ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 4, Number 3, Summer 1974

LIMIT THEOREMS FOR A CLASS OF MULTIPLICATIVE OPERATOR FUNCTIONALS OF BROWNIAN MOTION

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1. **Introduction.** Multiplicative operator functionals (MOF's) of Markov processes have proved to be an important extension of the concept of real valued multiplicative functionals as developed, for example, in Dynkin's treatise [2]. MOF's provide a unifying concept for the solution of many concrete problems encountered in transport theory, operations research, wave propagation in random media, and systems of partial differential equations. Pinsky's survey article [9] contains a fairly comprehensive account of the theory and applications of MOF's. The main results concerning MOF's consist of representation theorems and limit theorems. In this note we establish some limit theorems for certain MOF's of one and two dimensional Brownian motion; in addition, connections with differential equations are discussed. These results differ from previous limit theorems for MOF's in that the limiting operators associated with these MOF's do not define semigroups in contrast to similar results for MOF's of Markov chains and other Markov processes as proved in previous works. The author wishes to thank Reuben Hersh and Steve Rosencrans for helpful discussions during the preparation of this paper.

2. **Multiplicative Operator Functionals.** We use the notation of Dynkin [2] for Markov processes. Let $X = (x(t), \zeta, \mathcal{M}_t, P_x, x \in E)$ be a Markov process with state space $(E, \mathcal{B})$. Let L be a fixed Banach space and let $\mathcal{L}$ be the space of all bounded linear operators on L. Following Pinsky [8] we say a *multiplicative operator functional* (MOF) of (X, L) is a mapping $(t, \omega) \to M(t, \omega)$ of $[0, \infty] \times \Omega \to \mathcal{L}$ so that

(i) $\omega \to M(t, \omega)f$ is $\mathcal{M}_t$ — measurable for each $t \geqq 0, f \in L$.
(ii) $t \to M(t, \omega)f$ is right continuous a.s. for each $f \in L$.
(iii) $M(t + s, \omega)f = M(t, \omega)M(s, \theta_t\omega)f$ a.s. for each $t \geqq 0, f \in L$.
(iv) $M(0, \omega)f = f$ a.s. for each $f \in L$.

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[1]Research supported by NSF Grant GP-31811.
Received by the Editors June 22, 1973 and in revised form on June 29, 1973.
AMS 1970 *subject classifications.* 35K15, 47D05, 60J55.
Key words and phrases. Random evolutions, semigroups of operators, continuous additive functionals of Brownian motion, limit theorems, probabilistic solution of differential equations.

Given a continuous additive functional $\{\alpha(t), t \geqq 0\}$ of a Markov process X and a strongly continuous semigroup $\{T(t), t \geqq 0\}$ of operators on a Banach space L then $M(t) = T(\alpha(t))$, $t \geqq 0$, defines a (continuous) MOF of (X, L) since

$$\begin{aligned} M(t+s, \omega) &= T(\alpha(t+s, \omega)) = T(\alpha(t, \omega) + \alpha(s, \theta_t\omega)) \\ &= T(\alpha(t, \omega))T(\alpha(s, \theta_t\omega)) = M(t, \omega)M(s, \theta_t\omega)). \end{aligned}$$

This method furnishes an easy way to manufacture MOF's for any Markov process. In what follows we choose certain continuous additive functionals of one or two dimensional Brownian motion in this procedure in order to obtain limit theorems.

3. **Limit Theorems for MOF's of Brownian Motion.** If $M_\epsilon(t)$ is an MOF depending on a parameter $\epsilon > 0$, then it is of interest to investigate the behavior of $M_\epsilon(t)$ as $\epsilon \to 0$ and $t \to \infty$. It is known ([3], [4], and [5]) that for a large class M_ϵ of MOF's of finite, irreducible Markov chains that

$$\lim_{\epsilon\to 0} E_x[M_\epsilon(t/\epsilon)f] = e^{tW}f \text{ and } \lim_{\epsilon\to 0} E_x[M_\epsilon(t/\epsilon^2)f] = e^{tV}f$$

for a class of operators W and V so that the limits define semigroups. These two results are analogous to the law of large numbers and the central limit theorem respectively. Kurtz [7] uses operator theoretic techniques to obtain limiting semigroups for MOF's of a wide class of Markov processes possessing invariant measures. More recent results [1] extend these limit theorems to a large class of stochastic processes satisfying mixing conditions. Nevertheless, the limiting operators form semigroups in all these results. The situation for Brownian motion is quite different as the following results show.

If X is one dimensional Brownian motion then all continuous additive functionals $\{\alpha(t), t \geqq 0\}$ of X are of the form $\alpha(t) = \int \gamma(x, t)\mu(dx)$ where $\gamma(x, t)$ is the local time at x and μ is a Borel measure on the real line R so that $\mu([a, b)) > 0$ for all $a < b$. See [6, p. 190].

THEOREM 1. *Let $\alpha(t) = \int \gamma(x, t)\mu(dx)$ be a continuous additive functional of one dimensional Brownian motion X with $0 < k = \mu(R)/2 < \infty$. Let $T(t)$ be a strongly continuous contraction semigroup on a Banach space L with infinitesimal generator A. For $\epsilon > 0$ define an MOF M_ϵ of (X, L) by $M_\epsilon(t) = T(\epsilon\alpha(t))$. Then for each $t > 0, f \in L$, and $x \in R$,*

$$(3.1)\quad \lim_{\epsilon\to 0} E_x[M_\epsilon(t/\epsilon^2)f] = \left(\frac{2}{\pi k^2 t}\right)^{1/2} \int_0^\infty T(s)f e^{-s^2/2k^2t}\, ds \equiv w(t).$$

The limit is taken in the strong topology of L. Furthermore, if f is in the domain of A^2, then

$$\frac{dw}{dt} = \frac{k^2}{2} A^2 w + \frac{k}{\sqrt{2\pi t}} Af, \quad w(0+) = f. \tag{3.2}$$

Proof. By a limit theorem of Ito and McKean [6, p. 229] that generalizes a result of Kallianpur and Robbins in one dimension, we have that $\alpha(t)/(k\sqrt{t})$ converges in distribution as $t \to \infty$ with respect to P_x for all x to a random variable Z distributed on $[0, \infty)$ with density $(2/\pi)^{1/2} e^{-z^2/2}$, $z \geqq 0$. Hence for fixed $t > 0$, $\epsilon\alpha(t/\epsilon^2)$ converges in distribution to $k\sqrt{t}\, Z$ as $\epsilon \to 0$. Since $T(t)$ is a contraction semigroup a strong version of the Helly-Bray theorem applies [3, Lemma 3] and we have

$$\begin{aligned}
\lim_{\epsilon \to 0} \quad E_x[M_\epsilon(t/\epsilon^2)f &= E[T(k\sqrt{t}\, Z)f] \\
&= (2/\pi k^2 t)^{1/2} \int_0^\infty T(s) f e^{-s^2/2k^2 t}\, ds \\
&= w(t).
\end{aligned}$$

To complete the proof we need to show that $w(t)$ solves (3.2). Hence, let f belong to the domain of A^2 and let $u(s, t) = (2/\pi k^2 t)^{1/2} e^{-s^2/2k^2 t}$, then

$$\begin{aligned}
A^2 w(t) &= \int_0^\infty A^2 T(s) f\, u(s, t)\, ds \\
&= \int_0^\infty T''(s) f\, u(s, t)\, ds \\
&= -\left(\frac{2}{\pi k^2 t}\right)^{1/2} Af + \int_0^\infty T(s) f \frac{\partial^2 u}{\partial s^2}\, ds \\
&\qquad \text{(integrate by parts twice)} \\
&= -\left(\frac{2}{\pi k^2 t}\right)^{1/2} Af + \int_0^\infty T(s) f \frac{2}{k^2} \frac{\partial u}{\partial t}\, ds \\
&= -\left(\frac{2}{\pi k^2 t}\right)^{1/2} Af + \frac{2}{k^2} \frac{dw}{dt},
\end{aligned}$$

or, $dw/dt = (k^2/2)A^2 w + k/\sqrt{2\pi t}\, Af$. Passages of operations through the integrals are justified since $\|T(s)\| \leqq 1$ and thus the integral in (3.1) converges uniformly. The condition $w(0+) = f$ is satisfied since the exponential kernel in the integral defining $w(t)$ behaves like a delta function as t approaches 0. This completes the proof.

We note that the limiting operators $W(t)f = \lim E_x[M_\epsilon(t/\epsilon^2)f]$ do not form a semigroup on all of L but they do on the kernel of A, that is, on the set of f so that $Af = 0$. The singular equation (3.2) has been derived by Rosencrans [**10**] in connection with transforms of Brownian motion. Also, since $\epsilon\alpha(t/\epsilon)$ converges in distribution to zero, we have that $\lim_{\epsilon\to 0} E_x M_\epsilon(t/\epsilon)f = f$ for each f in L.

If $\mu(dx) = c(x)\,dx$ in Theorem 1, then $M_\epsilon(t)$ solves the stochastic operator equation

$$\frac{dM_\epsilon}{dt} = \epsilon c(x(t)) M_\epsilon A \tag{3.3}$$

and M_ϵ is then termed a *random evolution*; see [4]. Furthermore by an operator version of the Feynman-Kac formula [4], it can be seen that $u^\epsilon(t, x) = E_x[M_\epsilon(t/\epsilon^2)f]$ solves

$$\frac{\partial u^\epsilon}{\partial t} = \frac{1}{\epsilon} c(x) A u^\epsilon + \frac{1}{2\epsilon^2}\frac{\partial^2 u^\epsilon}{\partial x^2}, \quad u^\epsilon(0+, x) \equiv f. \tag{3.4}$$

Hence, Theorem 1 gives the convergence as $\epsilon \to 0$ of a solution $u^\epsilon(t, x)$ of (3.4) to $w(t)$ (independent of x) that solves (3.2).

Let us now consider a continuous additive functional $\alpha(t)$, $t \geqq 0$, of two dimensional Brownian motion. There then exists a measure μ on R^2 so that

$$E_x[\alpha(\tau_D)] = \int_D G_D(x, y)\mu(dy), \quad x \in D,$$

where τ_D is the first exit time for the Greenian domain D and G_D is the Green function of D. See [**6**, p. 277] for a discussion of these facts.

THEOREM 2. *Let $\alpha(t)$ be a continuous additive functional of two dimensional Brownian motion X associated with a measure μ as above. Suppose $0 < k = \mu(R^2)/4\pi < \infty$. Let $T(t)$ be a strongly continuous contraction semigroup on a Banach space L with infinitesimal generator A. For $\epsilon > 0$ define an* MOF *M_ϵ of (X, L) by $M_\epsilon(t) = T(\epsilon\alpha(t))$. Then for each $t > 0$, $f \in L$, and $x \in R^2$,*

$$\lim_{\epsilon\to 0} E_x[M_\epsilon(e^{1/t\epsilon})f] = \frac{t}{k}\int_0^\infty T(s)f e^{-ts/k}\,ds \equiv v(t). \tag{3.5}$$

Also,

$$t\frac{dv}{dt} = v(t) - \frac{t}{k} R\left(\frac{t}{k}\right)v(t) \tag{3.6}$$

or

$$t\frac{dv}{dt} = -R\left(\frac{t}{k}\right)Av, \tag{3.7}$$

where $R(t) \equiv \int_0^\infty T(s)f e^{-ts}\,ds$, $t > 0$, *defines the resolvent operators of the semigroup* $T(t)$.

PROOF. Again by an extension of the Kallianpur-Robbins limit theorem in two dimensions [6, p. 277], $\alpha(t)/k \log t$ converges in distribution as $t \to \infty$ with respect to P_x for each $x \in R^2$ to a random variable Y distributed on $[0, \infty)$ with density e^{-y}, $y \geqq 0$. For fixed $t > 0$, $\epsilon\alpha(e^{1/t\epsilon})$ then converges in distribution to $(k/t)Y$ as $\epsilon \to 0$. As in Theorem 1 we obtain for each $t > 0, f \in L$,

$$\begin{aligned}\lim E_x[M_\epsilon(e^{1/t\epsilon})f] &= \frac{t}{k}\int_0^\infty T(s)f e^{-ts/k}\,ds \\ &= \frac{t}{k} R\left(\frac{t}{k}\right)f \equiv v(t),\end{aligned}$$

thereby proving (3.5). By the resolvent equation, $R(t) - R(s) = (s - t)R(t)R(s)$, so that $dR/dt = -R^2(t)$. Hence, since $v(t) = (t/k)R(t/k)f$, we easily obtain (3.6). Moreover by properties of the resolvent, $R(t) = (tI - A)^{-1}$, where I is the identity operator. Equation (3.7) follows from this observation and (3.6), completing the proof.

The initial condition for v above is given by $v(0+) = \lim_{t\downarrow 0} tR(t)f$ whenever this limit exists. In particular if A^{-1} exists with domain dense in L then $v(0+) = 0$ for all $f \in L$. Again the limiting operators do not form a semigroup and the associated differential equation is singular.

4. **An Application.** Consider a particle that can move on the line with any one of n constant velocities $c_j \geqq 0$, $j = 1, \cdots, n$. The particle changes velocities according to the motion of a Markov chain with state space $\{1, \cdots, n\}$, so that, if the chain starts out in the state i, the particle moves with velocity c_i until the chain jumps to a new state, say j, then the particle moves with velocity c_j until the next jump, and so on. We can consider this process to be determined by the MOF $M(t) = T(\sum_{j=1}^n c_j \cdot \gamma_j(t))$, where $\gamma_j(t)$ is the occupation time of the

chain in state j up to time t and $T(t)$ is the translation semigroup $T(t)f(x) = f(x + t)$. See [3, Theorem 3]. If we allow a continuum of possible velocities $c(x) \geqq 0$, $-\infty < x < \infty$, and allow the particle to change velocities continuously according to one-dimensional Brownian motion instead of with respect to a chain as above, then the MOF associated with this motion is $M(t) = T(\int_{-\infty}^{\infty} c(x)\gamma(x, t)\, dx)$, where $\gamma(x, t)$ is the local time of Brownian motion at the point x up to time t.

The scaling $M(t) \to M_\epsilon(t/\epsilon^2)$ of Theorem 1 has the effect of speeding up the velocity of the particle by a factor of $1/\epsilon$ and speeding up the time scale by a factor of $1/\epsilon^2$. The limiting equation (3.2) becomes

$$\frac{\partial w}{\partial t} = \frac{k^2}{2}\frac{\partial^2 w}{\partial x^2} + \frac{k}{\sqrt{2\pi t}}\frac{df}{dx}, \quad w(0+, x) = f(x). \tag{4.1}$$

Thus the limiting motion of the particle is governed by the inhomogeneous heat equation (4.1), where the inhomogeneous term becomes unbounded at the origin. The unbounded term can be interpreted as a heat source from which the rate of heat flow is infinite initially and then slows down with time.

Cogburn and Hersh [1] consider MOF's of Brownian motion on the unit circle. It is worthwhile to compare their results to ours. Let $\theta(t)$ be the Brownian motion on the unit circle with generator $(1/2)d^2/d\theta^2$. Suppose $\tilde{M}_\epsilon(t)$ is the MOF given by

$$\frac{d\tilde{M}_\epsilon}{dt} = \epsilon c(\theta(t), z)\tilde{M}_\epsilon \frac{d}{dz}, \quad 0 \leqq \theta \leqq 2\pi, -\infty < z < \infty, \tag{4.2}$$

where it is assumed that

$$\int_0^{2\pi} c(\theta, z)\, d\theta = 0 \text{ for all } z. \tag{4.3}$$

By the operator Feynman-Kac formula, $v^\epsilon(t, \theta, z) = E_\theta[\tilde{M}_\epsilon(t/\epsilon^2)f(z)]$ solves

$$\frac{\partial v^\epsilon}{\partial t} = \frac{1}{\epsilon} c(\theta, z)\frac{\partial v^\epsilon}{\partial z} + \frac{1}{2\epsilon^2}\frac{\partial^2 v^\epsilon}{\partial \theta^2}, \quad v^\epsilon(0+, \theta, z) \equiv f(z). \tag{4.4}$$

Equation (4.4) has the same form as (3.4) with $A = d/dz$ and θ playing the role of x, the state variable for the Brownian motion. Cogburn and Hersh prove the remarkable result that $v^\epsilon(t, \theta, z)$ converges as $\epsilon \to 0$ to $v(t, z)$, a solution of

$$\frac{\partial v}{\partial t} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \gamma(\theta - \theta')c(\theta, z)\frac{\partial}{\partial z}\left(c(\theta', z)\frac{\partial v}{\partial z}\right) d\theta\, d\theta', \tag{4.5}$$

$$v(0+, z) = f(z),$$

where

$$\gamma(\theta) = \frac{\theta^2}{4\pi} - \frac{|\theta|}{2} + \frac{\pi}{6}, \quad -\pi \leqq \theta \leqq \pi, \; \gamma(\theta + 2\pi) = \gamma(\theta).$$

Note that M_ϵ in (3.3) (with $A = d/dz$) and $\tilde{M}_\epsilon$ in (4.2) have exactly the same form, as do u^ϵ given by (3.4) and v^ϵ given by (4.4), the only difference being that the "directing process" is Brownian motion on the real line in the first case and Brownian motion on the unit circle in the second case. Yet the limiting differential equations (4.1) and (4.5) are radically different. This striking difference is due to two factors. First, Brownian motion on the real line has only a trivial invariant measure (the zero measure), whereas Brownian motion on the unit circle has normalized Lebesgue measure as an invariant measure. Secondly, the function c is required to have the property that $k = \int c(x)\, dx > 0$ in the first case, while $k = \int c(\theta, z)\, d\theta = 0$ in the second. Apparently then, one cannot take the limit as $k \to 0$ to obtain one procedure from the other.

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