = e\}$ and encoded by:

$$x \leftrightarrow j(n, 0)$$
$$x^2 \leftrightarrow j(n, 2)$$
$$\vdots$$
$$x^{k-1} \leftrightarrow j(n, 2k - 4)$$
$$e \leftrightarrow 1$$

(b) If $n \notin \alpha$, C_n is an infinite cyclic group with generator z and encoded by:

$$\begin{array}{c} \vdots \\ z^m \leftrightarrow j(n, 2(m-1)) \\ \vdots \\ z^2 \leftrightarrow j(n, 2) \\ z \leftrightarrow j(n, 0) \\ e \leftrightarrow 1 \\ z^{-1} \leftrightarrow j(n, 1) \\ z^{-2} \leftrightarrow j(n, 3) \\ \vdots \\ z^{-m} \leftrightarrow j(n, 2m-1) \\ \vdots \end{array}$$

Let G be a recursive direct product of the C_n, i.e., $G = \otimes_{n \in \epsilon - \{0,1\}} C_n$ where $x \in G$ if and only if x is a Gödel number of a member of the direct product of the C_n's which has only a finite number of coordinates which are different from 1. If we can show that the mulitplication in G is effective we will be done, since then G will be an ω-semigroup that is a group but not an ω-group. For, we have for $n > 1$,

$$n \in \alpha \iff \text{the inverse of } j(n, 0) \text{ is not } j(n, 1).$$

Thus if we could effectively find inverses for elements in G, we could effectively decide if $n \in \alpha$, for $n \in \epsilon$. This is impossible since α is not recursive.

In order to show multiplication is effective in G, it suffices to consider the effectiveness of coordinate multiplication, i.e., it suffices to show that multiplication is effective in C_n, for an arbitrary n. Thus let $j(n, k)$ and $j(n, w)$ be given:

(a) If k is even and w is odd we know that we are in a C_n which is infinite cyclic. Then $j(n, k) \leftrightarrow z^{k/2}$ and $j(n, w) \leftrightarrow z^{-(w+1)/2}$ and the product of $j(n, k)$ and $j(n, w)$ is the code number of $z^{[k-(w+1)]/2}$.

(b) If both k and w are odd, then C_n is infinite cyclic and we proceed as in (a).

(c) If both k and w are even, then no matter whether C_n is finite or infinite, $j(n, k) \leftrightarrow z^{k/2}$ and $j(n, w) \leftrightarrow z^{w/2}$, where z is the generator of C_n. Thus $j(n, k) \cdot j(n, w)$ is the code number of $z^{(k+w)/2}$; but if C_n is finite we might have to reduce $(k + w)/2$ modulo the order of C_n. However, this is easy. We just compute $a(0), \cdots, a((k + w)/2)$.

If n does not appear in this list, the product is $j(n, (k + w)/2)$, since order $(C_n) > (k + w)/2$. If n does appear, $n = a(j)$ for some $j < (k + w)/2$ and $j =$ order C_n. Then we reduce $(k + w)/2$ modulo j. In either case we can effectively compute $j(n, k) \cdot j(n, w)$.

Since multiplication is effective in C_n and the Gödel numbering in G is effective and one-to-one, it follows that the multiplication in G can be extended to a partial recursive function of two variables. Hence G is an ω-semigroup that is a group but not an ω-group.

DEFINITION. An element i of a semigroup S is an *idempotent* if $i^2 = i$.

REMARK. By [8, p. 113], every element x of a periodic semigroup has a power of itself, x^n, such that x^n is an idempotent.

DEFINITION. A semigroup S is a *left* (*right*) *cancellation* semigroup if for $x, y, z \in S$, $x \cdot y = x \cdot z \Rightarrow y = z$ $(y \cdot x = z \cdot x \Rightarrow y = z)$. A semigroup S is a *cancellation* semigroup if it is both a left and right cancellation semigroup.

THEOREM T3. *A periodic ω-semigroup with left cancellation is an ω-group if and only if it has a single idempotent.*

PROOF. Let S be a periodic ω-semigroup with left cancellation. If S is an ω-group then S has only one idempotent, namely its identity. Conversely if S has a single idempotent then by [8, p. 113], S is a group. Hence by T1, S is an ω-group.

COROLLARY 1. *An isolic semigroup with left cancellation is an isolic group if and only if it has a single idempotent.*

COROLLARY 2. *A periodic ω-semigroup which is not an ω-group cannot be embedded in an ω-group.*

PROOF. Let S be a periodic ω-semigroup. Thus S has an idempotent. If S has more than one idempotent then any group that S is embedded in has more than one idempotent. This is impossible since a group has only one idempotent. Hence if S has more than one idempotent S cannot be embedded in an ω-group. Also if S has a single idempotent then by T3, since S is not an ω-group, S is not left cancellative. Hence there exist $x, y, z \in S$ such that $x \cdot y = x \cdot z$ but $y \neq z$. But then if S is embedded in an ω-group the above will violate the cancellation property in that ω-group. This completes the proof.

The following three examples of ω-semigroups which are not ω-groups together show that the conditions of T3 are all necessary.

Examples. (1) $(\epsilon, +)$ is a non-periodic ω-semigroup which is cancellative and contains a single idempotent.

(2) Define $S = (\alpha, p)$ for $\alpha \subset \epsilon$ by $p(x, y) = k$ for k a fixed element of α and $x, y \in \alpha$. S is a periodic ω-semigroup with a single idempotent, but S does not have a left cancellation property.

(3) Define $S = (\alpha, p)$ for $\alpha \subset \epsilon$ by $p(x, y) = y$ for $x, y \in \alpha$. S is a periodic ω-semigroup with left cancellation. But S has no unique idempotent, since every element of S is an idempotent.

Remark. Every ω-semigroup of the form of example (3) is called a *right zero ω-semigroup.*

Remark. In [7], Hassett showed that there are continuum many (c) RET's which are not the order of an ω-group. This is not the case for ω-semigroups.

Theorem T4. *For every $A \in \Omega$, there exists an ω-semigroup* S *such that $o(S) = A$.*

Proof. Examples (2) and (3) above are such ω-semigroups.

Remark. Since every periodic cancellation semigroup is a group, T1 allows us to prove the following theorem.

Theorem T5. *Every periodic cancellation ω-semigroup is an ω-group.*

Corollary. *Every cancellation isolic semigroup is an isolic group.*

§4. ω-Divisors.

Definition. Let S be an ω-semigroup and P a property of S. We say that an element $y \in S$ can be *effectively found given* $x_1, \cdots, x_n$ (or *from* $x_1, \cdots, x_n$) *such that* $P(x_1, \cdots, x_n, y)$, for $x_1, \cdots, x_n \in S$, $n \geqq 1$, if there exists a function f such that:

(i) δf is the set of all n-tuples $(w_1, \cdots, w_n)$, $w_1, \cdots, w_n \in S$, such that there exists a $z \in S$ for which $P(w_1, \cdots, w_n, z)$,

(ii) for all $(w_1, \cdots, w_n) \in \delta f$, $P(w_1, \cdots, w_n, f(w_1, \cdots, w_n))$,

(iii) $f(x_1, \cdots, x_n) = y$,

(iv) f can be extended to a partial recursive function of n variables.

Remark. The purpose of the above definition is to allow us to talk about certain effective properties of individual n-tuples of elements of S in such a way that if the property holds for all n-tuples of elements of S then it holds uniformly for S. The need for the definition should become clear to the reader as he proceeds through this paper.

REMARK. If the property P is clear from the context, we may just say an element $y \in S$ can be effectively found given $x_1, \cdots, x_n$.

DEFINITION. An element b of an ω-semigroup S is called a *right ω-divisor* of the element $a \in S$ if there exists an $x \in S$ which can be effectively found from a and b such that $x \cdot b = a$. An element $b \in S$ is called a *left ω-divisor* of $a \in S$ if there exists a $y \in S$ which can be effectively found from a and b such that $b \cdot y = a$.

NOTATION. If b is a right (left) ω-divisor of a, we say a is *ω-divisible on the right* (*left*) by b.

REMARK. The following three theorems relate ω-divisibility to ω-groups.

THEOREM T6. *An ω-semigroup S is an ω-group if and only if each of its members is ω-divisible both on the right and the left by every element of* S.

PROOF. Left to the reader.

REMARK. The conditions in T6 for S to be an ω-group can be weakened as follows.

THEOREM T7. *If an ω-semigroup S possesses an element which is both a right and left ω-divisor of every element of S and at the same time, itself is ω-divisible both on the right and left by every element of* S, *then* S *is an ω-group, and conversely.*

PROOF. See [8, p. 46].

REMARK. Suppose that $G_1 = (\alpha_1, p_1)$ and $G_2 = (\alpha_2, p_2)$ are ω-groups, and $\alpha_1 | \alpha_2$. We can define an ω-semigroup $S = (\alpha, p)$ as follows. Let $\alpha = \alpha_1 \cup \alpha_2$ and for $x, y \in \alpha$ define $p(x, y)$ to be $p_1(x, y)$, if $x, y \in \alpha_1$; $p_2(x, y)$, if $x, y \in \alpha_2$, and if $x \in \alpha_1$ and $y \in \alpha_2$ then $p(x, y) = p(y, x) = y$. It is straightforward to show that S is an ω-semigroup and that S has both left and right ω-divisors. Also it is clear that if α_1 and α_2 are non empty, that S is not a group. Let B be the set of all elements of S that are both left and right ω-divisors of every element of S, and let C be the set of all elements of S that are ω-divisible both on the right and the left by every element of S.

THEOREM T8. *Following the above notation, $G_1 = B$ and $G_2 = C$.*

PROOF. See [8, p. 46].

REMARK. We recall from semigroup theory that for $a, b \in S$, S a semigroup, if $a \cdot b = a$ $(b \cdot a = a)$ then a is called a *left* (*right*) *zero* of b

and b is called a *right* (*left*) *unit* of a. Also an element of S that is a two-sided unit (zero) of S (i.e., every element of S) is called a *unit* (*zero*) of S. The following is a theorem of semigroup theory.

THEOREM T9. *A semigroup possesses at most one unit and one zero.*

REMARK. By T9, we also refer to a unit of S as the *identity* of S.

THEOREM T10. *Let* S *be an* ω*-semigroup which possesses left and right* ω*-divisors.*

(i) *A unit of* S *is a right and left* ω*-divisor of every element of* S.

(ii) *Every element of* S *is both a right and left* ω*-divisor of a zero of* S.

(iii) *The zero of* S *is neither a right nor left* ω*-divisor of any element of* S, *except itself.*

PROOF. Left to the reader.

THEOREM T11. *The following three conditions on an* ω*-semigroup* S *are equivalent*:

(i) S *possesses a unit and each element* x *of* S *possesses a multiplicative inverse which can be effectively found given* x.

(ii) S *possesses a right unit* i, *and every element of* S *is a left* ω*-divisor of* i.

(iii) S *is an* ω*-group.*

PROOF. Left to the reader.

§5. ω-Regular ω-semigroups.

REMARK. The following are analogues of the concepts of regular semigroup and completely regular semigroup as used in [8].

DEFINITION. Let S be an ω-semigroup. An element a of S is said to be *ω-regular* if we can effectively find an $x \in$ S, given a, such that $a \cdot x \cdot a = a$.

DEFINITION. An ω-semigroup S is said to be *ω-regular* if every element of S is ω-regular.

REMARK. We see that S is ω-regular if and only if there is a function f defined on S, such that for all $a \in$ S, $a \cdot f(a) \cdot a = a$ and f has a partial recursive extension.

REMARK. We also note that if S has an ω-regular element, then every regular element is ω-regular.

DEFINITION. An element a of an ω-semigroup S is said to be *completely ω-regular,* if we can effectively find, given a, an $x \in$ S such that

$a \cdot x \cdot a = a$ and this x also satisfies the condition that $a \cdot x = x \cdot a$.

REMARK. The above definition of $a \in S$ being completely ω-regular is to be read so that a is ω-regular and in addition $a \cdot x = x \cdot a$.

REMARK. Although this definition may seem to be unnatural and it would appear the definition should read "an element a is completely ω-regular if we can effectively find, given a, an $x \in S$ such that $a \cdot x \cdot a = a$ and $a \cdot x = x \cdot a$," there is a good reason for not using this second definition. By using the second definition, it would be possible to have a completely ω-regular element which is not ω-regular. This would occur for an ω-semigroup S in which there are regular elements a and no general effective procedure for finding an x such that $a \cdot x \cdot a = a$, but for the restricted set of regular elements which are completely regular there is such an effective procedure. The first definition does not have this problem. Also, both definitions are equivalent for ω-semigroups in which all elements are completely ω-regular.

DEFINITION. An ω-semigroup S is said to be *completely ω-regular* if every element of S is completely ω-regular.

REMARKS. (i) For a commutative ω-semigroup S, S is ω-regular if and only if S is completely ω-regular.

(ii) An ω-group is a completely ω-regular ω-semigroup.

(iii) Every completely ω-regular element (ω-semigroup) is an ω-regular element (ω-semigroup).

DEFINITION. A *finite partial* (*f.p.*) *function*, f, is a function such that δf and ρf are finite sets.

REMARK. It is clear that if f is an *f.p.* function then f is a partial recursive function.

NOTATION. Let $q(n)$ denote the $(n + 1)^{\text{st}}$ odd prime, for $n \in \epsilon$.

DEFINITION. Let α and β be sets. A function f is an *f.p. function from β into α*, if f is an *f.p.* function such that $\delta f \subset \beta$ and $\rho f \subset \alpha$.

NOTATION. Let $\alpha \subset \epsilon$. $\xi(\alpha) = \{f \mid f$ is an *f.p.* function from α into $\alpha\}$. If $\alpha = \epsilon$, then $\xi(\alpha) = \xi$.

NOTATION. Let $f \in \xi$. If $\delta f = \emptyset$, then $f^* = 1$. Suppose δf has $n + 1$ members, say $\{x_0, \cdots, x_n\}$, then

$$f^* = 2^{n+1} \prod_{i=0}^{n} q(x_i)^{f(x_i)+1}.$$

REMARK. Let $f, g \in \xi$. If $\rho f \cap \delta g = \emptyset$, then $\delta(g \circ f) = \emptyset$. If $\rho f \cap \delta g \neq \emptyset$, then $\delta(g \circ f) = f^{-1}(\rho f \cap \delta g)$. Hence we have a multiplication of $f, g \in \xi$ by

$$g \circ f(x) = \begin{cases} g(f(x)), \text{ if } x \in \delta(g \circ f), \\ \text{undefined, otherwise.} \end{cases}$$

We see that $g \circ f \in \xi$, if g and f are in ξ. If we are given g and f we can effectively find $g \circ f$. Also the above multiplication in ξ is clearly associative. Given $f \in \xi$, i.e., given $\delta f, \rho f$ and $f(x)$ for each $x \in \delta f$, we can effectively find f^* and vice versa. We see that $f \leftrightarrow f^*$ is a one-to-one Gödel numbering of ξ.

NOTATION. Let $\alpha \subset \epsilon$. We denote the semigroup $(\xi(\alpha), \circ)$ by $\tau(\alpha)$. Also $\xi^*(\alpha) = \{f^* \mid f \in \xi(\alpha)\}$ and the semigroup $(\xi^*(\alpha), \circ)$ is denoted by $T(\alpha)$, where $f^* \circ g^* = (f \circ g)^*$, for $f^*, g^* \in \xi^*(\alpha)$.

REMARK. $T(\epsilon)$ is a universal *r.e.* super semigroup for all $T(\alpha)$, $\alpha \subset \epsilon$.

THEOREM T12. *Let $\alpha \subset \epsilon$. $T(\alpha)$ is an ω-regular ω-semigroup.*

PROOF. Multiplication in $T(\alpha)$ is effective since given $f^*, g^* \in T(\alpha)$, we can effectively find $f, g \in \tau(\alpha)$. Then we can compute $f \circ g$ in $\tau(\alpha)$ and effectively find $(f \circ g)^*$. Thus it follows that $T(\alpha)$ is an ω-semigroup. Given $f \in \tau(\alpha)$ we can effectively find all of its inverse functions such that $\delta f^{-1} = \rho f$, since δf and ρf are finite. We select a unique one by: $f^{-1}(y) = (\mu x)[x \in \delta f \,\&\, f(x) = y]$ for $y \in \rho f$. Clearly $f \circ f^{-1} \circ f = f$, for $f \in \tau(\alpha)$. Thus for $f^* \in T(\alpha)$, $f^* \circ (f^{-1})^* \circ f^* = f^*$. It follows that $T(\alpha)$ is an ω-regular ω-semigroup.

REMARK. For $\alpha \subset \epsilon$, $\tau(\alpha)$ is not a group, since every partial identity function on α, i.e., $f(x) = x$, for $x \in \delta f \subset \alpha$, is an idempotent, and there are $2^{\mathrm{Req}(\alpha)}$ such partial identity functions. Thus $T(\alpha)$ is not an ω-group.

THEOREM T13. *Let $\alpha \subset \epsilon$ and $Req(\alpha) = A$. Then $o(T(\alpha)) = (A + 1)^A$.*

PROOF. We assume without loss of generality that $0 \notin \alpha$. Hence $A + 1 = \mathrm{Req}(\alpha \cup \{0\})$. We recall from [**6**] that $(A + 1)^A = \mathrm{Req}(\alpha \cup \{0\})^\alpha_*$ where $(\alpha \cup \{0\})^\alpha = \{n \mid r_n$ is a finite function from α into $\alpha \cup \{0\}\}$. Define a map ψ from $\xi^*(\alpha)$ into $(\alpha \cup \{0\})^\alpha$ as follows. Let $f^* \in \xi^*(\alpha)$. Form the finite function r_n with $\delta_e r_n = \delta f$ and

$$r_n(x) = \begin{cases} f(x), & \text{if } x \in \delta_e r_n, \\ 0, & \text{if } x \notin \delta_e r_n. \end{cases}$$

Put $\psi(f^*) = n$. It is easy to see that ψ maps $\xi^*(\alpha)$ one-to-one, onto $(\alpha \cup \{0\})^\alpha$ and that ψ and ψ^{-1} have partial recursive extensions. Hence by [5, Proposition 1], $\xi^*(\alpha) \simeq (\alpha \cup \{0\})^\alpha$. Thus $o(T(\alpha)) = (A + 1)^A$.

THEOREM T14. *For* $\alpha, \beta \subset \epsilon$,

$$\alpha \simeq \beta \Longrightarrow T(\alpha) \cong_\omega T(\beta).$$

PROOF. Left to the reader.

THEOREM T15. *For* α, β *non empty isolated sets,*

$$\alpha \simeq \beta \Longleftrightarrow T(\alpha) \cong_\omega T(\beta).$$

PROOF. It suffices to show by T14 that $T(\alpha) \cong_\omega T(\beta) \Longrightarrow \alpha \simeq \beta$. However it is easy to show that the function $f(n) = (n + 1)^n$ is a one-to-one recursive combinatorial function. Hence by [4, p. 54] and T13, if $\mathrm{Req}(\alpha) = A$ and $\mathrm{Req}(\beta) = B$, then

$$T(\alpha) \cong_\omega T(\beta) \Longrightarrow (A + 1)^A = (B + 1)^B \Longrightarrow A = B \Longrightarrow \alpha \simeq \beta.$$

COROLLARY. *There are c non ω-isomorphic ω-regular isolic semigroups which are not isolic groups.*

THEOREM T16. *Let* $\beta \subset \alpha$. *Then*

$$T(\beta) \leqq_{rec} T(\alpha) \text{ if and only if } \beta \mid \alpha - \beta.$$

PROOF. If $\beta \mid \alpha - \beta$, it is clear that $\xi^*(\beta) \mid \xi^*(\alpha) - \xi^*(\beta)$. Hence $T(\beta) \leqq_{rec} T(\alpha)$. Conversely if $\xi^*(\beta) \mid \xi^*(\alpha) - \xi^*(\beta)$, then let b be a fixed element of β. For $x \in \alpha$, let f_x be the *f.p.* function such that $\delta f_x = \rho f_x = \{x, b\}$ and $f(b) = x$, $f(x) = b$. But we see that $x \in \beta$ if and only if $f_x^* \in \xi^*(\beta)$. It follows that $\beta \mid \alpha - \beta$.

THEOREM T17. *If S is an ω-regular ω-semigroup that is a group, it is an ω-group.*

PROOF. Let $a \in S$. We can effectively find $x \in S$ such that $a \cdot x \cdot a = a$. But x is clearly a^{-1}, since S is a group. Hence S is an ω-group.

COROLLARY. *If S is an ω-semigroup that is a group but not an ω-group, then S is a regular ω-semigroup that is not ω-regular.*

THEOREM T18. *There exist c non ω-isomorphic, completely ω-regular, non-abelian, isolic semigroups which are not isolic groups.*

PROOF. Let α and β be immune, separable sets, and let γ be an infinite subset of β. For any set τ, we recall from [7] that $P(\tau)$ is the ω-group of Gödel numbers of finite permutations of τ and that

$o(P(\tau)) = T!$, where $\mathrm{Req}(\tau) = T$. Also from the remark following T7, we have that if $G_1 = (\alpha_1, p_1)$ and $G_2 = (\alpha_2, p_2)$ are any two ω-groups with $\alpha_1 \mid \alpha_2$, then $G_1 \cup G_2$ forms an ω-semigroup with the given multiplication p. Now, since $\alpha \mid \gamma$, we can effectively recode $P(\alpha)$ and $P(\gamma)$ so that they have no elements in common, instead of having the identity element, 1, in common. For example, let $P_1(\alpha) = \{2x \mid x \in P(\alpha)\}$ and $P_2(\gamma) = \{2x + 1 \mid x \in P(\gamma)\}$, and adjust their multiplications accordingly. Let S_γ be the ω-semigroup $P_1(\alpha) \cup P_2(\gamma)$ with the appropriate multiplication, p. Clearly S_γ is not abelian. Also if $z \in S_\gamma$ then $z \in P_1(\alpha)$ or $z \in P_2(\gamma)$. In either case, z has a group inverse z^{-1} such that $p(z, p(z^{-1}, z)) = z$ and $p(z, z^{-1}) = p(z^{-1}, z)$. Thus S_γ is a completely ω-regular ω-semigroup. Now $o(S_\gamma) = o(P_1(\alpha)) + o(P_2(\gamma))$. Thus if $\mathrm{Req}(\alpha) = A$ and $\mathrm{Req}(\gamma) = C$ then $o(S_\gamma) = A! + C!$. But $A, C \in \Lambda$ implies $A! + C! \in \Lambda$. Therefore S_γ is a completely ω-regular isolic semigroup. Now if γ_1 and γ_2 are two subsets of β with $\mathrm{Req}(\gamma_1) = C_1$ and $\mathrm{Req}(\gamma_2) = C_2$, then

$$S_{\gamma_1} \cong_\omega S_{\gamma_2} \Rightarrow A! + C_1! = A! + C_2! \Rightarrow C_1! = C_2! \Rightarrow C_1 = C_2.$$

Hence $S_{\gamma_1} \cong_\omega S_{\gamma_2} \Rightarrow \gamma_1 \simeq \gamma_2$. But β has c immune subsets which are mutually nonrecursively equivalent. Thus there are c completely ω-regular isolic semigroups of the form S_γ.

Remark. The following are analogues of regular left (right, two-sided) unit as used in [8].

Definition. Let S be an ω-semigroup. An element a of S has an *ω-regular left* (*right*) *unit* i if, given a, we can effectively find an $i \in S$ and $x \in S$ such that $i \cdot a = a$ and $a \cdot x = i$ ($a \cdot i = a$ and $x \cdot a = i$). An element i of S is *ω-regular two-sided unit of* a if i is both an ω-regular left unit of a and an ω-regular right unit of a.

Remark. We see that i is an ω-regular left (right, two-sided) unit implies i is a regular left (right, two-sided) unit.

Theorem T19. (i) *Every idempotent of an ω-semigroup containing ω-regular elements is completely ω-regular.*

(ii) *A regular left* (*right, two-sided*) *unit of an arbitrary element of a semigroup is always an idempotent.*

Proof. Left to the reader.

Theorem T20. *Let S be an ω-semigroup.*

(i) *If an element a of S has either an ω-regular left unit or an ω-regular right unit, then a is ω-regular.*

(ii) *An ω-regular element a of S has both an ω-regular left unit and* *an ω-regular right unit.*

(iii) *An element a of S is completely ω-regular if and only if a has an ω-regular two-sided unit.*

PROOF. (i) Let i be an ω-regular left unit of a. That is, given a we can effectively find $i, x \in S$ such that $i \cdot a = a$ and $a \cdot x = i$. Thus given a we can effectively find $x \in S$ such that $a \cdot x \cdot a = i \cdot a = a$. Also suppose we have a $b \in S$ for which there exists a $y \in S$ such that $b \cdot y \cdot b = b$. But $i_b = b \cdot y$ satisfies $i_b \cdot b = b$ and $b \cdot y = i_b$, and hence given b we can effectively find $j, z \in S$ such that $j \cdot b = b$ and $b \cdot z = j$. However, $b \cdot z \cdot b = j \cdot b = b$. In other words, given b we can effectively find z such that $b \cdot z \cdot b = b$. Thus a is ω-regular. Similarly if i is an ω-regular right unit of a then a is ω-regular.

(ii) Let a be an ω-regular element. Thus given a we can effectively find $x \in S$ such that $a \cdot x \cdot a = a$. Put $i_a = a \cdot x$ and $i^a = x \cdot a$. Thus i_a is a regular left unit and i^a is a regular right unit. Suppose for $b \in S$, there exist $i, y \in S$ such that $i \cdot b = b$ and $b \cdot y = i$. Then, $b \cdot y \cdot b = b$ and b is regular. Thus b is ω-regular and given b we can effectively find $z \in S$ such that $b \cdot z \cdot b = b$. Letting $i_b = b \cdot z$, we have i_b is a regular left unit of b. It follows that i_a is an ω-regular left unit of a. Similarly i^a is an ω-regular right unit of a.

(iii) See [8, p. 73].

REMARK. We know from semigroup theory that an element of a semigroup S may have no more than one regular two-sided unit, hence *an element of an ω-semigroup S may not have more than one ω-regular two-sided unit.*

DEFINITION. Let S be an ω-semigroup and $a \in S$. If, given a, we can effectively find a $b \in S$ such that $a \cdot b \cdot a = a$ and $b \cdot a \cdot b = b$, then b is called an *ω-inverse* of a.

REMARK. (i) An element of an ω-semigroup may have several ω-inverses.

(ii) An element that possesses an ω-inverse is ω-regular. The converse is also true.

THEOREM T21. *Every ω-regular element of an ω-semigroup S has an ω-inverse. A completely ω-regular element of S has an ω-inverse which commutes with it.*

PROOF. Let a be an ω-regular element of S. Hence we can effectively find, given a, an $x \in S$ such that $a \cdot x \cdot a = a$. Put $b = x \cdot a \cdot x$. Thus it is easy to check that b is an ω-inverse of a. Also if $a \cdot x = x \cdot a$ then $a \cdot b = a \cdot x \cdot a \cdot x = x \cdot a \cdot x \cdot a = b \cdot a$.

THEOREM T22. *The following are necessary and sufficient conditions that an ω-semigroup S is an ω-group.*

(i) *S has a unit which is an ω-regular left unit of every element of S.*

(ii) *S is ω-regular and has only one idempotent.*

(iii) *S is ω-regular with two-sided cancellation.*

PROOF. The three conditions are clearly necessary. It suffices to show they are sufficient. But (i) follows from T11, and (ii) and (iii) follow from [8, p. 76], T17, and the fact that any ω-regular ω-semigroup is a regular semigroup.

§6. Inverse ω-semigroups.

DEFINITION. An *inverse ω-semigroup* is an ω-regular ω-semigroup S in which for each $a \in S$, there exists a unique $b \in S$ such that $a \cdot b \cdot a = a, b \cdot a \cdot b = b$ and b is an ω-inverse of a.

REMARK. (i) We see that an inverse ω-semigroup is an inverse semigroup in which each element has an ω-inverse.

(ii) If an ω-regular ω-semigroup is also an inverse semigroup then by T21, it is an inverse ω-semigroup.

NOTATION. If S is an inverse ω-semigroup, we denote the ω-inverse of $a \in S$ by a^{-1}.

THEOREM T23. *Let S be an inverse ω-semigroup.*

(i) *For all $a, b \in S$, b is an ω-inverse of a if and only if a is an ω-inverse of b.*

(ii) *For all $a \in S$, $(a^{-1})^{-1} = a$.*

(iii) *If i is an idempotent of S then $i^{-1} = i$.*

(iv) *For all $a \in S$, $a \cdot a^{-1}$ is the only ω-regular left unit of a and $a^{-1} \cdot a$ is the only ω-regular right unit of a.*

PROOF. Left to the reader.

NOTATION. Let $\alpha \subset \epsilon$. $\zeta(\alpha) = \{f \in \xi(\alpha) \mid f$ is one-to-one on its domain$\}$. $\mathcal{I}(\alpha)$ is the subsemigroup of $\tau(\alpha)$, $(\zeta(\alpha), \circ)$. Also $\zeta^*(\alpha) = \{f^* \mid f \in \zeta(\alpha)\}$ and $I(\alpha)$ is the ω-subsemigroup of $T(\alpha)$, $(\zeta^*(\alpha), \circ)$.

REMARK. $I(\alpha)$ is called the *symmetric inverse ω-semigroup on α.*

THEOREM T24. *For $\alpha \subset \epsilon$, $I(\alpha)$ is an inverse ω-semigroup.*

PROOF. If $f^* \in I(\alpha)$ then f^{-1}, the inverse function of f, is the only *f.p.* function which satisfies $f \circ f^{-1} \circ f = f$ and $f^{-1} \circ f \circ f^{-1} = f^{-1}$. Also $(f^{-1})^*$ is an ω-inverse of f^*. Hence $I(\alpha)$ is an inverse ω-semigroup.

REMARK. By the remark following T12, we see that $I(\alpha)$ is not an ω-group.

REMARK. $I(\alpha)$ is not completely ω-regular, since if $f \in \zeta(\alpha)$ and $\delta f \neq \rho f$, then $\delta(f^{-1} \circ f) = \delta f$ and $\delta(f \circ f^{-1}) = \rho f$. Hence $(f^{-1} \circ f)^* \neq (f \circ f^{-1})^*$.

THEOREM T25. For $\alpha, \beta \subset \epsilon, \alpha \simeq \beta \rightarrow I(\alpha) \cong_\omega I(\beta)$.

PROOF. Left to the reader.

REMARK. We recall from [4] that $\rho_0 = \emptyset$ and $\rho_n = \{a_1, \cdots, a_p\}$, where $n = 2^{a_1} + \cdots + 2^{a_p}$, for $n \geqq 1$, is a one-to-one enumeration of Q, the finite subsets of ϵ.

REMARK. If α is a finite set and card $(\alpha) = n$, then it is easy to show that card $I(\alpha) = \chi(n)$, where $\chi(n) = \sum_{i=0}^{n} \binom{n}{i} \binom{n}{i} i!$. Let us define for $\alpha \in V, \Phi(\alpha) = \zeta^*(\alpha)$. We can see that:

(1) $\alpha \in Q \rightarrow \Phi(\alpha) \in Q$,

(2) $\alpha, \beta \in Q$ and $\alpha \sim \beta \rightarrow \Phi(\alpha) \sim \Phi(\beta)$,

(3) $\alpha \in Q$ and card $(\alpha) = n \rightarrow$ card $\Phi(\alpha) = \chi(n)$.

To prove that Φ is a combinatorial operator which induces $\chi(n)$, it only remains to show that Φ has a quasi-inverse. Clearly $\Phi^\epsilon = \zeta^*(\epsilon)$. Consider the function $m(x)$ with domain $\zeta^*(\epsilon)$ such that if $f \in \zeta(\epsilon)$ and $x = f^*$, $m(x)$ is the unique canonical index such that $\rho_{m(x)} = \delta f \cup \rho f$. Thus for $\alpha \in V$ and $x \in \zeta^*(\epsilon)$,

$$x \in \Phi(\alpha) \Longleftrightarrow \rho_{m(x)} \subset \alpha.$$

Hence $\rho_{m(x)}$ is a quasi-inverse for Φ and Φ is a combinatorial operator. Also the function $g(n)$ such that $\Phi(\rho_n) = \rho_{g(n)}$ is recursive so that Φ is a recursive combinatorial operator which induces the recursive combinatorial function $\chi(n)$. Let for $A \in \Omega$, $C_\chi(A) = \text{Req}\, \Phi(\alpha), \alpha \in A$, be Myhill's canonical extension of $\chi(n)$ to Ω. We now have the following theorem.

THEOREM T26. $o(I(\alpha)) = C_\chi(X), \alpha \in X, X \in \Omega$.

THEOREM T27. *For* α, β nonempty isolated sets $\alpha \simeq \beta \leftrightarrow I(\alpha) \cong_\omega I(\beta)$.

PROOF. By [4, p. 54] and T26, for $\alpha \in A$ and $\beta \in B$, $A, B \in \Lambda$,

$$\begin{aligned} I(\alpha) \cong_\omega I(\beta) \rightarrow o(I(\alpha)) = o(I(\beta)) \rightarrow C_\chi(A) \\ = C_\chi(B) \Longrightarrow A = B \Longrightarrow \alpha \simeq \beta. \end{aligned}$$

Thus we are done by T25.

COROLLARY. There exist c non ω-isomorphic inverse isolic semigroups, which are not completely ω-regular.

LEMMA 1. [3, vol. 1, p. 28]. *Let S be a semigroup. Then S is an inverse semigroup if and only if S is regular, and any two idempotent elements of S commute with each other.*

THEOREM T28. *Let S be an ω-semigroup. Then S is an inverse ω-semigroup if and only if S is ω-regular and any two idempotent elements of S commute with each other.*

PROOF. If S is an inverse ω-semigroup, then S is ω-regular and S is an inverse semigroup. Thus by lemma 1, any two idempotents of S commute with each other. Conversely if S is ω-regular and any two idempotents commute with each other, then by lemma 1, S is an inverse semigroup. Hence by a previous remark, S is an inverse ω-semigroup.

COROLLARY 1. *A mapping of an inverse ω-semigroup S onto itself, which carries every element a of S onto its ω-inverse a^{-1}, is an ω-anti-automorphism.*

PROOF. See [8, p. 80].

COROLLARY 2. *If S is an inverse ω-semigroup and $a, b \in S$, then* $(ab)^{-1} = b^{-1} \cdot a^{-1}$.

COROLLARY 3. *If S is an inverse ω-semigroup and H is a subsemigroup of S such that if $a \in H$ then $a^{-1} \in H$, it follows that H is an inverse ω-semigroup.*

PROOF. If $a \in H$ then we can effectively find $a^{-1} \in H$, given a. Hence H is an ω-regular ω-semigroup. But all idempotents of H commute since $H \subset S$. Thus H is an inverse ω-semigroup.

DEFINITION. Let S be an inverse ω-semigroup and e be an idempotent in S. $G_e = \{x \in S \mid e \text{ is an } \omega\text{-regular two sided unit of } x\}$.

THEOREM T29. *Let S be an inverse ω-semigroup. Then G_e is an ω-group and $G_e \leqq_{\text{rec}} S$.*

PROOF. We have $e \in G_e$. Let $a \in G_e$, then $a \cdot a^{-1} = e$ and $a^{-1} \cdot a = e$ by T20 and the fact that a may have only one ω-regular two-sided unit. Hence $a^{-1} \in G_e$ by T20 and the fact that $(a^{-1})^{-1} = a$. Also if $a, b \in G_e$ then

$$(ab) \cdot (ab)^{-1} = (a \cdot b)(b^{-1} \cdot a^{-1}) = a \cdot (b \cdot b^{-1})a^{-1}$$
$$= a \cdot e \cdot a^{-1} = a \cdot a^{-1} = e.$$

Similarly $(ab)^{-1}(ab) = e$. Thus $a \cdot b \in G_e$. Hence G_e is a group.

Therefore by T17, G_e is an ω-group. Finally, given e, we can effectively test if $a \cdot a^{-1} = e$, for $a \in S$. Thus $G_e \leqq_{\text{rec}} S$.

§7. **ω-Homomorphisms of ω-semigroups.**

Definition. Let S_1 and S_2 be semigroups. Then ϕ is an *ω-homomorphism* of S_1 onto S_2 if
(i) ϕ is a homomorphism of S_1 onto S_2,
(ii) ϕ is an ω-function from S_1 onto S_2.

Notation. If ϕ is an ω-homomorphism from S_1 onto S_2, we say S_1 is *ω-homomorphic* to S_2, [written: $S_1 \geqq_\omega S_2$].

Theorem T30. (i) *Let $S_1 = (\alpha_1, p_1)$ be an ω-semigroup and ϕ be an ω-homomorphism from S_1 onto S_2. Then S_2 is an ω-semigroup.*
(ii) *If ϕ_1 is an ω-homomorphism from an ω-semigroup S_1 onto an ω-semigroup S_2 and ϕ_2 is an ω-homomorphism from S_2 onto an ω-semigroup S_3 then $\phi_2 \circ \phi_1$ is an ω-homomorphism from S_1 onto S_3.*
(iii) *If ϕ is an ω-homomorphism from an ω-semigroup S_1 onto an ω-semigroup S_2 and ϕ is one-to-one on S_1 then ϕ is an ω-isomorphism.*

Proof. (i) Let S_2 be the semigroup (α_2, p_2). Thus $\alpha_2 = \phi(\alpha_1)$. Also let g be a function with a one-to-one partial recursive extension such that $\delta g = \alpha_2$ and for all $y \in \alpha_2$, $g(y) \in \phi^{-1}(y)$. Thus $p_2(y_1, y_2) = \phi p_1(g(y_1), g(y_2))$, for all $y_1, y_2 \in \alpha_2$. Hence p_2 can be extended to a partial recursive function of two variables. Therefore S_2 is an ω-semigroup.
(ii) Use [2, p. 5].
(iii) Use [2, p. 4].

Theorem T31. *If S_1 is an isolic semigroup and $S_1 \geqq_\omega S_2$ then S_2 is an isolic semigroup.*

Proof. Since $S_1 \geqq_\omega S_2$ then $o(S_1) \geqq o(S_2)$ by [2, p. 18]. Thus $o(S_1) \in \Lambda$ implies $o(S_2) \in \Lambda$.

Remark. We recall that an equivalence relation ρ is a congruence on a semigroup S, if for $a, b \in S$ $a\rho b$ implies that $a \cdot c \rho b \cdot c$ and $c \cdot a \rho c \cdot b$ for every $c \in S$.

Notation. If ρ is a congruence on a semigroup S then $[S/\rho]$ is the factor semigroup of S induced by ρ. Also $\rho^\natural$ is the canonical mapping of S onto $[S/\rho]$.

Remark. If ρ is a congruence on a semigroup S, then $\rho^\natural$ is a homomorphism of S onto $[S/\rho]$.

Definition. A congruence ρ on an ω-semigroup S is an *ω-congruence* if $[S/\rho]$ is a gc-decomposition of S.

REMARK. By [2, p. 12], if m_1 and m_2 are gc-functions of $[S/\rho]$ and $\gamma_1 = m_1(S)$ and $\gamma_2 = m_2(S)$ are the associated gc-sets, then $\gamma_1 \simeq \gamma_2$.

DEFINITION. Let $S = (\alpha, p)$ be an ω-semigroup and ρ a congruence on S. With every choice function m of $[S/\rho]$ we associate a semigroup $C_m = (\gamma, q_m)$, defined by,

(i) $\gamma = m(\alpha), \delta q_m = \gamma \times \gamma$,

(ii) $q_m(x, y) = mp(x, y)$, for $x, y \in \gamma$.

REMARK. Note that (ii) can be phrased as

(iii) $q_m[m(x), m(y)] = mp(x, y)$, for $x, y \in \alpha$.

DEFINITION. If ρ is an ω-congruence on an ω-semigroup S then the *factor ω-semigroup* of S [written: S/ρ] is C_m, where m is a gc-function of $[S/\rho]$.

REMARK. By the previous two remarks we see that S/ρ is a well-defined ω-semigroup.

THEOREM T32. *If ρ is an ω-congruence on an ω-semigroup S and m is a gc-function of $[S/\rho]$ then m is an ω-homomorphism from S onto S/ρ.*

PROOF. Clearly for $x \in S$, $m(x) \in \rho^\natural(x)$, thus m is a homomorphism. Further by [2, p. 9 and p. 16], m is an ω-function. Thus m is an ω-homomorphism from S onto S/ρ.

THEOREM T33. *Let ϕ be an ω-homomorphism of an ω-semigroup S_1 onto an ω-semigroup S_2. Define a relation ρ by $a\rho b$ if and only if $\phi(a) = \phi(b)$, for $a, b \in S_1$.* Then *ρ is an ω-congruence on S and there exists an ω-isomorphism ψ from S_1/ρ onto S_2 such that $\phi = \psi \circ f$, where f is a gc-function of $[S_1/\rho]$.*

PROOF. We know from semigroup theory that ρ is a congruence. Since ϕ is an ω-homomorphism then by [2, p. 15], $[S/\rho]$ is a gc-class, i.e., ρ is an ω-congruence, and there exists a mapping ψ from C_f onto S_2 such that $\phi = \psi \circ f$, where f is a gc-function of $[S_1/\rho]$, and ψ has a one-to-one partial recursive extension. Thus ψ is an ω-isomorphism from S_1/ρ onto S_2, by T32 and the fact that ϕ is an ω-homomorphism.

REMARK. From T32 and T33 we see that each ω-congruence induces an ω-homomorphism and vice versa.

DEFINITION. We call the ω-congruence ρ associated with the ω-homomorphism ϕ in T33, the *ω-congruence induced by ϕ*. We sometimes denote ρ by $\phi^{-1} \circ \phi$.

THEOREM T34. *Let ϕ_1 and ϕ_2 be ω-homomorphisms of an ω-semi-*

group S *onto* ω-*semigroups* S_1 *and* S_2 *respectively such that* $\phi_1{}^{-1} \circ \phi_1 \subset \phi_2{}^{-1} \circ \phi_2$. *Then there exists a unique* ω-*homomorphism* θ *of* S_1 *onto* S_2 *such that* $\theta \circ \phi_1 = \phi_2$.

PROOF. For $a_1 \in S_1$, define $\theta(a_1) = \phi_2(a)$, where $a \in S$ and $\phi_1(a) = a_1$. If $a, b \in S$ and $\phi_1(a) = \phi_1(b)$ then $a(\phi_1{}^{-1} \circ \phi_1)b$. Hence by hypothesis, $a(\phi_2{}^{-1} \circ \phi_2)b$ and $\phi_2(a) = \phi_2(b)$. Thus θ is well-defined. It is easy to check that θ is a homomorphism from S_1 onto S_2. Since ϕ_1 and ϕ_2 are ω-homomorphisms there exists functions p_1 and p_2 with partial recursive extensions such that $\delta p_1 = S_1$, $\delta p_2 = S_2$ and for all $a_1 \in S_1$, $p_1(a_1) \in \phi_1{}^{-1}(a_1)$ and for all $a_2 \in S_2$, $p_2(a_2) \in \phi_2{}^{-1}(a_2)$. Thus $\theta(a_1) = \phi_2 p_1(a_1)$, for all $a_1 \in S_1$ and θ has a partial recursive extension. Also if $a_2 \in S_2$ then $p_3(a_2) = \phi_1 p_2(a_2) \in \theta^{-1}(a_2)$ and p_3 has a partial recursive extension. Hence θ is an ω-homomorphism from S_1 onto S_2. The uniqueness of θ follows immediately from $\theta \circ \phi_1 = \phi_2$.

COROLLARY. *If* ρ_1 *and* ρ_2 *are* ω-*congruences on an* ω-*semigroup* S *such that* $\rho_1 \subset \rho_2$, *then* $S/\rho_1 \geqq_\omega S/\rho_2$.

PROOF. Left to the reader.

LEMMA. [8, p. 270]. *If* ϕ *is a homomorphism of an inverse semigroup* S, *and if for* $a \in S$, $\phi(a)$ *is an idempotent of* $\phi(S)$, *then* S *contains an idempotent* i *for which* $\phi(i) = \phi(a)$.

REMARK. The idempotent of the above lemma is $a^{-1} \cdot a$.

THEOREM T35. [8, p. 271]. *If* ϕ *is a homomorphism of an inverse semigroup* S, *then* $\phi(S)$ *is an inverse semigroup.*

THEOREM T36. *If* ϕ *is an* ω-*homomorphism of an* ω-*regular* ω-*semigroup* S_1 *onto an* ω-*semigroup* S_2, *then* S_2 *is* ω-*regular.*

PROOF. Let p be a function associated with ϕ such that for all $y \in S_2$, $p(y) \in \phi^{-1}(y)$ and p has a partial recursive extension. Let $b \in S_2$. Then $p(b) \in S_1$ and there exists an $x \in S_1$ such that $p(b) \cdot x \cdot p(b) = p(b)$ and x can be effectively found given $p(b)$. Thus $\phi(p(b) \cdot x \cdot p(b)) = \phi p(b) = b$. But $\phi(p(b) \cdot x \cdot p(b)) = \phi p(b) \cdot \phi(x) \cdot \phi p(b) = b \cdot \phi(x) \cdot b$. Hence $b \cdot \phi(x) \cdot b = b$ and given b we can effectively find $\phi(x)$. Therefore S_2 is ω-regular.

THEOREM T37. *If* ϕ *is an* ω-*homomorphism of an inverse* ω-*semigroup* S, *then* $\phi(S)$ *is an inverse* ω-*semigroup.*

PROOF. Let S be an inverse ω-semigroup; then by T35 $\phi(S)$ is an inverse semigroup. Hence by Lemma 1 of T28, all the idempotents of $\phi(S)$ commute. Also by T28, S is ω-regular. Thus by T36, $\phi(S)$ is ω-regular. It follows by T28 that $\phi(S)$ is an inverse ω-semigroup.

Corollary 1. *If ϕ is an ω-homomorphism of an inverse ω-semigroup S then for each $x \in S$, $(\phi(x))^{-1} = \phi(x^{-1})$.*

Proof. Left to the reader.

Corollary 2. *If ϕ is an ω-homomorphism of an inverse ω-semigroup S, and if $\phi(a)$, for $a \in S$, is an idempotent, then given $\phi(a)$ we can effectively find an idempotent $i \in S$ such that $\phi(i) = \phi(a)$.*

Proof. Given $\phi(a)$ we can effectively find $b \in S$ such that $\phi(a) = \phi(b)$. Set $i = b^{-1} \cdot b$ which we can effectively find given $\phi(a)$. By lemma to T35, since $\phi(b)$ is an idempotent, i is an idempotent. Also by this lemma, $\phi(i) = \phi(b) = \phi(a)$.

Corollary 3. *Suppose that ϕ is an ω-homomorphism of an inverse ω-semigroup S and that $\lambda = \phi(a)$, $a \in S$, is an idempotent of $\phi(S)$. Then the set B_λ of all $b \in S$ such that $\phi(b) = \lambda$ is an inverse ω-semigroup and $B_\lambda \leqq_{\text{rec}} S$.*

Proof. If $b_1, b_2 \in B_\lambda$, then $\phi(b_1 \cdot b_2) = \phi(b_1) \cdot \phi(b_2) = \lambda \cdot \lambda = \lambda$. Thus $b_1 \cdot b_2 \in B_\lambda$. Hence B_λ is closed under multiplication. It follows that $B_\lambda \leqq S$. Furthermore $x \in B_\lambda \leftrightarrow \phi(x) = \lambda$. Thus $B_\lambda \leqq_{\text{rec}} S$. Suppose $b \in B_\lambda$. Then $\phi(b^{-1}) = (\phi(b))^{-1} = \lambda^{-1} = \lambda$. Therefore $b^{-1} \in \lambda$. Hence by Corollary 3 of T28, B_λ is an inverse ω-semigroup.

Remark. We recall that for a semigroup S and an ideal I of S, the *Rees congruence modulo I, ρ,* is defined by $a\rho b$ if and only if $a = b$ or $a, b \in I$, for $a, b \in S$.

Definition. If S is an ω-semigroup and I is an ideal of S, then I is a *recursive ideal* of S if $I \leqq_{\text{rec}} S$.

Theorem T38. *If I is a recursive ideal of an ω-semigroup S then the Rees congruence modulo I, ρ, is an* ω-congruence.

Proof. Let a be fixed element of I. For $x \in S$, define

$$c(x) = \begin{cases} a, & \text{if } x \in I, \\ x, & \text{if } x \notin I. \end{cases}$$

Then it is easy to see that $[S/\rho]$ is a gc-class, with gc-function, $c(x)$, since $I \mid S - I$.

Notation. If S is an ω-semigroup, I is a recursive ideal of S, and ρ is the Rees congruence modulo I, then we denote S/ρ by S/I.

Theorem T39. *Let A be a recursive ideal of the ω-semigroup S. Then S is an inverse ω-semigroup if and only if A and S/A are both inverse* ω-semigroups.

PROOF. Suppose S is an inverse ω-semigroup. Since S/A is an ω-homomorphic image of S, by T37, S/A is an inverse ω-semigroup. Also if $a \in A$, then $a^{-1} \cdot a \cdot a^{-1} \in A$, since A is an ideal. But $a^{-1} \cdot a \cdot a^{-1} = a^{-1}$. Hence $a^{-1} \in A$. It follows by Corollary 3 of T28 that A is an inverse ω-semigroup. Conversely, suppose A and S/A are both inverse ω-semigroups. Let $a \in S$. If $a \in A$, there exists a unique $x \in A$ such that $a \cdot x \cdot a = a$ and $x \cdot a \cdot x = x$ and we can effectively find x given a. Further since A is an ideal, $x \cdot a \cdot x \in A$ for any $x \in S$. Hence there exists a unique $x \in S$ satisfying $a \cdot x \cdot a = a$ and $x \cdot a \cdot x = x$. If $a \in S - A$ then since S/A is an inverse ω-semigroup, there is a unique $x \in S/A$ such that $a \cdot x \cdot a = a$ and $x \cdot a \cdot x = x$ and we can effectively find x given a. Furthermore any $x \in S$ satisfying $a \cdot x \cdot a = a$ must belong to $S - A$, and hence S/A, since if $x \in A$ then $a \cdot x \cdot a = a \in A$. But this contradicts $a \in S - A$. Hence there exists a unique $x \in S$ such that $a \cdot x \cdot a = a$ and $x \cdot a \cdot x = x$. Since $A \mid S - A$ it follows that S is an inverse ω-semigroup.

§8. ω-Right groups.

REMARK. We recall that a semigroup S is called right (left) simple if it contains no proper right (left) ideal. Also from [3, vol. 1, p. 37], a semigroup S is called a right group if it is right simple and left cancellative. This is equivalent to the statement that for every $a, b \in S$, there exists a unique $x \in S$ such that $a \cdot x = b$.

DEFINITION. An ω-semigroup S is an *ω-right group* if given $a, b \in S$ there exists a unique $x \in S$ such that $a \cdot x = b$ and we can effectively find x given a and b.

REMARK. We see that every ω-right group is a right group.

REMARK. We recall that an ω-semigroup S is called a right zero ω-semigroup if $x \cdot y = y$, for all $x, y \in S$. Clearly every right zero ω-semigroup is an ω-right group.

We need the following lemma from [3, vol. 1, p. 37].

LEMMA. *Every idempotent of a right simple semigroup S is a left unit of S.*

DEFINITION. Let $S_1 = (\alpha_1, p_1)$ and $S_2 = (\alpha_2, p_2)$ be ω-semigroups. We define the *direct ω-product* of S_1 and S_2 [written: $S_1 \times_\omega S_2$] as the semigroup which consists of the set $j(\alpha_1 \times \alpha_2)$ and the semigroup operation:

$$j(x_1, x_2) \cdot j(y_1, y_2) = j[p_1(x_1, y_1), p_2(x_2, y_2)],$$
$$\text{for } x_1, y_1 \in \alpha_1 \text{ and } x_2, y_2 \in \alpha_2.$$

REMARK. It is readily seen that the direct ω-product of two ω-semigroups is again an ω-semigroup.

THEOREM T40. *The direct ω-product of two ω-right groups is an ω-right group.*

PROOF. Left to the reader.

THEOREM T41. *An ω-isomorphic image of an ω-right group is an ω-right group.*

PROOF. Left to the reader.

THEOREM T42. *An ω-semigroup S is an ω-right group if and only if S is ω-isomorphic to the direct ω-product $G \times_\omega E$ of an ω-group G and a right zero ω-semigroup E.*

PROOF. Let S be an ω-right group and $a \in S$. Then there is a unique solution to $a \cdot x = a$. Call this solution e. But then $a \cdot e^2 = a \cdot e = a$. Hence by uniqueness $e^2 = e$. Now let E be the set of idempotents of S. Since $e \in E$, E is not empty. Also by the previous lemma, if $x \in E$, then x is a left unit of S. In particular $e \cdot f = f$, for all $e, f \in E$. Hence E is a right zero ω-subsemigroup of S. Also $x \in E$ if and only if $x^2 = x$, for all $x \in S$. Thus $E \leqq_{\text{rec}} S$.

Now if $e \in E$, $S \cdot e$ is an ω-subsemigroup of S with unit e. Also if $a \in S \cdot e$, we have a solution for $a \cdot x = e$. But $a(x \cdot e) = e^2 = e$, so a has a right inverse $x \cdot e$ in $S \cdot e$. Hence $S \cdot e$ is a subgroup of S. However given a and e we can effectively find $a^{-1} = x \cdot e$. Thus for fixed e, $S \cdot e$ is an ω-group. So let g be a fixed element of E and let G be the ω-group $S \cdot g$. Next, form the direct ω-product $G \times_\omega E = \{j(x, y) \mid x \in G \,\&\, y \in E\}$. We define a map ϕ from $G \times_\omega E$ into S by $\phi j(a, e) = a \cdot e$, for $j(a, e) \in G \times_\omega E$. It is easy to check that ϕ is a homomorphism. Also if $\phi j(a, e) = \phi j(b, f)$ then $a \cdot e = b \cdot f$. But g is the identity of G. Hence

$$a = a \cdot g = a(e \cdot g) = (a \cdot e) \cdot g = (b \cdot f) \cdot g = b \cdot (f \cdot g) = b \cdot g = b.$$

It follows that $a \cdot e = a \cdot f$. But S is left cancellative, so $e = f$. Thus ϕ is one-to-one. Furthermore, if $a \in S$ then $a \cdot e = a$, for some $e \in S$. But $a \cdot e^2 = a \cdot e = a$. Hence $e^2 = e$ and $e \in E$. It follows that

$$\phi j(ag,\ e) = (a \cdot g) \cdot e = a \cdot e = a.$$

Thus ϕ is onto S. Now clearly if $x = j(a, e) \in G \times_\omega E$, then $\phi(x) = k(x) \cdot \ell(x)$, and ϕ has a partial recursive extension. Also if $a \in S$, given a, we can effectively find $e \in E$ such that $a \cdot e = a$. Therefore $\phi^{-1}(a) = j(a \cdot g, e)$ has a partial recursive extension. It follows from

[5, Proposition 1] that ϕ has a one-to-one partial recursive extension and $S \cong_\omega G \times_\omega E$.

Conversely, the direct ω-product $G \times_\omega E$ of an ω-group G and a right zero ω-semigroup E is an ω-right group by T40.

COROLLARY. *If* S *is an* ω*-right group and* e *is an idempotent of* S, *then* $S \cdot e$ *is an* ω*-group.*

THEOREM T43. *An* ω*-semigroup* S *is an* ω*-right group if and only if* S *is* ω*-regular and left cancellative.*

PROOF. Let S be an ω-right group; then S is a right group and hence left cancellative. Now we can effectively find a solution to $a \cdot x = a$ given a. Call this solution e. By left cancellation we get $e^2 = e$. But we can find $y \in S$ such that $e \cdot y = a$. Hence $e \cdot a = e^2 \cdot y = ey = a$. Finally, given a, we can effectively find $z \in S$ such that $a \cdot z = e$. However $a \cdot z \cdot a = e \cdot a = a$. Thus S is ω-regular.

Conversely, suppose S is ω-regular and left cancellative. Then given $a, b \in S$, we can effectively find $y \in S$ such that $a \cdot y \cdot a = a$. Thus $x = y \cdot b$ is a solution to $a \cdot x = b$. For $(ay) \cdot b = (aya) \cdot y \cdot b = (ay) \cdot a \cdot y \cdot b$. Thus by left cancellation $a \cdot y \cdot b = b$. Finally this solution is unique by left cancellation.

REMARK. If S is an ω-semigroup which is a group but not an ω-group, then S is right simple, left cancellative and contains a unique idempotent. But S is not an ω-right group by T17 and T43. Hence the following theorem.

THEOREM T44. *If* S *is an* ω*-semigroup which is a group but not an* ω*-group then* S *is not an* ω*-right group.*

REMARK. In the case that S is a periodic ω-semigroup or an r.e. semigroup, then we have nice behavior.

THEOREM T45. *The following assertions concerning an r.e. semigroup or a periodic* ω*-semigroup* S *are equivalent:*

(i) S *is an* ω*-right group,*
(ii) S *is right simple and left cancellative,*
(iii) S *is right simple and contains an idempotent.*

PROOF. By [3, p. 38], (ii) and (iii) are equivalent and we already have (i) implies (ii); hence to complete the proof, i.e., to show (ii) implies (i), it suffices to show that if S is a right group then given $a, b \in S$ we can effectively find the unique $x \in S$ such that $a \cdot x = b$. Clearly if S is an r.e. semigroup we can do this. So suppose S is a periodic ω-semigroup. The key to the proof is to show that given

$a, b \in S$, there exists an $n \in \epsilon$ such that $a^{n+1} \cdot b = b$, i.e., that $a \cdot x = b$, where $x = a^n \cdot b$. Hence, suppose we are given $a, b \in S$. If a is an idempotent then by a previous lemma, a is a left unit of S, and hence $a \cdot b = b$. So assume a is not an idempotent of S. But since S is right simple and left cancellative, there is a unique idempotent $e \in S$ such that $a \cdot e = a$. However, by [3, p. 38], since e is an idempotent, $S \cdot e$ is a subgroup of S. Since S is a periodic ω-semigroup, $S \cdot e$ is a periodic ω-semigroup. Thus, it follows by T1, that $S \cdot e$ is a periodic ω-group. Hence, given a, we can effectively find the inverse of $a \in S \cdot e$, of the form $y \cdot e$, since $y \cdot e = a^n$, where $n \in \epsilon - \{0\}$ and $a^{n+2} = a$. In other words, we can effectively find a^n, such that $a \cdot a^n = e$. But $a \cdot (a^n \cdot b) = (a \cdot a^n) \cdot b = e \cdot b = b$, since, as above, e is a left unit of S. Therefore, given $a, b \in S$, we can effectively find $x = a^n \cdot b$ such that $a \cdot x = b$. This completes the proof.

COROLLARY 1. *An r.e. semigroup or periodic ω-semigroup is an ω-right group if and only if it is a right group.*

COROLLARY 2. *An isolic semigroup is an isolic right group if and only if it is a right group.*

REFERENCES

1. C. H. Applebaum, *ω-Homomorphisms and ω-groups*, J. Symbolic Logic 36 (1971), 55-65. MR 44 #2603.

2. C. H. Applebaum and J. C. E. Dekker, *Partial recursive functions and ω-functions*, J. Symbolic Logic 35 (1970), 559–568. MR 43 #3116.

3. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vols. I, II, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R. I., 1961; 1967. MR 24 #A2627; 36 #1558.

4. J. C. E. Dekker, *Les fonctions combinatoires et les isols*, Collection de Logique Math. Ser. A, no. 22, Gauthier-Villars, Paris, 1966. MR 35 #6552.

5. ——, *Infinite series of isols*, Proc. Sympos. Pure Math., vol. 5, Amer. Math. Soc., Providence, R. I., 1962, pp. 77–96. MR 26 #16.

6. J. C. E. Dekker and J. Myhill, *Recursive equivalence types*, Univ. California Publ. Math. 3 (1960), 67–213. MR 22 #7938.

7. M. J. Hassett, *Recursive equivalence types and groups*, J. Symbolic Logic 34 (1969), 13–20. MR 39 #2634.

8. E. S. Ljapin, *Semigroups*, Fizmatgiz, Moscow, 1960; English transl., Transl. Math. Monographs, vol. 3, Amer. Math. Soc., Providence, R. I., 1963. MR 22 #11054; 29 #4817.

9. J. Myhill, *Recursive equivalence types and combinatorial functions*, Bull. Amer. Math. Soc. 64 (1958), 373–376. MR 21 #7.

10. ——, *Recursive equivalence types and combinatorial functions*, Logic, Methodology and Philosophy of Science (Proc. 1960 Internat. Congr.), Stanford Univ. Press, Stanford, Calif., 1962, pp. 46–55. MR 27 #1102.

11. A. Nerode, *Extension to isols*, Ann. of Math. (2) 73 (1961), 362–403. MR 24 #A1215.

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