ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 4, Number 3, Summer 1974

RANDOM EVOLUTIONS: A SURVEY OF RESULTS AND PROBLEMS

REUBEN HERSH

1. **Introduction and Summary.** Consider the following three physical models:

— a particle moves in a straight line with constant speed, until it suffers a random collison; then it changes velocity, and again moves in a straight line with a new constant speed.

— a radio signal propagates through a turbulent medium, in which the index of refraction is changing at random.

— a population of bacteria evolve in an environment that is subject to random fluctuations.

These are all examples of a single abstract situation, in which an evolving system changes its "mode of evolution" or "law of motion" because of random changes in the "environment" or the "medium." (In the first example, the mode of evolution is prescribed by the speed and direction of the particle; in the second, by the refractive index of the medium; and so on.)

Such situations arise in every branch of science. Recently, a general mathematical theory of such problems has been developed, the theory of "random evolutions." It is the purpose of this article to summarize the literature so far.

In physical language, a random evolution is a model for a dynamical system whose equation of state is subject to random variation. In mathematical language, a random evolution is an operator M satisfying a linear differential equation of the form

$$\frac{dM}{ds}(s, t) = -V(x(s))M(s, t) \tag{1.1}$$

or, equivalently,

$$\frac{dM}{dt}(s, t) = M(s, t)V(x(t)). \tag{1.2}$$

The coefficient V is an operator depending on a parameter x, and this parameter is stochastic. (That is, $x(t)$ is an abbreviation for $x(t, w)$, where w is a sample point in some probability space Ω.)

In this generality our model includes any homogeneous linear

Research supported in part by NSF Grant No. GP-34188.

evolving system. For example, $V(x(t))$ can be a random Hamiltonian in applications to quantum mechanics. In applications to electromagnetic wave-propagation, (1.1) is a random system of first-order partial differential equations (Maxwell's equations, with random refractive index). A particularly instructive example is to take each $V(x)$ as a single first-order linear differential operator, with random coefficients. Then (1) is a "transport equation," associated with the trajectory of a particle whose speed and direction change at random. This corresponds to our first example above.

Particular examples of such equations have long been studied in physical application, but their mathematical study, in the generality presented here, is very recent.

Progress has been rapid, and some key questions have attained a reasonably definitive answer. In this survey I attempt to show how various results in the literature are related. Throughout the exposition I point out open questions and inviting areas that are still untouched.

To begin with, let us denote by $u(t, x)$ the expected value of the solution of (1.1), conditioned on the initial value of $x(s)$, $x = x(0)$:

$$u(t, x) = E_x[M(0, t)]. \tag{1.3}$$

It turns out that if the random parameter $x(t)$ is Markovian, $u(t, x)$ satisfies a simple deterministic equation. Denote by Q the generator of the process. (For example Q is an n-by-n matrix if $x(t)$ is an n-state chain; $Q = (1/2)\, d^2/dx^2$ if $x(t)$ is Brownian motion.) Then $u(t, x)$ satisfies

$$\frac{du}{dt} = V(x)u + Qu. \tag{1.4}$$

(See [19]).

This is a generalization of the Feynman-Kac formula of potential theory. By appropriate choice of $V(x)$ and Q, (1.3) yields a stochastic solution of certain systems of partial differential equations, of either the parabolic or the hyperbolic type. The connection with hyperbolic equations is one of the most intriguing aspects of random evolutions. As a special case, we recover Kac's solution of the telegrapher's equation in terms of the Poisson process (see [16, 22]). These and related topics are the content of § 2 of this paper.

This part of the subject could be called "representation theory." By requiring $x(t)$ to be Markovian, we obtain an exact equation for the expected values of M, valid for all $t > 0$.

Most of the paper presents results of a different type, asymptotic

theorems which are valid for small stochastic disturbances and large times, in an appropriate scaling.

It turns out that there are limit theorems which enable one to say with certainty what happens in the long run, even without knowing exactly what happens over finite time intervals.

To get a first idea of what to expect, one can write down a formal "solution" to (1.1),

$$M(s, t) \sim \exp\left(\int_s^t V(s(r))\, dr\right).$$

(This can in fact be justified if the $V(x)$ commute with each other.) Now, $\int V(x(r))\, dr$ is (approximately) a sum of operator-valued random variables, $\sum V(x(r_j))\, \Delta_j r$. For such sums, under appropriate hypotheses, there are two important classes of limit theorems: the laws of large numbers, and the central limit theorems. One is thereby led to seek analogous limit theorems for M, the random evolution.

In fact, it is these limit theorems which are the deepest and most useful parts of the theory. (But see the paper by Griego in this issue [13] for a limit theorem of quite a different character.)

Formal derivations for specific physical models were obtained by R. Kubo [28] and M. Lax [31].

To prove limit theorems in the generality needed for applications, one must overcome three distinct obstacles: the coefficients $V(x)$ may be unbounded; different values of the variable operator $V(x)$ need not commute with each other; and the stochastic structure of the process $x(t)$ should be as unrestricted as possible. Only recently (see [6] and [41]) have theorems been proved which meet these demands. A series of earlier works [16, 19, 20, 25, 30, 38, 39] proved limit theorems under one or another special restrictive hypotheses. These more special results are still important, because they provide more explicit formulas and simpler arguments.

In § 3, we present two types of asymptotic theorems. First-order theorems are those which generalize the classical laws of large numbers. Roughly speaking, these theorems say that if the evolution coefficients V are multiplied by ϵ, and the process $x(t)$ is speeded up to $x(t/\epsilon)$, then as $\epsilon \to 0$, M converges to $\exp(t\bar{V}_1)$ where $\bar{V}_1$ is the average (over both the sample parameter w and the time-parameter t) of the random coefficients $V(x(t, w))$.

If $\bar{V}_1 = 0$, the first-order limit $\exp(t\bar{V}_1)$ is just the identity. In this case, by speeding up the process still more to $x(t/\epsilon^2)$, one can prove second-order theorems, which generalize the classical central limit theorem.

Here one can prove convergence for the expected value of M. Its limit is $\exp(t\bar{V}_{11})$ where $\bar{V}_{11}$ is an operator-valued covariance, doubly averaged in time:

$$\bar{V}_{11} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \int_0^s E(V(x(s))V(x(r))\,dr\,ds. \tag{1.5}$$

The order of the factors under the integral is important, since they do not commute in general.

Simple expressions for $\bar{V}_{11}$ can be written down if $x(t)$ is Markovian. On the other hand, a more complicated form is necessary in certain cases where $M(s, t)$ depends on ϵ in a more complicated way. The canonical example of (1.5) is in particle transport, where $V(x(t))$ is a random first-order differential operator. Then (see [20]) the parameter ϵ is proportional to the mean free path between collisions, $\bar{V}_{11}$ is a second-order elliptic differential operator, and $E[M(0, t)]$ goes in the limit to a solution of a parabolic equation — i.e., a diffusion. This shows that our second-order asymptotic theorems are generalizations of the classical diffusion approximation for linear transport theory, for which they also serve as a rigorous foundation.

It is important to recognize that the hypothesis $\bar{V}_1 = 0$, which is necessary for the validity of the second-order theorem, is a genuine restriction on the random coefficients $V(x)$.

This is most easily seen if we suppose that $V(x)$ is a scalar multiple of a single operator, $V(x) = c(x)V$. $\bar{V}_1$ is an average of $V(x)$ with positive weights, and it can vanish only if $c(x)$ takes on both positive and negative values. Now, for existence of $M(0, t)$, $t > 0$, it is necessary that $\exp(tV(x))$ be well-defined; that is, the $V(x)$ should generate semigroups. This will be the case for positive and negative scalars $c(x)$ if and only if V generates a *group*. For example $V = d/dz$ generates a group of translation operators, and so $V(x) = c(x)\,d/dz$ can satisfy $\bar{V}_1 = 0$. On the other hand, if $V = d^2/dz^2$, then $V(x) = c(x)V$ generates a semigroup only for $c(x) \geqq 0$ and so in this case $\bar{V}_1$ cannot vanish, and a second-order theorem cannot hold. Related to this is the fact that for this example $\bar{V}_{11}$ would be a positive multiple of d^4/dz^4, and for such a choice of $\bar{V}_{11}$, $\exp(t\bar{V}_{11})$ does not exist (as a C_0 semigroup on a Banach space).

In case $x(t)$ is Markovian, there is a very useful connection between the asymptotic theory of § 3 and the representation theory of § 2. This is the topic of § 4. The parameter ϵ, which measures the "speeding up" of the random evolution, appears now as a factor in the Feynman-Kac formula (1.4), either as

$$\frac{du}{dt} = V(x)u + \frac{1}{\epsilon}Qu \tag{1.6}$$

in the case of the first-order asymptotics, or

$$\frac{du}{dt} = \frac{1}{\epsilon}V(x)u + \frac{1}{\epsilon^2}Qu \tag{1.7}$$

in the second-order asymptotics.

In either case, the limit as $\epsilon \to 0$ is a highly singular perturbation; by using the probabilistic limit theorems for $M(s, t)$ one proves that the solution of (1.6) or (1.7), with initial value $u_0(x)$, converges, respectively to

$$e^{tV_1} \int u_0(x)d\mu(x)$$

or

$$e^{tV_{11}} \int u_0(x)d\mu(x)$$

where $d\mu$ is the invariant measure associated with the ergodic Markov process $x(t)$ and where $t > 0$.

In particular, one obtains in this manner the convergence of solutions of first-order hyperbolic systems to a single second-order diffusion equation. Further specializing this result, one concludes that solutions of the abstract "telegrapher's equation"

$$\epsilon u_{tt} + u_t = V^2u$$

converge to solutions of the "diffusion equation"

$$u_t = V^2u.$$

If V^2 is chosen as $(d/dz)^2$, we have the classical telegrapher's equation; it can also be chosen, for example, as a general higher-order elliptic operator, in several dimensions, with variable coefficients.

Thus the limit theorems of random evolutions can be used as tools in analysis, to prove perturbation theorems arising quite independently of any probabilistic models.

Conversely, one way to prove limit theorems for random evolutions, at least in the Markovian case, is to prove an equivalent singular perturbation theorem for equation (1.6) or (1.7). This was done by Pinsky [42] in case $x(t)$ is an n-state chain, and $V = v(x)\, d/dz$; it was done by Kurtz for general Q and V (see § 4 below).

A good review of much of the work on Markovian random evolutions is contained in the expository article by Pinsky [45].

Pinsky also has studied random evolutions from a somewhat different point of view:

In the theory of Markov processes an important role is played by real-valued "multiplicative functionals," which satisfy the equation

$$M(s, t)M(t, u) = M(s, u) \tag{1.8}$$

if $s < t < u$. This equation is also satisfied by the random evolutions $M(s, t)$ defined by (1.1); now, of course, the multiplication is in the sense of operator-multiplication. This observation led Pinsky to study the question of finding representations for the most general operator-valued solution of (1.8), at least for certain particular types of Markov process.

If $M(s, t)$ is a functional of a diffusion process, he obtained a representation using an Ito integral [44]. If $M(s, t)$ is a functional of a jump process, he found that M is a product of the form

$$M(0, t) = T_{x(0)}(\tau_1)P_{x(0)x(\tau_1)}T_{x(\tau_1)}(\tau_2 - \tau_1)P_{x(\tau_1)x(\tau_2)} \cdots T_{x(\tau_{N(t)})}(t - \tau_{N(t)}). \tag{1.9}$$

Here the τ_j are the jump times of $x(t)$, $N(t)$ is the number of jumps up to time t, $T_x(t)$ is the semigroup generated by $V(x)$, and $P_{x,y}$ are "jump operators" that act instantaneously when $x(t)$ is making a transition. If $M(0,t)$ is continuous, all P_{xy} drop out, and $M(0, t)$ reduces to a formula found earlier in [15]. The asymptotics for the discontinuous case were developed in detail by R. Kertz and are discussed at the end of § 4.

Many physical applications, especially of the second-order limit theorems, have been made by Papanicolaou and his co-workers. In [4] the problem considered is propagation in a wave guide where random inhomogeneities cause transfer of energy from one mode to another. In [40] the application is to a beam in a strongly focussing medium. In [33] transmission coefficients and reflection coefficients are computed for scalar waves in a slab of random refractive index. In [34] there is a generalization of the Ornstein-Uhlenbeck theory for a particle in a random field. In [39] there is a study of transmission through a random slab, and also a study of a harmonic oscillator with spring constant a random function of time. An application of learning theory is presented in [52]. In many of these works a second-order asymptotic theorem is used to obtain an equation satisfied by the physical quantity of interest. Then in some cases an explicit solution can be given by suitable use of special functions and symmetry arguments. See also [32] for the related work of Morrison and McKenna.

A type of application with a history of its own is to random prod-

ucts of matrices. As (1.9) shows, a random evolution in the case of a jump process is precisely a random product of operators. If these operators are specialized to be finite dimensional, we are considering random products of matrices. This application has interesting geometrical interpretations. For instance, one can study random rotations of a sphere, which, of course, are represented by products of unitary matrices. Applications of this type are given in [38] and in [41].

By use of the theory of weak convergence of measures, it is possible in these cases to prove convergence of distributions, not just convergence of expectations. Results of this type have been obtained by Gorostiza [10, 11] and by Griego and Gorostiza [12], by relying on a general theory of Rosen. In [12] a particle moves in a straight line for a random time (not necessarily exponentially distributed.) Then it changes direction; the new direction is distributed uniformly over a cone making a random vertex angle α with the previous direction as center line. The random times and angles are independent and identically distributed. The paths converge weakly to Brownian motion, and the variance parameter is obtained explicitly.

Branching processes in random environments have been studied as an application of random evolutions in the recent thesis of J. Corona Burgueno [51]. Another important area of application is to control theory. In spite of its obvious practical importance, very little has been done as yet in relating random evolutions to problems of optimal control. Rishel has treated [47] the special case that $x(t)$ is a finite state chain and $V(x)$ are first-order one-dimensional differential operators. He establishes a maximal principle and obtains optimality conditions, for the conditional expectations of a general functional of the position of a particle moving on a line with one of n random position-dependent velocities.

For a general reference on the theory of random operators, the recent book of Bharucha-Reid [2] is very useful.

2. **The Operator-Valued Feynman-Kac Formula; Applications to Hyperbolic Equations.** Suppose the random coefficient V is parametrized by a Markov process $x(t, w) : V(t, w) \equiv V(x(t, w))$. We let Q denote the infinitesimal generator for $x(t, w)$. (For example, if x is an n-state chain, Q is an n-by-n matrix, the derivative at $t = 0$ of the matrix of transition probabilities for $x(t, w)$. If $x(t, w)$ is a diffusion process, Q is a 2nd-order elliptic differential operator.)

Define $M(s, t)$ as the solution (assumed to exist) of

$$(2.1)\quad \frac{dM}{ds} = -V(x(s))M,\ \frac{dM}{dt} = MV(x(t)),\ M(t, t) = I,\ 0 \leqq s \leqq t.$$

Let E_x denote the expected value, conditioned on the value of $x(t, w)$ at $t = 0$. Given a function $f(x)$ with values in a Banach space L, we define

$$u(t,x) = E_x(M(0, t))f(x(t)). \tag{2.2}$$

A direct calculation (see [19], p. 361) quickly shows that if du/dt exists,

$$\begin{aligned} \frac{du}{dt} &= V(x)u + Qu \\ u(0, x) &= f(x). \end{aligned} \tag{2.3}$$

(2.3) requires a little interpretation. u is an L-valued function of t and x; $V(x)$ is, for each x, an operator on L, so $V(x)$ is a pointwise operation on L-valued functions. Q, as the infinitesimal generator of $x(t, w)$, is in the first place defined as operating on real-valued functions of x, but it has an obvious interpretation as operating on L-valued functions of x: if $\lim_{t\to 0}(1/t)[E_x(f(x(t, w)) - f(x)]$ exists, then it is defined to be Qf.

In order to recognize (2.3) as an operator version of the Feynman-Kac formula, suppose all values of the variable coefficient $V(x)$ commute with each other. Then we can write

$$M = \exp\left(\int_s^t V(r)dr \right). \tag{2.4}$$

Substituting (2.4) into (2.2), we have precisely the classical formula of Feynman-Kac where $x(t, w)$ is Brownian motion, $Q = (1/2)\, d^2/dx^2$, and $V(x)$ is multiplication by a scalar.

On the other hand, if $x(t, w)$ is an n-state chain, and Q is an n-by-n matrix, (2.3) is a system of n equations for $u(t, x)$, $x = 1, \cdots, n$. $V(x)$ is a diagonal matrix, operating on the vector $u(t,x)$.

We will consider several different applications to partial differential equations. To begin with, we can specialize (2.3) to obtain a parabolic system of differential equations.

We choose $x(t, w)$ as an n-state Markov chain. Now Q is an n-by-n matrix, whose row-sums are zero, and whose off-diagonal terms are non-negative. For each $x = 1, \cdots, n$, let $V(x)$ be a second-order elliptic operator on, say, R_1, $-\infty < z < \infty$. In this case, (2.3) is a 2d-order parabolic system. The principal part can vary from one row to the next in our system of n equations. On the other hand, there is the restriction that in the xth equation, $1 \leqq x \leqq n$, only the xth unknown is differentiated. The coupling of the equations is

only through the matrix Q, i.e., through the zero-order terms. Chabrowski [5] has shown that only for such weakly coupled systems can one obtain a non-negative fundamental solution.

To obtain a hyperbolic system, we again take Q as the generator of an n-state Markov chain, but now restrict $V(x), x = 1, \cdots, n$, to be first-order differential operators. Now (2.3) is a first-order system of hyperbolic equations, with variable or constant coefficients.

In all these cases, of course, (2.2) yields a solution, where E_x and M are interpreted in the appropriate way.

Once we specialize $x(t)$ to be a jump process — in particular, a finite-state chain — we can solve (2.1) "explicitly", since now $V(x(s))$ is piecewise constant. If, as we assume, the solution $M(s, t)$ exists, then each $V(x)$ must generate a semi-group $\exp(tV(x))$, and we can represent $M(s, t)$ as a random product of "exponentials." If τ_j are the jump times of the process $x(t, w)$

$$M(s,t) = \exp[(\tau_1 - s)V(x(s))]\exp[(\tau_2 - \tau_1)V(x(\tau_1))] \cdots \exp[(t - \tau_N)V(x(\tau_N))] \tag{2.5}$$

where N is the number of jumps performed by the chain between the epochs s and t, and τ_j, $1 \leqq j \leqq N$, are the jump times of the sample path $x(\tau^*)$, $s \leqq \tau^* \leqq t$.

In this case, a standard renewal-theoretic argument shows that $u(t, x)$ satisfies the integral equation

$$u(t,x) = \exp(tV(x))f(x)\text{prob}(\tau > t \mid x(0) = x) + \int_0^t \exp(rV(x)) \sum_{x \neq y} u(t - r, y)q_{xy}p_{xx}(r)\, dr \tag{2.6}$$

(see [16], p. 411). Here τ is the first jump of the chain $x(t)$, and $p_{xy}(r)$ is the matrix of transition probabilities generated by Q. Differentiating (2.6), one obtains again the Feynman-Kac equation (2.3). In the general case, (2.3) is valid only if f is assumed a priori to be in the domain of Q, but in the case of a jump process, Q is an integral operator or a matrix, and the renewal argument shows that (2.3) is valid without any special restrictions on f.

One case had already been considered by several authors [3, 9, 22, 42] before the general notion of random evolution was introduced. This is the case where

$$V(x) = v(x)(d/dz), v(x) \in R.$$

That is to say, as x ranges over $1, \cdots, n$, $V(x)$ is one of n different

scalar multipliers of d/dz. Our model is a particle moving on the z-axis at one of n possible speeds; it changes speed at random, according to a Markov chain with generator Q.

In this case, the "exponentials" of $V(x)$, i.e., the semi-groups they generate, are just translation at speed $v(x)$,

$$\exp(tV(x))f(z) = f(z + tv(x)). \tag{2.7}$$

Furthermore, the generators, and therefore the semi-groups, all commute with each other, and so (2.5) reduces to

$$M(s, t) = \exp\Big(\sum_x \gamma(x, t)V(x)\Big) \tag{2.8}$$

where $\gamma(x, t)$ is the occupation time of the chain in state x up to time t.

Combining (2.7) and (2.8), we get

$$u(t, x, z) = E_x f\Big(x(t),\ z + \sum_y v(y)\gamma(y, t)\Big) \tag{2.9}$$

as the solution of the first-order hyperbolic system

$$\frac{d}{dt}\ u(t, x,z) = v(x)\frac{d}{dz}u(t, x, z)$$

$$+ \sum_{y=1}^{n} q(x, y)u(t, y, z),\ 1 \leqq x \leqq n. \tag{2.10}$$

$$u(0, x, z) = f(x, z)$$

The fact that systems of the form of (2.10) arise in connection with the above-described motion of a particle with random velocity had been noted by Pinsky [42] and earlier by Birkhoff-Lynch [3]. However, the concise formula (2.9) was first discovered only as a special case of the general theory of random evolutions.

If $x(t)$ is a two-state chain, and

$$Q = \begin{pmatrix} -a & a \\ a & -a \end{pmatrix}$$

then $N(t)$, the number of jumps performed by x up to time t, is a Poisson process with intensity a.

If $V(1) = vd/dz$ and $V(2) = -vd/dz$, we have a particle moving on the line at speed v, and reversing direction according to a Poisson process.

In this case (2.10) is a system of two hyperbolic equations of first

order. Differentiation and addition show that both $u(t, 1, z)$ and $u(t, 2, z)$ satisfy the single second-order equation

$$u_{tt} + 2au_t = v^2 u_{zz}. \tag{2.11}$$

This is the telegrapher's equation, which was derived by Kac, following earlier work of S. Goldstein [9]. By going to the continuous limit from a discrete random walk (see [22], reprinted in this issue), Kac found an elegant solution formula for (2.11): if $u(0, z) = f(z)$ and $u_t(0, z) = 0$, then

$$u = E(w(\tau, z)) \tag{2.12}$$

where $\tau = \int_0^t (-1)^{N(s)}\, ds$, N is a Poisson process with intensity a, and w is a solution of the wave equation,

$$w_{tt} = v^2 w_{zz},$$

$$w(0, z) = f \qquad w_t(0, z) = 0.$$

Formula (2.12) was remarkable for at least two reasons. First of all, it is a stochastic solution of a second-order hyperbolic equation. Secondly, it is still valid, if in (2.11) and (2.12) the operator d^2/dz^2 is replaced by a two or three-dimensional Laplacian, ∇^2. Yet there is no random translation in the plane or in space which yields the higher-dimensional "telegrapher's equation" in the manner in which Kac obtained the one-dimensional equation.

The operator-theoretic viewpoint of random evolutions supplies the missing link. If we return to the system (2.3), still with

$$Q = \begin{pmatrix} -a & a \\ a & -a \end{pmatrix}$$

and choose $V(1)$ as the generator of any continuous group of operators, and $V(2)$ as its negative,

$$V(1) = V$$

$$V(2) = -V,$$

then again the system is equivalent to a single higher order equation,

$$u_{tt} + 2au_t = V^2 u. \tag{2.11$'$}$$

This becomes a higher-dimensional telegrapher's equation if

$$V = (d^2/dz_1{}^2 + d^2/dz_2{}^2 + d^2/dz_3{}^2)^{1/2}.$$

Such a V indeed exists, as a pseudo-differential operator, and it generates a group. The Fourier transformation,

$$\mathcal{F}f = \int e^{i\xi z} f(z)\, dz,$$

yields

$$e^{tV} f = \mathcal{F}^{-1}[\exp(t(\xi_1^2 + \xi_2^2 + \xi_3^2)^{1/2})\mathcal{F}f].$$

So Kac's formula (2.12) can be obtained, in any number of dimensions, by considering a random evolution which switches, according to a Poisson process, between forward and backward evolution according to a square root of the Laplacian.

Now formula (2.8) reduces to

$$\sum_x \gamma(x, t)V(x) = [\gamma(1, t) - \gamma(2, t)]\, V.$$

One easily checks that

$$\gamma(1, t) - \gamma(2, t) = \int_0^t (-1)^{N(s)}\, ds,$$

and so formula (2.12) is obtained as a special case of (2.2). See [16] for more details.

Formulas (2.2), (2.9) and (2.12) are remarkable because they yield, for certain *hyperbolic* equations, a solution expressed as the expectation of a stochastic process — a type of representation that was often thought to be attainable only for parabolic and elliptic equations.

The question naturally arises to extend these formulas to as general as possible a class of hyperbolic equations. I know of three papers that have results of this type.

The telegraph equation with a *time-dependent* coefficient was solved by Stanley Kaplan [23]. His idea was to introduce a process $N(t)$, the Poisson process with variable intensity $a(t)$. Its frequency function is given by

$$P\{N(t) - N(s) = m\} = \left[\int_s^t a(\tau)d\tau\right]^m (m!)^{-1} \exp\left[-\int_s^t a(\tau)d\tau\right].$$

He shows by a moment calculation and an induction argument, that if now the random time τ is again defined as in Kac's paper for the case when a is constant,

$$\tau = \int_0^t (-1)^{N(s)}\, ds$$

and if $v(t)$ satisfies

$$v_{tt} = Lv,$$

then

$$u(t) = E\{v(\tau)\}$$

satisfies $u_{tt} + 2a(t)u_t = Lu$. Moreover, $u(0) = v(0)$, $u_t(0) = v_t(0)$. L can be an n-dimensional Laplacian, or, more generally, "any reasonable linear operator."

This theorem should be open to generalization. It seems more than likely that by suitable consideration of a non-stationary Markov process, one could prove a Feynman-Kac formula (2.2) for a variable operator $Q(t)$. But so far this has not been done, even for the case of an n-state chain. (Kaplan's result, of course, is closely related to the special case $n = 2$.)

An extension of the Kac-Kaplan formula in a different direction has recently been given by Rosencrans [48].

He uses a random time τ which is given, not by a Poisson process but by a diffusion,

$$d\tau = f(\tau)\,dt + e(\tau)\,db, \tau(0) = 0$$

where $b(t)$ is a standard Brownian motion process, and e, f are given smooth functions. Then he finds that if v satisfies

$$\tfrac{1}{2}\, e^2(t)v_{tt} + f(t)v_t = Av,$$

then $u(t) = E[v(\tau)]$ satisfies

$$u_t = Au, u(0) = v(0).$$

As a formula for solving hyperbolic systems, (2.9) has restricted application, for equation (2.10) is somewhat special: the coefficients $v(x)$ and $q(x, y)$ are independent of z, and $q(x, y) \geqq 0$ if $x \neq 0$, $\sum_{y=1}^{n} q(x, y) = 0$.

The case $V = v(x, z)$ is covered by formula (2.5). But to allow Q to be z-dependent and arbitrary-valued requires a genuine extension of the theory.

This was done in 1969 by David Heath [18]. He was able to modify (2.5) so that he could solve a general first-order linear hyperbolic system:

$$\frac{\partial u_i}{\partial t} = v_i(z)\,\frac{\partial u_i}{\partial z} + \sum g_{ij}(z)u_j, \tag{2.13}$$

$$u_i = f(z) \text{ at } t = 0.$$

Here we use the usual notation for a system of equations; the subscripts i, j of course, correspond to the variables x, y in (2.10) and above.

To allow a general zero-order term $g_{ij}(z)$ instead of just the restricted constants q_{ij}, Heath used a method of "piecing out" to construct a process in which a particle moving on the z-axis according to one of n different position-dependent velocities $v_i(z)$ undergoes a random jump in velocity from $v_i(z)$ to $v_j(z)$ with a probability $|g_{ij}(z)|$. Then he introduces a multiplicative functional, $m(t, w)$ which depends on the signs of the off-diagonal elements g_{ij}, and on the sign and magnitude of g_{ii}:

$$(2.14) \qquad m(t, w) = \exp\left[\int_0^t d(z(s, w))ds\right] \cdot \prod_{\ell=1}^{K} \operatorname{sign} g_{i_\ell j_\ell}(z(\tau_\ell)).$$

Here $d(z) = g_{ii}(z) + \sum_{j \neq i} |g_{ij}(z)|$ and $\tau_k \leqq t < \tau_{k+1}$.

The jumps of the process are at time τ_ℓ, $\ell = 1, \cdots$, and at the ℓth jump it goes from velocity $v_{i_\ell}(z)$ to velocity $v_{j_\ell}(z)$, if the particle at the jump time has position z.

With this functional, Heath is able to show that

$$E_{(x,i)}\{m(t, w)f(z, t, w)\}$$

satisfies (2.13).

By constructing an equivalent first-order system, he can solve second-order equations of the form

$$u_{tt} = u_{zz} + 2r(z)u_z.$$

He also gives a theory of generalized solutions, analogous to Doob's theory of "parabolic" functions.

Heath's work was followed up by Griego and Moncayo [17]. They showed that if n Markov processes on a common state space are "pieced together" to yield a new process, the semigroup of the "pieced-out process" is the same as that constructed in [16] by the method of random evolutions, as the expected value of a random product of semigroups.

This means, in particular, that if Heath's method and the Griego-Hersh method are both specialized so that they are comparable, then they yield two different constructions of the same semigroup.

It would be interesting to see if Heath's technique could be extended to more general cases.

For arbitrary second-order elliptic operators V_i one should be able to solve parabolic systems of the form

$$\frac{\partial u_i}{\partial t} = V_i u_i + \sum g_{ij}(z)u_j$$

by piecing together diffusions and using the same multiplicative functional $m(t, w)$. There should be a generalization to the case where the random velocities and diffusion coefficients are parametrized, not just by a finite set, but by an arbitrary probability space. This would yield a generalization of the Feynman-Kac formula (2.3), permitting an additional term of very general form, provided the operators $V(x)$ are themselves generators of Markov processes on a common state space.

Even more to the point would be a study of limit theorems for Heath's solutions. Heath's thesis contains no asymptotic results. The asymptotic formulas we discuss below are in the main restricted to the cases included in the "random differential equation" or "Feynman-Kac formula" cases.

Presumably there is a connection between Heath's work and that of Pinsky and Kertz on multiplicative functionals. In particular, some of Kertz' asymptotic theorems may be applicable to Heath's equations. If so, this would add considerably to the applicability of the limit theorems in the study of hyperbolic equations. Kertz' work is described at the end of § 4.

Also relevant to hyperbolic systems is the work of M. Keepler [24]. Keepler's work can be regarded as an extension to the operator-valued case of the renewal-theoretic approach to the forward and backward equations satisfied by the transition probabilities of an n-state chain.

If the chain has generator Q, and transition matrix $p_{ij}(t) = e^{tQ}$, then

$$p_{ik}'(t) = -q_i p_{ik} + \sum_{j \neq i} q_{ij} p_{jk}(t)$$

(the backward equation) and

$$p_{ik}'(t) = -p_{ik}(t) q_k + \sum_{j \neq k} p_{ij}(t) q_{jk} \tag{2.15}$$

(the forward equation).

These two equations are obtained by differentiation from the formulas

$$p_{ik}(t) = \delta_{ik} e^{-q_i t} + \sum_{j \neq i} \int_0^t e^{-q_{ij} s} q_{ij} p_{jk}{}^{(t-s)}\, ds \tag{2.16}$$

and

$$p_{ik}(t) = \delta_{ik} e^{-q_k t} + \sum_{j \neq k} \int_0^t e^{-q_{jk}(t-s)} q_{jk} p_{ij}(s)\, ds. \tag{2.17}$$

If one formally replaces $-q_i$ by $V_i - q_i$, where V_i generates a group $\exp(tV_i)$, then (2.16) becomes a new solution formula for equation (2.3), and (2.17) solves a different equation, in which Q has been replaced by Q^T. Keepler verifies that these formulas actually are valid. His work is so far the only study of the forward-backward duality in the random evolution context. (Except for an announcement [49] of Schay.)

Keepler finds that the solution of the "transposed" equation

$$\frac{du}{dt} = V_i u_i + Q^T u$$

is *not* the transpose of the solution of (2.3), except in the special case that the V_i commute with each other. He also studies the effect of "time-reversal" of the chain, in case of a countable-state chain with a finite "explosion time"; a relationship is established between time-reversal and the substitution of Q^T for Q in (2.3).

It remains to extend these investigations to more general random evolutions, where renewal theory is not directly applicable.

The interesting paper [3] of Birkhoff and Lynch also deserves mention. Perhaps because its main concern is the construction of difference approximations, it has not been influential in probabilistically-oriented work. Nevertheless, it uses probabilistic ideas in a very suggestive way, particularly to motivate certain formal asymptotic expansions for first-order hyperbolic systems of the type we are discussing here, and for the telegraph equation. It would be interesting to see whether some of the heuristic discussion in this paper can be substantiated on the basis of more recent rigorous work, such as that of Pinsky [42] and others, which we discuss in § 4 below.

3. **Limit Theorems in the General Case.** With a single exception (a theorem of Pinsky which we discuss below) the asymptotic theorems on random evolutions all involve a balance between two limits — one takes the limit of *small* stochastic disturbances over long *times*.

Indeed, one should expect that for a *fixed* time, and small disturbance, the system would remain approximately at rest; while disturbances of finite size, acting over a very long period of time, might be expected to produce arbitrarily large effects, precluding the existence of any kind of limit. (By making a hypothesis strong enough to exclude such misbehavior, in this latter situation, Pinsky obtained a limit theorem which he calls an "ergodic-type theorem." (See [45], § 3.6). He assumed also that the random parameter was a finite state Markov chain.)

To say that the stochastic disturbances are "small", we substitute $\epsilon V(x)$ for $V(x)$ in (1.1), (1.2) and therefore also in (2.3), (2.5), and any other formulas which follow from (1.1) – (1.2) or special cases thereof. It turns out that the random evolution generated by $\epsilon V(x(t))$ approaches a limit as $\epsilon \to 0$ if t/ϵ is kept fixed. We call results of this type "first-order theorems"; they generalize the classical laws of large numbers, as was indicated in § 1.

If $M(0, t)$ converges to the identity in this first-order limit, one can consider a second limiting process, in which t/ϵ^2 is kept fixed as $\epsilon \to 0$. Results of this type are "second-order theorems," and generalize the classical central limit theorems.

In discussing questions of asymptotic behavior, we do not have to assume that $x(t)$ is Markovian. Whereas the representation theory of §2 is valid only if the random coefficients are controlled by a Markovian random parameter, the asymptotic theory is valid in much more generality; the main thing needed is a "mixing condition" — i.e., $V(x(t))$ and $V(x(t'))$ should be approximately independent if $(t - t')$ is large.

Nevertheless, the additional structure available in the Markov case makes possible simpler proofs and more explicit formulas. A great deal of interesting work has been devoted to the case where $x(t)$ is Markovian.

For purposes of exposition, we treat the two situations separately. In this section, we present those results which do not require $x(t)$ to be a Markov process.

First of all we should refer to Stratonovich and Khasminski. Stratonovich obtained limit formulas for solutions of random nonlinear ordinary differential equations. His arguments were incomplete, and Khasminski [26, 27] succeeded in making them rigorous. Using Prohorov's theorem on weak convergence of measures, he was able [26] to prove under suitable hypotheses that, as $\epsilon \to 0$ and t/ϵ stays fixed, the solution of his nonlinear random system converges in distribution to the solution of an "averaged equation"; and, under stronger hypotheses, that as $\epsilon \to 0$ and t/ϵ^2 stays fixed, it converges in distribution to a certain diffusion, whose generator is given explicitly in terms of the right-hand side of the random ordinary differential equation [27].

The precise conditions are too complicated to repeat here.

In certain respects, as we will explain shortly, this work of Khasminski is included in the more recent results on random evolutions.

In order to relate Khasminski's work with random evolutions, it is

necessary to associate to a nonlinear ordinary differential equation a linear operator equation. This can be done, as follows: if $x(t)$, $x \in R^n$, is a solution of a nonlinear o.d.e. $dx/dt = F(x)$, and $f(x)$ is an "arbitrary" real-valued function of x, then define a one-parameter family of operators $T(t)$ by

$$T(t)f(x) = f(x(t)), x(0) = x.$$

$T(t)$ is a group if the trajectories $x(t)$ are uniquely defined for all $-\infty < t < \infty$, $x(0) = x$ arbitrary in R^n. (Only in this case is there any asymptotic behavior to study as $t \to \infty$.)

The generator of $T(t)$ is the linear differential operator

$$F(x) \cdot \frac{\partial}{\partial x} \equiv \sum F_i(x) \frac{\partial}{\partial x_i}.$$

Now if F depends on a sample point w and a small parameter ϵ, so do $T(t)$ and its generator. In this way, the asymptotics of a stochastic nonlinear o.d.e. is connected to the asymptotics of a linear operator equation with a random coefficient (see [38, p. 830] for details).

A formal but quite general solution to the asymptotics of a random linear operator equation such as (1.1) or (1.2) was obtained by Papanicolaou and Keller [39]. They used a formal perturbation expansion, the "two-time" method of Julian Cole. In this formalism, one introduces a second time-variable $\tau = \epsilon^r t$, and regards the unknown function $u(t, \epsilon)$ as a function of three variables, $u(t, \epsilon) = v(t, \tau, \epsilon)$. The equation

$$\frac{du}{dt} = V(t, \epsilon)u$$

$$u(0, \epsilon) = f(\epsilon)$$

becomes

$$\frac{\partial v}{\partial t} + \epsilon^r \frac{\partial v}{\partial \tau} = V(t, \epsilon)v.$$

Now a formal expansion for v in powers of ϵ is obtained by matching coefficients of equal powers of ϵ on the left and right side of the equation.

This calculation yields an expansion $v \sim \sum \epsilon^n v_n$, where the v_n are functions of t with certain arbitrary or indeterminate terms. These are chosen so as to prevent $v_n(t)$ from growing too rapidly as $t \to \infty$. This choice also determines the value of the integer r used to define τ.

If $V(t,\epsilon) \sim \sum_1^\infty V_n(t)\epsilon^n$ and

$$\bar{V}_n = \lim_{t\to\infty} t^{-1} \int_0^t V_n(\sigma)\, d\sigma \tag{3.1}$$

the result if $\bar{V}_1 \neq 0$ is $r = 1$,

$$u(t,\epsilon) = \exp(\tau \bar{V}_1) f(0) + 0(\epsilon). \tag{3.2}$$

If $\bar{V}_1 = 0$, then $r = 2$, and

$$u(t,\epsilon) = \exp(\tau(\bar{V}_2 + \bar{V}_{11}))f(0) + 0(\epsilon) \tag{3.3}$$

where

$$\bar{V}_{11} = \lim_{t\to\infty} t^{-1} \int_0^t \int_0^\sigma V_1(\sigma)V_1(s)\, ds\, d\sigma. \tag{3.4}$$

In order to apply these formulas to the case where V is a random function $V(t,\epsilon,w)$, w in some probability space, Papanicolaou and Keller assumed that $\bar{V}_1, \bar{V}_2$ and $\bar{V}_{11}$, which are defined as time averages of random variables $V_1(t,w)$ and $V_2(t,w)$, are independent of w. Then they are equal to the time averages of the mean values of $V_1(t)$, $V_2(t)$, or, respectively,

$$\int_0^\sigma V_1(\sigma)V_1(s)\, ds,$$

and (3.4) becomes (1.5).

In order to justify this assumption, Papanicolaou and Keller assumed the operators V were finite, numerical-valued matrices.

The use of the method of matching powers depends, also, of course, on an assumption that the solution $u(t,\epsilon)$ is a smooth function of ϵ near $\epsilon = 0$. This would be difficult to verify if the coefficients $V(t,\epsilon,w)$ are unbounded operators, as in some of the examples considered in § 2. Within the framework of matrix coefficients, Papanicolaou and Keller were able to give complete treatments of two significant physical problems: a harmonic oscillator with spring constant a random function of time, and one-dimensional wave propagation through a layer with refractive index a random function of position.

In [38] a partial success was achieved in making mathematically rigorous the second-order asymptotic formula (3.4).

The hypothesis is made that $x(t)$ is a renewal process, with a random renewal time that is independent of state. Assuming V is a function only of $x(t)$ and

$$dy/dt = \epsilon Vy, y(0) = f,$$

one finds,

$$\lim_{\epsilon \to 0} E[y(t)] = e^{\tau \bar{V}},$$

where $t = \tau/\epsilon^2$. $\bar{V}$ satisfies (1.5), but in this case can be expressed much more simply,

$$\bar{V} = \frac{\mu^2 + z^2}{2\mu} E\{V^2(\xi))\}$$

where μ and z^2 are the mean and variance of the renewal time, which is assumed to have also a finite third moment. It is assumed also that $\bar{V}$ generates a semigroup, and several other technical hypotheses are used, which are easily checked in the main special cases where $V(x)$ is for each x a differential operator with well-behaved coefficients.

In order to cover certain important applications, this theorem was extended [38, Theorem 3] to certain cases where V is explicitly time-dependent,

$$V = V(x(t), t).$$

But in order to accomplish this, the process $x(t)$ was restricted still more strongly, to be a renewal process with a deterministic renewal time.

These restrictions on $x(t)$ were removed in [**6**]. In this paper, and in a subsequent paper by Papanicolaou and Varadhan [**41**], the only restrictions on the stochastic structure of the coefficients are that the values of V in the remote future should be almost independent of their values in the past. Such a condition is called a "mixing condition." Some condition of this type is always needed in proving a central limit theorem.

In [**6**] the equation considered is

$$\frac{dM}{dt} = (\epsilon V_1(t) + \epsilon^2 V_2(t) + \epsilon^3 V_3(t))M, s \leqq t, M(t,t) = I \tag{3.5}$$

where all the V_i are random functions of t, and V_3 may also depend on ϵ.

The absence of a zero-order term in ϵ on the right of (3.5) is not a real restriction. If a term $V_0 M$ were present, the substitution $M = e^{-tV_0}\tilde{M}$ removes it.

The inclusion of the higher-order terms in ϵ on the right is important. For some applications it is essential to include V_2, which

has an effect that is felt in the second-order asymptotic limit (as can be seen from the Papanicolaou-Keller formula (3.3)).

Abbreviating $V(t, x(t))$ to $V(t)$, we define

$$\bar{V}_1 f = \lim_{\ell \to \infty} \frac{1}{\ell} \int_{t_0}^{t_0+\ell} E(V_1(t))\, dt\, f$$

$$\bar{V}_2 f = \lim_{\ell \to \infty} \frac{1}{\ell} \int_{t_0}^{t_0+\ell} E(V_2(t))\, dt\, f \tag{3.6}$$

$$\bar{V}_{11} f = \lim_{\ell \to \infty} \frac{1}{\ell} \int_{t_0}^{t_0+\ell} \int_{s}^{t_0+\ell} E(V_1(s)V_1(r))\, dr\, ds\, f$$

where the limits are assumed to be independent of t_0.

We abbreviate $\bar{V} = \bar{V}_2 + \bar{V}_{11}$. Then we have two theorems: a first-order theorem, which says that $M(t/\epsilon)f$ converges in probability to $e^{t\bar{V}_1}f$; and a second-order theorem, which says that if $\bar{V}_1 = 0$, the expected value of $M(t/\epsilon^2)f$ converges to $e^{t\bar{V}}f$.

In order to prove these theorems, it is necessary to assume some regularity about the domains of the random operators, about their measurability and t-dependence, and about the existence in a reasonably strong sense of the limits by which $\bar{V}_1$ and $\bar{V}$ are defined. It is also necessary to assume that $\bar{V}_1$ and $\bar{V}$ do generate semigroups, and that these semigroups have certain regularity properties.

We comment here on the key assumptions; for full and precise statements, it is necessary to refer to [**6**].

The principal hypothesis is the mixing condition. Let $P_s(A \mid w)$ denote a regular version of the conditional probability of an event A, given the past up to time s. Let $\rho_t = \sup(P_s(A \mid w) - PA)$, where the sup is taken over all w and s, and all events A which depend on the processes V_i from time $s + t$ onward. Thus ρ_t measures the influence of the past upon future events distant by a time-interval of t or more.

If the random coefficient is Markovian, and time-homogeneous, it is known that either $\rho_t \equiv 1$ or ρ_t goes to zero with exponential speed. To prove the first-order limit theorem in [**6**], ρ_t is merely required to go to zero as $t \to \infty$. For the second-order theorem, we assume $\int_0^\infty \rho_s^{1/2}\, ds < \infty$. (Roughly speaking, $\rho_t = 0(t^{-2-\epsilon})$).

We have to assume that M exists, and that it is bounded uniformly in s, t and ϵ. It would be desirable to remove this latter assumption; the second-order theorem was proved for random evolutions having exponential growth with respect to t, in the special case of a commutative, finite-state Markovian equation [**20**].

As in any second-order theorem, a necessary condition is that $\bar{V}_1 = 0$ (the first-order limit will overpower the second-order limit unless it is absent). But unlike earlier proofs, it is not assumed that $E(V_1(t)) \equiv 0$; the expectation of V_1 need vanish only after averaging over time. But it must do this fast enough; we require that in a suitable norm it decays faster than $t^{-1/2}$([6], p. 1084).

We also have to assume, especially for the second-order theorem, that the limit operators $\bar{V}_1$ and $\bar{V}$ generate semigroups with certain regularity properties with respect to the operators $V(t)$. Roughly speaking [6, p. 1084] we need to know that expressions such as

$$V(t_1)V(t_2)V(t_3)e^{r\bar{V}}f$$

are meaningful, for a certain dense set of f, and are bounded functions of r for $r \leqq t$.

In applications, $V(t)$ is often a differential operator, and $\bar{V}$ an elliptic differential operator (because $\bar{V}_{11}$ is a quadratic expression, essentially an operator-valued covariance). For example, if $V(t) = c(t, z, w)\, d/dz$, and if $c(t, z, w)$ is sufficiently smooth with respect to z (of class C^4) then $\bar{V}$ is a second-order elliptic operator with coefficients in c^3, and the hypotheses needed for the limit theorem can be verified by known theorems on parabolic partial differential equations.

The second-order theorem of [6] has been refined by Papanicolaou and Varadhan to yield a rate of convergence [41]. Using the added condition that the limit (3.6) is approached at a rate of $0(t^{-1})$, they prove that as $\epsilon \to 0$ and $t = \tau/\epsilon^2$, $E(y(t))$ approaches $\exp(\tau V_{11})$ at a rate of $0(\epsilon)$. This is known to be best possible. It had been obtained earlier in a few special cases [50, 42]. ([41] starts with the equation $dy/dt = \epsilon V y$, so, in terms of (3.5), $V \equiv V_1$, and V_2 and V_3 do not appear.)

Their hypotheses are different in some respects from those of [6], and their proof is somewhat simpler. They carry out in full the verification of their conditions in the special case of

$$V(t) = \sum_{j=1}^{n} F_j(z, t, w)\frac{\partial}{\partial z_j}$$

which comes from the linear operator equation associated to a nonlinear ordinary differential equation $dz/dt = F(z, t, w)$. In this way they carry out in detail a limit theorem comparable to Khasminski's. (This had been done earlier, under more restrictive conditions, in [38].)

4. **Singular Perturbation Theorems and Asymptotics for Markovian Evolutions.** A very fruitful and illuminating connection can be established between the representation theory of § 2 and the asymptotic theory of § 3. To this end, we now return to equations (2.1) and set

$$V(x) = \epsilon V_1(x) + \epsilon^2 V_2(x) + \epsilon^3 V_3(x, \epsilon). \tag{4.1}$$

If $u(t, x) = E_x[M(0, t)]$, (2.2) and (2.3) yield

$$\frac{du}{dt} = (\epsilon V_1(x) + \epsilon^2 V_2(x) + \epsilon^3 V_3(x, \epsilon))u + Qu, u(0) = I. \tag{4.2}$$

Here $u(t)$ is , for each value of t and ϵ, an operator on the space of L-valued functions of x. In fact, it is, for each ϵ the "solution operator" for (4.2). The limit theorems of § 3 can be applied to equation (4.2).

To apply the first-order theorems, let $\tau = \epsilon t$. Then (4.2) becomes

$$\frac{du}{d\tau} = (V_1 + \epsilon(V_2 + \epsilon V_3))u + \frac{1}{\epsilon}Qu, u(0) = I. \tag{4.3}$$

Now, if $x(t, w)$ satisfies the "mixing conditions" of [6], namely, that $\rho_t \to 0$, then as $\epsilon \to 0$, $M(0, \tau/\epsilon)$ converges in probability to $\exp(\tau \bar{V}_1)$. In particular, the expected value of $M(0, \tau/\epsilon)$ converges to $\exp(\tau \bar{V}_1)$. But this means precisely that $u(\tau)$, the solution of (4.3), converges, as $\epsilon \to 0$ and τ is fixed, to the solution of

$$\frac{dv}{d\tau} = \bar{V}_1 v, v(0) = I.$$

This is a perturbation theorem about solutions of a certain class of operator equations; such equations may arise quite apart from any probabilistic interest.

The simplest possible example is if $x(t, w)$ is a 2-state chain, with generator

$$Q = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}$$

and states $\{1, 2\}$. Then the invariant measure of Q is $(q_2/(q_1 + q_2), q_1/(q_1 + q_2))$ and so $\bar{V}_1$, the average value of $V_1(x)$ is

$$\frac{q_2 V_1(1) + q_1 V_1(2)}{q_1 + q_2}.$$

For example, consider the coupled heat and wave equations

$$\frac{du_1}{d\tau} = \frac{d^2u_1}{dz^2} - \frac{q_1u_1}{\epsilon} + \frac{q_1u_2}{\epsilon}$$

$$\frac{du_2}{d\tau} = \frac{du_2}{dz} + \frac{q_2u_1}{\epsilon} - \frac{q_2u_2}{\epsilon} \tag{4.4}$$

$$u_1(0, z) = u_2(0, z) = f(z).$$

Here $V_1(1) = d^2/dz^2$, $V_1(2) = d/dz$, $V_2 \equiv 0$, $V_3 \equiv 0$.

We can conclude that as $\epsilon \to 0$ both u_1 and u_2 converge to the solution of

$$\frac{dv}{d\tau} = \left(\frac{q_2}{q_1 + q_2}\right)\frac{d^2v}{dz^2} + \left(\frac{q_1}{q_1 + q_2}\right)\frac{dv}{dz}, v(0, z) = f(z).$$

If the initial data for u_1 and u_2 in (4.4) are not equal, an application of the Markov property and the ergodicity of $x(t)$ (along the lines of the Corollary, p. 357, [19]) shows that

$$v(0, z) = \frac{q_2f_1 + q_1f_2}{q_1 + q_2}.$$

The initial values are mixed together according to the invariant measure of $x(t)$.

If $\bar{V}_1 \equiv q_2V_1(1) + q_1V_1(2) = 0$, we can use the second-order theorem. For this purpose, define $T = \epsilon^2t$. Then (4.2) becomes

$$\frac{du}{dT} = \left(\frac{1}{\epsilon}V_1(x) + V_2(x) + \epsilon V_3(x)\right)u + \frac{1}{\epsilon^2}Qu, u(0) = I. \tag{4.5}$$

Our general theory tells us that u converges, as $\epsilon \to 0$, to the solution of

$$\frac{dv}{dT} = (\bar{V}_2 + \bar{V}_{11})v = 0, v(0) = I.$$

If Q is the same as before, it is clear that

$$\bar{V}_2 = \frac{q_2V_2(1) + q_1V_2(2)}{q_1 + q_2}.$$

What is $\bar{V}_{11}$? In view of (1.5), (3.4) or (3.6) it is easy to see that it must be of the form $cV_1(1)V_1(2)$ where c is some function of q_1 and q_2. We will shortly give general formulas for $\bar{V}_{11}$. (See (4.9) and (4.11) for the commutative case, (4.12)-(4.17) for the general case.) In this case, we get

$$\bar{V}_{11} = \frac{-2V_1(1)V_1(2)}{q_1 + q_2}.$$

For example, consider the system of equations

$$
\begin{aligned}
\frac{du_1}{dT} &= \frac{g(z)}{\epsilon}\frac{du_1}{dz} + a_1(z)\frac{d^2u_1}{dz^2} - \frac{q}{\epsilon^2}u_1 + \frac{q}{\epsilon^2}u_2 \\
\frac{du_2}{dT} &= -\frac{g(z)}{\epsilon}\frac{du_2}{dz} + a_2(z)\frac{d^2u_2}{dz^2} + \frac{q}{\epsilon^2}u_1 - \frac{q}{\epsilon^2}u_2
\end{aligned} \tag{4.6}
$$

$$u_1(0, z) = f_1$$

$$u_2(0, z) = f_2.$$

Here we have specialized $q_1 = q_2 = q$, and taken

$$V_1(1) = -V_1(2) = v(z)\frac{d}{dz}$$

$$V_2(1) = a_1(z)\frac{d^2}{dz^2}, V_2(2) = a_2(z)\frac{d^2}{dz^2}$$

$$V_3 \equiv 0.$$

We assume $a_1(z)$ and $a_2(z) > 0$ and in C^3, $g(z)$ in C^4.

Then we find that u_1 and $u_2$both converge, as $\epsilon \to 0$, to the solution of

$$
\begin{aligned}
\frac{dv}{dT} &= \frac{1}{q}g(z)\frac{d}{dz}\left(g(z)\frac{dv}{dz}\right) + \frac{a_1(z) + a_2(z)}{2}\frac{d^2v}{dz^2} \\
v(0, z) &= \tfrac{1}{2}f_1(z) + \tfrac{1}{2}f^2(z).
\end{aligned} \tag{4.7}
$$

On the right-hand side of (4.7), the first term is $\bar{V}_{11}v$ and the second is $\bar{V}_2v$.

This example shows the advantage of keeping the ϵ^2 term on the right of (3.5); it enables us to include on the right side of (4.6) terms which do not blow up as $\epsilon \to 0$, and yet which do influence the limiting value of u.

If in (4.5) we take

$$x = \{1, 2\}, Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix}, V_2 \equiv V_3 \equiv 0, V_1(1) = V, V_1(2) = -V,$$

then we again have a pair of equations which, as we saw in Section 2, is equivalent to the "abstract telegrapher's equation," (2.11′). But now the parameter ϵ appears in the leading term:

$$\epsilon u_{tt} + 2qu_t = V^2u.$$

From this equation it can be seen that as $\epsilon \to 0$, u should go to

$\exp(tV^2/2q)$; this is consistent with our claim above that $\bar{V}_{11} = V^2/(q_1 + q_2)$.

This special case of the abstract telegrapher's equation was treated in [15, 16]. It is particularly simple because we have only two values for our random operator, $V_1(1)$ and $V_1(2)$, and they must commute with each other in order to satisfy the condition that

$$\bar{V}_1 = \frac{V_1(1) + V_1(2)}{2} = 0,$$

Another commutative case which arises naturally is the motion of a particle whose random speed is independent of position. This is the case considered in formulas (2.7) – (2.10) above. The $V(x) = v(x)\, d/dz$ commute with each other because the speeds $v(x)$ are independent of the position z.

In these cases, the commutativity makes possible a much simpler convergence proof. As (2.8) shows, the random evolution M is now a functional of a finite-dimensional random vector — namely, the occupation times $\gamma(x, t)$, $1 \leqq x \leqq n$.

In [16] for the special case of the abstract telegrapher's equation (2.11′), and in [20] for the general n-state case, the second-order limit theorem is obtained by using the representation (2.8) and proving a central limit theorem for the occupation times of the chain.

If $\{\mu(x)\}$ is the invariant measure of $x(t)$, then

$$\frac{1}{\epsilon}(\gamma(x, t/\epsilon^2) - t\mu(x))$$

converges weakly to a degenerate Gaussian distribution with mean zero, and covariance matrix

$$c_{\alpha\beta} = \lim_{t\to\infty} \frac{1}{t}\operatorname{cov}(\gamma(\alpha, t)\gamma(\beta, t)) \tag{4.8}$$

(lemmas 2 and 3, [20]).

It follows that in this finite-state commutative case we have the simple formula for the limit generator,

$$\bar{V}_{11} = \tfrac{1}{2}\sum\sum c_{\alpha\beta} V(\alpha)V(\beta). \tag{4.9}$$

In the particular case $n = 2$, $Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix}$, we saw that the limiting operator $\bar{V}_{11}$ could be obtained formally by going over to the equivalent second-order equation (the telegrapher's equation) and then setting $\epsilon = 0$.

The same is true in the n-state commutative case. The system

$$\frac{du}{dt} = \frac{1}{\epsilon} V(x)u + \frac{1}{\epsilon^2} Qu, 1 \leqq x \leqq n,$$

is algebraically reducible to

$$\det\left(\epsilon^2 \frac{\partial}{\partial t} I - \epsilon \operatorname{diag}(V(x)) - Q\right)u = 0.$$

This is an nth order equation in $\partial/\partial t$. Because $\det Q = 0$, a factor ϵ can be divided out after the left-hand side is expanded. When this is done, the left-hand side has the form

$$u_t - 2\left(\sum_{\alpha \neq \beta} \hat{q}_{\alpha\beta} V(\alpha)V(\beta)u / \sum_{\gamma=1}^{n} \hat{q}_\gamma\right) + 0(\epsilon), \tag{4.10}$$

where $\hat{q}_\gamma$ is the determinant of the principal minor of Q with γth row and column deleted, $\hat{q}_{\alpha,\beta}$ is the determinant of the principal minor with αth row and column and βth row and column deleted, $\hat{q}_{\alpha,\beta} \equiv 1$ if $n = 2$.

Letting $\epsilon \to 0$ formally in (4.10), we conclude that

$$\bar{V}_{11} = 2 \sum_{\alpha \neq \beta} \hat{q}_{\alpha,\beta} V(\alpha)V(\beta) / \sum \hat{q}_\gamma. \tag{4.11}$$

The study of this determinantal equation was done in [42] for the particular case $V = v(x)d/dz$, but nothing changes if we go to the abstract commutative case. (In [42] the limit theorem is proved in a purely analytical way, by Fourier-transforming in the z-variable, solving the transformed system, and studying its ϵ-dependence.)

Although the limit theorems can now be proved for non-commutative $V(x)$, the results in [20] are in certain respects stronger than those which have been proved so far without commutativity. First of all, convergence of the expected value is actually obtained as a consequence of convergence in distribution. In the case of non-commutative operators which are not finite numerical matrices, all that is known in the second-order asymptotic (central limit theorem case) is convergence of the expected value of M. Nothing is known as yet about convergence in distribution except in the special case of transport theory ($V(x)$ a first-order partial differential operator) which is covered by the work of Khasminski [27] and the recent work [52] of Papanicolaou.

Moreover, in the commutative case, the random evolution is permitted to grow exponentially, whereas in the non-commutative case it must be uniformly bounded in t.

These discrepancies show room for improvement in the non-

commutative theory. They also show that there might be some value even to extending the asymptotics for commutative random evolutions beyond the n-state Makov chain considered by [2] to a general state-space. The results of Hitsuda and Shimizu [21] on compact spaces would be relevant for such a project.

One more special result in the commutative case can be mentioned here. Quiring [46] gave a construction of $M(s, t)$ where the random parameter $x(t, w)$ is a diffusion process and the $V(x)$ commute with each other and generate contractions $e^{tV(x)}$. He reduces the general diffusion to the special case of Brownian motion, which he then approximates by a suitably constructed jump process.

The asymptotic theory for Markovian random evolutions without any commutativity assumptions was obtained in [19] for the n-state case, and by Kurtz [29, 30] for general state spaces.

[19] uses a renewal-theoretic approach. The idea is to factor $E[M(0, \epsilon^2 T)]$ into a product of operators S_ϵ. Each factor is the expected value of an "orbit" of M between two successive entries into a particular ergodic state, say $\{1\}$. The S_ϵ are identical because the disjoint random orbits of the evolution are independent and identically distributed. Because the argument depends on a random return time, it makes use of Doob's theorem on stopping times for martingales. For the case $V \equiv V_1$, it is shown that if $\sum \mu_\alpha V(\alpha) = 0$, then $M(0, T/\epsilon^2)$ converges to $\exp(T\bar{V}_{11})$ where

$$\bar{V}_{11} = \sum V_1(\alpha)V_1(\beta)\mu_\alpha\gamma_{\alpha\beta}. \tag{4.12}$$

In (4.12) μ_α, $\alpha = 1, \cdots, n$ is the invariant measure of $x(t)$, and

$$\gamma_{\alpha\beta} = \int_0^\infty (P_{\alpha\beta}(t) - \mu_\beta)\, dt \tag{4.13}$$

where $P_{\alpha\beta}(t) = (e^{tQ})_{\alpha,\beta}$ are the transition probabilities.

As in the commutative case, $\bar{V}_{11}$ is a quadratic expression in $V(x)$; but now it need not be symmetric. In fact, as is explained in [19], (4.12) is a symmetric quadratic form, for non-commuting $V(x)$, if and only if the Markov chain $x(t)$ is invariant under time-reversal.

In the case of a general state-space, (4.12) becomes

$$\int\int V(\alpha)V(\beta)\mu(\alpha)\gamma(\alpha, \beta)\, d\alpha\, d\beta, \tag{4.14}$$

where $\mu(\alpha)d\alpha$ is the invariant measure and

$$\gamma(\alpha, \beta)d\beta = \int_0^\infty dt[\text{prob}\{x(t) \in d\beta \mid x(0) = \alpha\} - \mu(\beta)\, d\beta].$$

This can be rewritten [19, p. 342] as

$$\gamma(\alpha,\beta) = \lim_{\lambda\to 0}\left[(\lambda - Q)^{-1} - \frac{1}{\lambda}\mu(\beta)\right]. \tag{4.15}$$

In the n-state case, Cramer's rule and l'Hospital's rule yield

$$\bar{V} = \sum V(\alpha)V(\beta)\mu(\alpha)\frac{d/d\lambda \operatorname{cof}_{\beta,\alpha}(\lambda - Q)}{d/d\lambda \det(\lambda - Q)}\Bigg|_{\lambda=0} \tag{4.16}$$

where $\operatorname{cof}_{\beta,\alpha}(\cdot)$ is the cofactor of the β, α-th element of a matrix.

In the general case, (4.15) yields the equation

$$Q\Gamma = \pi - I, \Gamma\pi = 0 \tag{4.17}$$

for the operators $\Gamma = \int\gamma(x,y)f(y)\,dy$, $\pi f = \int \mu(y)f(y)\,dy$ (see [6, p. 1073].)

Formula (4.17) says that $-\gamma(x,y)$ is a Green's function for the operator Q, subject to the side condition $\int \gamma(x,y)\,dy = 0$.

The equation (4.17) can be used to compute $\gamma(x,y)$, thereby making (4.14) an effective explicit formula for $\bar{V}_{11}$.

It is evident that (4.16) must reduce to (4.11) if $V(\alpha)V(\beta) \equiv V(\beta)B(\alpha)$. This is a purely algebraic identity, which should be open to a direct algebraic proof. But so far as I know, no such proof has been given.

We give two specific examples where $\bar{V}_{11}$ has been computed.

If $x(t)$ is Brownian motion in the unit circle, $0 \leqq x < 2\pi$, normalized so that $Q = d^2/dx^2$, then the invariant measure $\mu(x)dx$ is just normalized Lebesgue measure, $dx/2\pi$. It turns out that

$$\gamma(x,y) = \frac{1}{4\pi}(x-y)^2 - \tfrac{1}{2}|x-y| + \pi/6 \quad \text{if} \quad 0 \leqq |x-y| < 2\pi,$$

and

$$\bar{V}_{11} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} V_1(x)V_1(y)\gamma(x,y)\,dx\,dy.$$

([6], p. 1074).

Another case with a simple formula for $\bar{V}_{11}$ is that where $x(t)$ is an n-state chain which is homogeneous in the sense that $p_{ij}(t)$ is independent of j. (The occupation time depends on the state, but once it jumps, it is equally likely to go to any other state.)

For this type of chain,

$$\bar{V}_{11} = \tfrac{1}{2}(1 - 1/n)\sum\left(\frac{V_i}{q_{ii}}\right)^2 \Big/ \sum |q_{ii}|^{-1}$$

([19], p. 344.)

The extension from the finite-state case to the case of general Markovian $x(t)$ was achieved by T. Kurtz [30].

For the case of a regular jump process, Kurtz also proved a first-order asymptotic theorem (law of large numbers) where convergence is *almost sure*; in this respect his result is stronger than the convergence in probability obtained in [6] for the general (non-Markovian) random evolution.

In both papers, he relies on his semi-group convergence theorem which generalizes the well-known Trotter-Kato theory of perturbations of semi-groups.

In fact, Kurtz's method involves a direct attack on the equations (4.3) or (4.5). The convergence of $E[M(0, t/\epsilon)]$ or $E[M(0, t/\epsilon^2)]$ is a corollary to a singular perturbation theorem for (4.3) or (4.5).

THEOREM (KURTZ). *Let $V(t)$ and $S(t)$ be strongly continuous semigroups of linear contractions on a Banach space $\mathcal{L}$ with infinitesimal operators Q and V respectively. Let the closure of $V + (1/\epsilon)Q$ generate a strongly continuous semigroup $T_\epsilon(t)$. Assume that Q is the closure of its restriction to the intersection of the domains of V and Q. Suppose $\lim_{\lambda \to 0} \lambda(\lambda - Q)^{-1} f \equiv Pf$ exists for every $f \in \mathcal{L}$. Let D be the intersection of the range of P and the domain of V. Let $PVf = 0$ for $f \in D$. Let D_0 be the intersection of D with the set of f such that Vf is in the range of Q, so that $Q^{-1}Vf$ exists (perhaps not uniquely) for $f \in D_0$. Suppose D_0 is contained in the closure of the range of $\lambda - PVQ^{-1}V$, for some $\lambda > 0$. Then the closure of $-PVQ^{-1}V$, restricted so that its range is in $\overline{D_0}$, is the generator of a strongly continuous contraction semigroup $T(t)$, and, for $f \in \overline{D_0}$,*

$$\lim T_\epsilon(t/\epsilon)f = T(t)f.$$

To connect this theorem with Markovian random evolutions, one chooses $\mathcal{L}$ as the space of L-valued functions of x, $T(t) = \exp(tV(x))$, and

$$S(t)f = \int f(y)p(t, x, dy)$$

where p is the transition function of a Markov process, with generator Q. Then Kurtz' P is the same as the operator π of formula (4.17). His condition $PVf = 0$ is our condition $\bar{V}_1 = 0$, and his operator $PVQ^{-1}V$ is a generalization of the quadratic expression $\bar{V}_{11}$ of (4.12), as (4.17) makes clear.

Kurtz points out that his limit theorem is applicable to the case where the generator Q is not constant. He calls this "random evolu-

tion with feedback." For the special case $V(x) = v(x, z)d/dz$, $Q = Q(z)$, this is precisely the type of process "pieced together" by Heath (see § 2 above.)

In [30] there is also a first-order theorem, but in the random evolutions application it is much weaker than the following theorem [29], which Kurtz calls a "random Trotter formula," because it specializes to give Trotter's product formula if $x(t)$ is a particular case of a two-state jump process.

THEOREM (KURTZ): *Let $X(t, w)$ be a pure jump process. Suppose S is a separable, locally compact metric space and there is a measure μ on the Borel subsets of S such that $\mu(S) = 1$ and*

$$P\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t g(X(s))\,ds = \int g(x)\mu(dx)\right\} = 1$$

for every real, bounded, continuous function g.

For each $x \in S$ let $T_x(t)$ be a semigroup of linear operators on a Banach space L with infinitesimal operator A_x satisfying $\|T_x(t)\| \leqq e^{\alpha t}$, for some α independent of x.

Let ξ_j be the jth state occupied by $X(t, w)$, let Δ_j *be the sojourn time there, and let $N(t)$ be the number of jumps up to time t.*

Let $T_\lambda(t, w)$ be the random product

$$T_\lambda(t, w) = T_{\xi_0}\left(\frac{1}{\lambda}\Delta_0\right)T_{\xi_1}\left(\frac{1}{\lambda}\Delta_1\right)\cdots T_{\xi_{N(\lambda t)}}\left(\frac{1}{\lambda}\Delta_{\lambda t}\right).$$

(This is the same as formula (2.5) above, with the scaling factor λ added.)

Let D be the set of $f \in L$ such that $A_x f: S \to L$ is a bounded continuous function of x. Define $Af = \int A_x f \mu(dx)$ for $f \in D$.

If D is dense in L and $\mathcal{R}(\mu - A)$ is dense in L for some $\mu > \alpha$, then the closure of A is the infinitesimal operator for a strongly continuous semigroup $T(t)$ defined on L and

$$P\{\lim_{\lambda\to\infty} T_\lambda(t, w)f = T(t)f\} = 1$$

for every $f \in L$.

All the methods described in this section are utilized successively in the thesis of R. Kertz [25]. Kertz considers "discontinuous" random evolutions, of the type (1.9) that Pinsky found in seeking the general solution of the equation $M(s, u) = M(s, t)M(t, u)$ where M is a functional of a Markov jump process. If M is not continuous, random "jump operators" may be inserted between the factors $\exp((\tau_{k+1} - \tau_k)V(x(\tau_k))$ in formula (2.5).

As a result, the "Feynman-Kac" equation (2.3) which is satisfied by the expected value of M is modified; each off-diagonal element q_{xy} in the probability-generating matrix Q is now multiplied by the associated operator π_{xy} which intervenes when the chain jumps from state x to state y.

This is really an abstract operator-valued version of the real-valued multiplicative functional $m(t, w)$ used by Heath (see formula (2.14) above). In order to obtain asymptotic theorems for his "discontinuous random evolutions," Kertz requires the "jump operators" to be ϵ-dependent; they must converge to the identity as $\epsilon \to 0$, at a suitable rate (depending on whether he is proving an asymptotic theorem of first-order or second-order type). Of course, his limiting equations now involve the jump operators π_{xy} as well as the $V(x)$ that occur in the "continuous" case. By so doing he is able to obtain some interesting applications; in particular, a limit theorem of Ilin and Khasminski about Brownian motion in phase space. Kertz systematically carries out his extension in the setting of [20] (commutative operators, limit theorems for occupation times and number of jumps); [19] (non-commutative operators, renewal methods using optional stopping) and [29, 30] (operator-theoretic perturbation techniques).

Recent work of R. Ellis should be mentioned in concluding this section. In his thesis [7], Ellis considered the second-order limit for equation (2.10), which becomes, after replacing v by ϵv and t by T/ϵ^2

$$\frac{du(t, x, z)}{dt} = \frac{1}{\epsilon} v(x) \frac{du(t, x, z)}{dz} + \frac{1}{\epsilon^2} \sum_{y=1}^{n} q(x, y) u(t, y, z), \qquad x = 1, \cdots, n, t > 0. \tag{4.18}$$

This particular equation was studied in Pinsky's thesis [42]. Assuming Q has a unique left null vector $\mu(x)$ such that $\sum \mu(x)v(x) = 0$, Pinsky proved that if $u(0)$ is smooth, $u(t)$ converges to the solution of a certain heat equation, at a rate of $0(\epsilon)$. Ellis obtains the same convergence rate for discontinuous $u(0)$.

In the notation of the more general theory described above, $\bar{V}_1 = \sum \mu(x)v(x) = 0$. The generator in the limiting diffusion equation is a special case of our $\bar{V}_{11}$.

If Q does not have a unique left null vector — that is, if the Markov chain it generates is not ergodic — the general theory is not applicable. However, in the special case of equation (4.18) Pinsky and Ellis [8] recently proved a limit theorem in which Q is allowed to

have a d-dimensional null space, $d > 1$. The proof uses Fourier transformation with respect to z, as in Pinsky's thesis. The limit theorem in this case has physical interest in connection with Boltzmann's equation and kinetic theory.

Bibliography

1. L. Baggett and D. Stroock, *An ergodic theorem for Poisson processes on a compact group, with applications to random evolutions*, preprint.

2. A. T. Bharucha-Reid, *Random Integral Equations*, Academic Press, 1972.

3. G. Birkhoff and R. E. Lynch, *Numerical solutions of the telegraph and related equations*, in *Numerical Solutions of Partial Differential Equations*, Proc. Symp. Univ. Md., 1965, New York, Academic Press, 1966.

4. R. Burridge and G. Papanicolaou, *The Geometry of coupled mode propagation in one-dimensional random media*, Comm. Pure and Appl. Math. **XXV** (1972), 715–757.

5. J. Chabrowski, *"Les solutions non negatives d'un systeme parabolique d'equations,"* Ann. Polon. Math. **19** (1967), 193–197.

6. R. Cogburn and R. Hersh, *Two limit theorems for random differential equations*, Indiana Univ. Math. J. **22** (1973), 1067–1089.

7. R. S. Ellis, *Limit theorems for random evolutions with explicit error estimates*, Z. Wahrscheinlichkeitstheorie verw. Geb. **28** (1974), 249–256.

8. R. S. Ellis and M. A. Pinsky, *Limit theorems for model Boltzmann equations with several conserved quantities*, Indiana Univ. Math. J. **23** (1973), 287–307.

9. S. Goldstein, *On diffusion by discontinuous movements and on the telegraph equation*, Quart. J. Mech. Appl. Math. **4** (1951), 129–156.

10. Luis G. Gorostiza, *An invariance principle for a class of d-dimensional polygonal random functions*, TAMS **177** (1973).

11. ——, *The central limit theorem for random motions of d-dimensional euclidean space*, Annals of Prob. **1** (1973), 603.

12. Luis G. Gorostiza and Richard J. Griego, *Convergence of d-dimensional transport processes with radially symmetric direction changes*, to appear.

13. R. J. Griego, *Limit theorems for a class of multiplicative operator functionals of Brownian motion*, this journal.

14. R. J. Griego, D. Heath and A. Ruiz-Moncayo, *Almost sure convergence of uniform transport processes to Brownian motion*, Ann. Math. Stat., **42** (1971), 1129–1131.

15. R. J. Griego and R. Hersh, *Random evolutions, Markov chains, and systems of partial differential equations*, Proc. of National Academy of Sciences **62** (1969), 305–308.

16. ——, *Theory of random evolutions with applications to partial differential equations*, TAMS **156** (1971), 405–418.

17. R. J. Griego and A. Moncayo, *Random evolutions and piecing out of Markov processes*, Boletin de la Sociedad Matematica Mexicana **15** (1970), 22–29.

18. D. C. Heath, *Probabilistic analysis of hyperbolic systems of partial differential equations*, Doctoral dissertation, University of Illinois, 1969.

19. R. Hersh and G. Papanicolaou, *Non-commuting random evolutions, and an operator-valued Feynman-Kac formula*, Comm. Pure and Appl. Math. **XXX** (1972), 337–367.

20. R. Hersh and M. Pinsky, *Random evolutions are asymptotically Gaussian,* Comm. Pure. and Appl. Math. **XXV** (1972), 33-44.

21. M. Hitsuda and A. Shimizu, *A central limit theorem for additive functionals of Markov processes and the weak convergence to Wiener measure,* J. Math. Soc. Japan **22** (4), 1970.

22. M. Kac, *A stochastic model related to the telegrapher's equation,* this journal.

23. S. Kaplan, *Differential equations in which the Poisson process plays a role,* Bull. AMS **70** (1964), 264-268.

24. M. Keepler, *Backward and forward equations for random evolutions,* Doctoral dissertation, University of New Mexico, 1973.

25. R. Kertz, *Limit theorems for discontinuous random evolutions,* Doctoral dissertation, Northwestern University, 1972. (To appear: Trans. A.M.S.; Annals of Prob.)

26. R. Z. Khas'minskii, *On stochastic processes defined by differential equations with a small parameter,* Theory of Prob. and its Appl. **XI** (1966), 211-228.

27. ——, *A limit theorem for the solutions of differential equations with random right-hand sides,* Theory of Prob. and its Appl. **XI** (1966), 390-406.

28. R. Kubo, *Stochastic Liouville equation,* J. Mathematical Phys. **4** (1963), 174-183.

29. T. G. Kurtz, *A random Trotter product formula,* Proc. AMS **35** (1972), 147-154.

30. ——, *A limit theorem for perturbed operator semigroups with applications to random evolutions,* J. Func. Anal. **12** (1973).

31. M. Lax, *Classical noise IV; Langevin methods,* Rev. Modern Phys. **38** (1966), 561-566.

32. J. A. Morrison and J. McKenna, *Analysis of some stochastic ordinary differential equations,* SIAM-AMS Proceedings Vol. VI, 1973, 97-161.

33. J. A. Morrison, G. C. Papanicolaou, and J. B. Keller, *Mean power transmission through a slab of random medium,* Comm. Pure and Appl. Math. XXIV (1971) 473-489.

34. G. C. Papanicolaou, *Motion of a particle in a random field,* J. Math. Physics. **12** (1971), 1494-1496.

35. ——, *Wave propagation in a one-dimensional random medium,* SIAM J. Appl. Math. **21** (1971), 13-18.

36. ——, *A kinetic theory for power transfer in stochastic systems,* J. Math. Phys. **13** (1972), 1912-1918.

37. ——, *Stochastic equations and their applications,* Amer. Math. Monthly, **80** (1973), 526-544.

38. G. C. Papanicolaou and R. Hersh, *Some limit theorems for stochastic equations and applications,* Indiana U. Math. J. **21** (1972), 815-840.

39. G. C. Papanicolaou and J. B. Keller, *Stochastic differential equations with applications to random harmonic oscillators and wave propagation in random media,* SIAM J. Appl. Math. **21** (1971), 287-305.

40. G. C. Papanicolaou, D. McLaughlin, and R. Burridge, *A stochastic Gaussian beam,* J. Math. Phys. **14** (1973), 84-89.

41. G. C. Papanicolaou and S. R. S. Varadhan, *A limit theorem with strong mixing in Banach space and two applications to stochastic differential equations,* Comm. Pure and Appl. Math. **26** (1973), 497-524.

42. M. Pinsky, *Differential equations with a small parameter and the central limit theorem for functions defined on a finite Markov chain,* Z. Wahrscheinlichkeitstheorie verw. Geb. **9** (1968), 101-111.

43. ——, *Multiplicative operator functionals of a Markov process,* Bull. Amer. Math. Soc. **77** (1971), 377–380.

44. ——, *Stochastic integral representation of multiplicative operator functionals of a Wiener process,* TAMS **167** (1972), 89–104.

45. ——, *Multiplicative operator functionals and their asymptotic properties,* to appear in Advances in Probability, V. 3.

46. Donald Quiring, *Random evolutions on diffusion processes,* Z. Wahrscheinlichkeitstheorie verw. Geb. **23** (1972), 230–244.

47. R. Rishel, *Dynamic programming and minimum principles for systems with jump Markov disturbances,* preprint.

48. S. I. Rosencrans, *Diffusion transforms,* J. Diff. Equat. **13** (1973), 457–467.

49. G. Schay, Notices, AMS, No. 147 (1973), Abstract 70-60-5.

50. A. Schoene, *Semi-groups and a class of singular perturbation problems,* Indiana U. Math. J., **20** (1970), 247–263.

51. J. Corona Burgueno, *Branching Processes with Cataclysmic Environmental Changes,* Doctoral Dissertation, University of New Mexico, 1974.

52. G. C. Papanicolaou, *Asymptotic Analysis of Transport Processes,* preprint.

UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, NEW MEXICO 87131