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SEQUENCES OF IRRATIONAL FRACTION APPROXIMANTS TO SOME HYPERGEOMETRIC FUNCTIONS

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I. **Introduction.** Let the Gauss continued fraction (C.F.) for $R(x) = \ln((x+1)/(x-1))$, with x real and $|x| > 1$, have convergents $L_s = N_s/D_s$, where for example, $N_0 = 0$, $D_0 = 1$; $N_1 = 2$, $D_1 = x$; $N_2 = 3x$, $D_2 = (3x^2 - 1)/2$, etc. Also let $u_s = D_sD_{s+2} - D^2_{s+1}$, $v_s = D_sN_{s+2} + D_{s+2}N_s - 2D_{s+1}N_{s+1}$, $w_s = N_sN_{s+2} - N^2_{s+1}$. Then I have shown [1] that if

$$R_s = \{(s+2)(s+1)v_s + 2\{(2s+3)^2x^2 - 4(s+2)(s+1)\}^{1/2}\}/\{(s+2)(s+1)u_s\}, \tag{1}$$

then for $x > 1$, $\{R_s\}$ is monotonic increasing, has the limit $R(x)$ and

$$L_{s+1} < R_{s-1} < R(x). \tag{2}$$

For example, $R(x) > (-x + (9x^2 - 8)^{1/2})/(x^2 - 1)$.

Similarly, if $R(t)$ is the Laplace C.F. for Mills's ratio for the normal integral, and

$$R(t) = \frac{1}{t+}\,\frac{1}{t+}\,\frac{2}{t+}\,\frac{3}{t+}\cdots \qquad , t > 0, \tag{3}$$

with convergents χ_s/ω_s, then [2] with u_s, v_s, w_s defined in terms of the convergents as before, if

$$R_{2s} = (v_{2s} + (2s)!\,(t^2 + 8s + 4)^{1/2})/(2u_{2s}), \tag{4a}$$

and

$$R_{2s+1} = 2w_{2s+1}/\{v_{2s+1} + (2s+1)!\,(t^2 + 8s + 8)^{1/2}\}, \tag{4b}$$

we have convergent monotonic sequences

$$R_0 < R_2 < R_4 \cdots < R < \cdots R_5 < R_3 < R_1,\ t \geqq 0. \tag{5}$$

II. **Irrational Approximants to the Confluent Hypergeometric Function.** With the usual notation, the Gauss C.F. [5] is

$$F(a, 1; c; t) = \frac{1}{1-}\,\frac{b_1t}{1-}\,\frac{b_2t}{1-}\cdots,$$

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with convergents N_s/D_s, and where

$$b_{2s+1} = \frac{(a+s)(s+c-1)}{(c+2s-1)(c+2s)}, \qquad b_{2s+2} = \frac{(s+1)(c-a+s)}{(c+2s)(c+2s+1)}.$$

I showed [3] that

$$D_{2s}F(a,1;c;t) - N_{2s} = b_1 b_2 \cdots b_{2s} t^{2s} F(s+a, s+1; 2s+c; t),$$

(6) $D_{2s+1}F(a,1;c;t) - N_{2s+1}$

$$= b_1 b_2 \cdots b_{2s+1} t^{2s+1} F(s+a+1, s+1; 2s+c+1; t).$$

Using the integral representation

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}\, dt$$

$$(R(c) > R(b) > 0)$$

followed by an appeal to the inequality of Schwarz, one shows for example, that under mild restrictions,

$$\theta_{s+1} y_{2s} y_{2s+4} - \theta_s y_{2s+2}^2 \geqq 0, \quad \left(\theta_s = \frac{(c+2s-1)(c+2s)}{t^2(s+a)(c-a+s)} \right)$$

where

$$y_s = D_s F(a,1;c;t) - N_s.$$

In particular, for the C.F. for the incomplete gamma function [5]

$$R(x,a) = e^x x^{-a} \int_x^\infty e^{-t} t^{a-1}\, dt \quad (x > 0, a > 0)$$

with N_s, D_s referring to the convergents of

(7) $$R(x,a) = \frac{1}{x+}\;\frac{1-a}{1}\;\frac{1}{+x+}\;\frac{2-a}{1}\;\frac{2}{+x+}\cdots,$$

one may prove [4], defining

(8a) $$R_{2s} = (v_{2s} + (v_{2s}^2 - 4u_{2s}w_{2s})^{1/2})/(2u_{2s}),$$

(8b) $$R_{2s+1} = (v_{2s+1} + (v_{2s+1}^2 - 4u_{2s+1}w_{2s+1})^{1/2})/(2u_{2s+1})$$

that these form increasing and decreasing monotonic sequences with limit R. It is interesting to note that the discriminants reduce to multiples of $(x+2s-a+3)^2 - 4(s+1)(s-a+2)$ and $(x+2s+4-a)^2 - 4(s+1)(s+3-a)$ respectively.

III. **Suggested Generalizations.** To set up irrational fraction sequences to $R = \sum_1^\infty \alpha_s/x^s$, for x in some region and α_s real, consider

$$R(x) = A_{r+2}(x)R^2 + B_{r+1}(x)R + C_r(x) \tag{9}$$

where A, B, C are rational polynomials of degrees $r + 2$, $r + 1$, r. We may always take one specified coefficient to be known; for example the highest coefficient in A_{r+2}. Then there are $3r + 5$ unknowns which can be determined in various ways. For example: (a) reduce the discriminant to linear form and make $R(x) = 0(x^{-4})$, (b) reduce the discriminant to some appealing canonical form and determine the remaining coefficients by a suitable choice of m in $R(x) = 0(x^{-m})$.

In passing, note that if we chose $R(x)$ to be given by the Schwarzian form

$$R(x) = \theta_{s+1}(R\omega_s - \chi_s)(R\omega_{s+2} - \chi_{s+2}) - \theta_s(R\omega_{s+1} - \chi_{s+1})^2, \tag{10}$$

then the discriminant is now

$$D(x) = \theta_{s+1}\{\theta_{s+1}d^2(s, s + 2) - 4\,\theta_s d(s, s + 1)d(s + 1, s + 2)\}, \tag{11}$$

where $d(s, r) \equiv \chi_s\omega_r - \chi_r\omega_s$. If χ_s, ω_s refer to the numerator and denominator of a C.F. then the usual determinant relation, along with a suitable choice of the parameter θ_s, would lead to a simple result for $D(x)$.

Questions of convergence and non-negativity of the discriminant might prove difficult problems.

References

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