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EXTRAPOLATION TO THE LIMIT BY USING CONTINUED FRACTION INTERPOLATION

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1. **The extrapolation problem.** Assume that a convergent sequence $\{a_0, a_1, a_2, \cdots\}$ of real numbers is given with A as limit. In order to find the limit A numerically one can form a new sequence $\{b_i\}$, which has also A as limit and whose convergence is faster. One way to perform the determination of $\{b_i\}$ is to use extrapolation methods.

Let $\{x_0, x_1, \cdots\}$ be a convergent sequence of points with z as limit. The essential idea in extrapolation is to define a sequence of interpolating functions $\{y_0(x), y_1(x), \cdots\}$ such that $y_n(x_i) = a_i$ for $i = 0, 1, \cdots n$ and $n = 0, 1, 2, \cdots$. The elements b_i can be defined as follows $b_i = \lim_{x \to z} y_i(x)$ for $i = 0, 1, 2, \cdots$, if these limits exist and are finite. The points x_i are called interpolation points and z is called the extrapolation point.

Let $R(\ell, m)$ be the class of ordinary rational functions $r_{\ell,m} = p/q$ where the degree of p is at most ℓ and the degree of q at most m. Under certain conditions it is possible to construct a set of rational functions $r_{\ell,m}$ for $\ell, m = 0, 1, 2, \cdots$, satisfying $r_{\ell,m}(x_i) = a_i$ for $i = 0, 1, \cdots, \ell + m$. This set of functions can be arranged in a table as follows

$r_{0,0}$	$r_{0,1}$	$r_{0,2}$	$r_{0,3}$	—
$r_{1,0}$	$r_{1,1}$	$r_{1,2}$	$r_{1,3}$	—
$r_{2,0}$	$r_{2,1}$	$r_{2,2}$	$r_{2,3}$	—
—	—	—	—	—

In the method of Neville (polynomial extrapolation) the first column is constructed. In the method of Bulirsch and Stoer (rational extrapolation) the "staircase" $r_{0,0}, r_{1,0}, r_{1,1}, r_{2,1}, \cdots$ is constructed. In both methods $z = 0$ is used as extrapolation point and this makes the calculation of $b_{\ell+m} = r_{\ell,m}(z)$ very simple.

The elements $r_{0,0}, r_{1,1}, r_{2,2}, \cdots$ can be found by using Thiele's method for continued fraction interpolation. If $z = \infty$ is taken as extrapolation point then the values of b_i can be computed by using a method similar to the ϵ-algorithm (see [1], p. 186 and [2]).

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The effectiveness of the above methods depends highly on the convergence properties of $\{a_i\}$ and the choice of the interpolation points x_i. If very little is known about the convergence of $\{a_i\}$ it might be of interest to construct the whole table. A method for constructing the lower half of the table is given in the next section. A similar method can be used to get the upper half of the table.

2. **An algorithm for rational interpolation.** Consider the following continued fraction

$$f_k(x) = c_0 \cdot p_0(x) + c_1 \cdot p_1(x) + \cdots c_{k-1} \cdot p_{k-1}(x)$$
$$+ \frac{c_k \cdot p_k(x)}{1} - \frac{q_1^k(x - x_k)}{1}$$
$$- \frac{e_1^k(x - x_{k+1})}{1} - \frac{q_2^k(x - x_{k+2})}{1} - \frac{e_2^k(x - x_{k+3})}{1} - \cdots$$

where $p_0(x) = 1$ and $p_k(x) = (x - x_{k-1}) \cdot p_{k-1}(x)$ for $k \geqq 1$. Under certain conditions the coefficients in this continued fraction can be defined such that the nth convergent $f_{k,n}$ of f_k satisfies $f_{k,n}(x_i) = a_i$ for $i = 0, 1, \cdots, k - 1 + n$. Using contraction it is possible to obtain a continued fraction f_k' whose convergents $f'_{k,n} = P_{k,n}/Q_{k,n}$ satisfy the relation $f'_{k,n} = f_{k,2n+1}$ for $n = 0, 1, \cdots$. If we also consider f_{k+1} and define a continued fraction f'_{k+1} whose convergents satisfy $f'_{k+1,n} = f_{k+1,2n}$ then we have $f'_{k,n}(x_i) = f'_{k+1,n}(x_i) = a_i$ for $i = 0, 1, \cdots, k + 2n$. This means that there exists a nonzero constant d_n^k satisfying $P_{k+1,n} = d_n^k \cdot P_{k,n}$ and $Q_{k+1,n} = d_n^k \cdot Q_{k,n}$. The coefficients c_i in f_k and f_{k+1} can be obtained by using divided differences. In order to get the other coefficients the following recurrence relations can be used. The starting values are

$$d_1^k = 1 + \frac{c_{k+1}}{c_k} \cdot [x_{k+1} - x_k];$$

$$q_1^k = \frac{1}{d_1^k} \cdot \frac{c_{k+1}}{c_k}; \quad e_1^k = \frac{1}{d_1^k} \cdot q_1^{k+1} - q_1^k.$$

For $i \geqq 2$ we have, with $y_{k,i} = x_{k+2i-2} - x_{k+2i-1}$,

$$d_i^k = d_{i-1}^k \cdot (1 + y_{k,i} \cdot e_{i-1}^{k+1}) - d_{i-2}^k \cdot y_{k,i} \cdot q_{i-1}^{k+1} \cdot \frac{e_{i-1}^{k+1}}{e_{i-1}^k};$$

$$q_i^k = \frac{d_{i-2}^k}{d_i^k}\left(q_{i-1}^{k+1} \cdot \frac{e_{i-1}^{k+1}}{e_{i-1}^k}\right); \quad e_i^k = \frac{d_{i-1}^k}{d_i^k}(e_{i-1}^{k+1} + q_i^{k+1}) - q_i^k.$$

This algorithm is similar to the *qd*-algorithm and much of the research done for the *qd*- algorithm can also be done for the above algorithm. The convergents of f_k for $k = 1, 3, 5, \cdots$, form the lower half of the table.

References

1. F. M. Larkin, *Some techniques for rational interpolation,* The Computer Journal **10** (1967), 178–187. MR 35 #6334.

2. L. Wuytack, *A new technique for rational extrapolation to the limit,* Numerische Mathematik **17** (1971), 215–221.

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