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CONTINUED FRACTION SOLUTION TO FREDHOLM INTEGRAL EQUATIONS

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I. **Introduction.** Since the solutions to Fredholm integral equations are meromorphic functions, it seems desirable to develop rational approximations for these solutions. In this paper we describe an algorithm to obtain the continued fraction representation of the solution to these equations.

We record some useful properties of the solution to Fredholm's equation in Section II. In Section III we describe the algorithm for constructing continued fraction solutions to the equation.

II. **The Fredholm Equation.** We concern ourselves with the equation

$$f(x) = g(x) + \lambda \int_0^1 K(x, t) f(t)\, dt, \tag{2.1}$$

in which it is assumed that $K(x, t)$ is continuous on its domain and that $g(x)$ is continuous on $[0, 1]$. The Fredholm solution and the Neumann series solution of (2.1) are, respectively,

$$f(x) = \sum_{j=0}^{\infty} r_j(x)\lambda^j \Big/ \sum_{j=0}^{\infty} s_j\lambda^j, \tag{2.2}$$

and

$$\begin{gathered} f(x) = \sum_{j=0}^{\infty} K_j(x)\lambda^j, \\ K_0(x) = g(x), K_{j+1}(x) = \int_0^1 K(x, t) K_j(t)\, dt. \end{gathered} \tag{2.3}$$

Here, (2.2) is a meromorphic function of λ and (2.3) converges for restricted values of λ.

We record an extension of a definition and a theorem from Wall [1, ch. 20]. Theorem 2 is proved in Taylor [2, p. 315]. The proof of Theorem 3 is straightforward and so is omitted. These Theorems give conditions under which the solution to (2.1) has a continued fraction expansion of a certain form.

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DEFINITION 1. The series (2.3) is normal, if $\Delta_{m,n} \not\equiv 0$, $m, n = 0, 1, \cdots$, where $x \in [0, 1]$ and

$$\Delta_{m,n} = \begin{vmatrix} K_{n-m} & K_{n-m+1} & \cdots & K_n \\ K_{n-m+1} & & & \vdots \\ \vdots & & & \vdots \\ K_n & \cdot & \cdot & K_{n+m} \end{vmatrix}, \quad K_{-r} \equiv 0. \tag{2.4}$$

THEOREM 1. *The series* (2.3) *is normal, if and only if, it has a continued fraction expansion of the form*

$$\frac{\beta_1(x)}{1} + \frac{\beta_2(x)\lambda}{1} + \frac{\beta_3(x)\lambda}{1} + \cdots. \tag{2.5}$$

THEOREM 2. *Equation* (2.1) *has a rational solution, if and only if* $K_0, K_1, \cdots, K_N$ *(see* (2.3)*) are linearly dependent for some* N.

THEOREM 3. *The solution to* (2.1) *is normal, if and only if, it does not represent a rational function of* λ.

Thus, the solution to (2.1) has an infinite continued fraction representation (2.5), if and only if, it does not represent a rational function of λ. Throughout the rest of this paper, we assume that this is the case, i.e., the solution to (2.1) is *not* a rational function.

III. **The Algorithm.** For notational convenience, we shall depress the functional notation by writing f for $f(x)$ etc. If a function appears under the integral sign, it is understood that the independent variable is that of the integration variable.

Under our assumption, f has the continued fraction expansion

$$f = \frac{\beta_1}{1} + \frac{\beta_2\lambda}{1} + \frac{\beta_3\lambda}{1} + \cdots. \tag{3.1}$$

Suppose now that $\alpha_i = \alpha_i(x)$ have been computed such that

$$\begin{aligned} f &= \frac{\alpha_1}{1 + f_1}, \\ f_k &= \frac{\lambda \alpha_{k+1}}{1 + f_{k+1}}, \quad k = 1, 2, \cdots, n-1, \end{aligned} \tag{3.2}$$

and

$$f - \left(\frac{\alpha_1}{1} + \frac{\alpha_2\lambda}{1} + \cdots + \frac{\alpha_n\lambda}{1} \right) = 0(\lambda^n). \tag{3.3}$$

Thus, $\alpha_i \equiv \beta_i$, $i = 1, 2, \cdots, n$, and

$$f = \frac{\alpha_1}{1+}\,\frac{\alpha_2\lambda}{1+}\,\cdots\,\frac{\alpha_i\lambda}{+\;1+f_i},\; i = 2, \cdots, n. \tag{3.4}$$

If A_n and B_n are the numerator and denominator, respectively, of the quantity in parentheses in (3.3), we have

$$f = \frac{A_n + f_n A_{n-1}}{B_n + f_n B_{n-1}} = \frac{A_n}{B_n} + \frac{(-\lambda)^{n-1}\alpha_1\alpha_2\cdots\alpha_n f_n}{B_n[B_n + f_n B_{n-1}]}. \tag{3.5}$$

Insertion of (3.5) into (2.1) yields

$$\begin{aligned} &\frac{A_n}{B_n} + (-\lambda)^{n-1} f_n U_n - g \\ &- \lambda \int_0^1 K(x,t)\frac{A_n}{B_n}\,dt + (-\lambda)^n \int_0^1 K(x,t) f_n U_n\,dt = 0, \end{aligned} \tag{3.6}$$

where

$$U_n = \frac{\alpha_1\alpha_2\cdots\alpha_n}{B_n(B_n + f_n B_{n-1})}. \tag{3.7}$$

After applying the transformation (3.2) with $k = n$ to f_n in (3.6) we get

$$\begin{aligned} &\frac{A_n}{B_n} - (-\lambda)^n \frac{B_{n+1}}{B_n} U_{n+1} - g - \lambda \int_0^1 K(x,t)\frac{A_n}{B_n}\,dt \\ &- (-\lambda)^{n+1} \int_0^1 K(x,t)\frac{B_{n+1}}{B_n} U_{n+1}\,dt = 0. \end{aligned} \tag{3.8}$$

Now

$$\frac{A_n}{B_n} - g - \lambda \int_0^1 K(x,t)\frac{A_n}{B_n}\,dt = 0(\lambda^n), \tag{3.9}$$

so, by equating powers of λ in (3.8), we can evaluate α_{n+1}. If we set

$$\frac{A_n}{B_n} - f = \frac{\lambda^n R_n}{B_n}, \tag{3.10}$$

after a few computations, we get

$$\alpha_{n+1} = \frac{(-1)^n}{\alpha_1\alpha_2\cdots\alpha_n}(K_n + b_{n,1}K_{n-1} + \cdots + b_{n,[n/2]}K_{n-[n/2]}) \tag{3.11}$$

in which

$$B_n = \sum_{j=0}^{[n/2]} b_{n,j}(x)\lambda^j \tag{3.12}$$

is the denominator polynomial defined in (3.5). One can easily show that $\alpha_{n+1} = \beta_{n+1}$.

Thus, in a straight forward fashion, one can determine the entries in the continued fraction (2.5).

References

1. H. S. Wall, *Analytic Theory of Continued Fractions,* D. Van Nostrand, 1948.
2. Angus E. Taylor, *Introduction to Functional Analysis,* John Wiley and Sons, 1967.

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