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A PRIORI TRUNCATION ERROR ESTIMATES FOR CONTINUED FRACTIONS $K(1/b_n)$

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The primary goal is to obtain *a priori* truncation error estimates of continued fractions of the form

$$K(1/b_n) = \frac{1}{b_1} + \frac{1}{b_2} + \cdots,$$

where for each $n = 1, 2, \cdots, b_n \in E_n$, and the E_n are subsets of the complex plane called element regions. The method employed is based upon a correspondence between sequences of element regions and sequences of value regions which determine a nested sequence of disks. Truncation error bounds are obtained by estimating the diameter of the nth disk which contains the nth approximant $f_n = A_n/B_n$ of the continued fraction; the A_n and B_n denote the nth numerator and denominator respectively.

The element regions E_n, can be disks, half-planes, and/or complements of disks. The following theorem, from which the results of Hillam, Sweezy and Thron ([2], [3]) are easily derived, is a typical result. In this theorem the E_n are complements of disks.

Let $\{c_n\}$ be a sequence of complex numbers and let $\{r_n\}$ and $\{\delta_n\}$ be sequences of real numbers such that

$$(1) \qquad 0 \leqq |c_n| < r_n, \delta_1 = 1, 0 < \delta_n \leqq 1, n \geqq 0.$$

Let $K(1/b_n)$ be a continued fraction with elements b_n satisfying the conditions

$$(2) \qquad \left| b_n + c_n + \frac{\bar{c}_{n-1}}{r_{n-1}^2 - |c_{n-1}|^2} \right| \geqq r_n + \frac{t_{n-1}}{\delta_n (r_{n-1}^2 - |c_{n-1}|^2)} \; .$$

If $f_n = A_n/B_n$ denotes the nth approximant of $K(1/b_n)$, where A_n and B_n are the nth numerator and nth denominator respectively, then for $n \geqq 2, p \geqq 0$

$$(3) \qquad |f_{n+p} - f_n| \leqq 2r_0 \prod_{j=2}^{n} g_j(\gamma_{j-1}, \delta_j)$$
$$\leqq 2r_0 \prod_{j=2}^{n} M_j(\delta_j) \leqq 2r_0 \prod_{j=2}^{n} \delta_j$$

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where

(4a)
$$g_j(\gamma_{j-1}, \delta_j) = \frac{\lambda_j(1-\gamma_{j-1}^2)}{2[(1/\delta_j)+\lambda_j-\gamma_{j-1}][(1/\delta_j)-\delta_{j-1}]}$$

(4b)
$$0 \leqq \gamma_{j-1} = \left| \frac{B_{j-2}}{B_{j-1}} - \frac{\bar{c}_{j-1}}{r_{j-1}^2 - |c_{j-1}|^2} \right| \left(\frac{r_{j-1}}{r_{j-1}^2 - |c_{j-1}|^2} \right)^{-1} \leqq 1$$

(4c)
$$\lambda_j = \frac{2r_j(r_{j-1}^2 - |c_{j-1}|^2)}{r_{j-1}}$$

and

(4d)
$$0 \leqq M(\delta_j) = \frac{1+\lambda_j\delta_j - \delta_j^2 - ((1-\delta_j^2)[(1-\lambda_j)^2 - \delta_j^2])^{1/2}}{\lambda_j\delta_j^2} \leqq \delta_j < 1.$$

The inequality (2) defines the element regions E_n, which by (1) cannot contain the origin. When $K(1/b_n)$ converges to a value f, truncation error estimates are obtained from (3) by replacing f_{n+p} by f. $K(1/b_n)$ will converge if $\prod M_j(\delta_j)$ (or $\prod \delta_j$) diverges to zero. If the product $\prod M(\delta_j)$ diverges to zero then the convergence of $K(1/b_n)$ is uniform over $\{E_n\}$. Although a simpler estimate of truncation error is obtained from $\prod \delta_j$ than from $\prod M(\delta_j)$, the latter estimate is in general much better. Furthermore, the error bounds in (3) are expressed directly in terms of the parameters which define E_n.

With simple geometric arguments, this theorem is useful in estimating truncation errors for continued fraction expansions of many functions of complex variables. Examples include: $\tan z$, $\tanh z$, $\arctan z$, $\operatorname{arctanh} z$, $\log(1+z)/(1-z)$, $\log(1+z)$, e^z, and $J_c(z)/J_{c-1}(z)$, the ratio of two consecutive Bessel Functions of complex order c, where $c \neq 0, -1, -2, \cdots$.

References

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