ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 4, Number 2, Spring 1974

SOME RECENT DEVELOPMENTS IN THE THEORIES OF CONTINUED FRACTIONS AND THE PADÉ TABLE

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1. **Introduction and notations.** We shall describe some new results in the theories of continued fractions and the Padé table. It is assumed that the reader is familiar with the elements of the theory of continued fractions (as given, for example, in [1] and [2]) and with the definition of the Padé quotient ([1] Ch. 5, [2] Ch. 20).

Use will be made of the following notations. The symbol $r \equiv I_i^j$ is used to indicate that an accompanying statement holds for $r = i, i+1, \cdots, j$; $r \equiv I_i$ indicates that the statement holds for $r = i, i+1, \cdots$; $r \equiv I$ indicates that it holds for $r = 0, 1, \cdots$; a symbol such as $r, m \equiv I$ is used in place of $r \equiv I$, $m \equiv I$. $r \in I$ $[r \in I_1]$ means that r is a fixed finite nonnegative [positive] integer. Single summation is tacitly understood to hold with respect to the dummy variable $\nu : \sum_i^j a_\nu$ represents $\sum_{\nu=i}^j a_\nu$; furthermore, $\sum_i a_\nu$ and $\sum a_\nu$ represent $\sum_{\nu=i}^\infty a_\nu$ and $\sum_{\nu=0}^\infty a_\nu$ respectively. Double summation is tacitly understood to hold with respect to the dummy variables ν and ν'; thus $\sum_i^j \sum_0^\nu a_{\nu,\nu'}$ represents $\sum_{\nu=i}^j \sum_{\nu'=0}^\nu a_{\nu,\nu'}$. Products are formed with respect to the variable $\tau : \prod_i^j a_\tau$ represents $\prod_{\tau=i}^j a_\tau$. $\mathfrak{D}_\mu$ denotes differentiation with respect to μ; thus $\mathfrak{D}_\mu{}^2 f(\mu)$ represents $d^2 f(\mu)/d\mu^2$. The continued fraction

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2 +} \cdots \frac{a_\nu}{b_\nu} + \cdots$$

is represented by the symbol $\{b_0 +; a_\nu : b_\nu +\}$; if b_0 is either missing or has the value zero, the symbol $\{a_\nu : b_\nu +\}$ is used; if, as sometimes occurs, a_1 and b_1 are not given by the same law of formation as that which determines the remaining $\{a_\nu\}$ and $\{b_\nu\}$, the extended symbol $\{b_0 +; a_1 : b_1 +; a_\nu : b_\nu +\}$ is used. The rth convergent of a prescribed continued fraction C is denoted by $C[C]_r$; thus $C[\{a_\nu : b_\nu +\}]_0 = 0$, $C[\{a_\nu : b_\nu +\}]_1 = a_1/b_1$, and so on. The Hankel determinant [3] of order $r+1$ $(r \geqq 0)$ whose $(\tau+1)$th row consists of the numbers $f_{m+\tau}, f_{m+\tau+1}, \cdots, f_{m+\tau+r}$ $(\tau \equiv I_0^r)$ is denoted by $H[f_{\tau+m}]_r$; we set $H[f_{\tau+m}]_{-1} = 1$. If $H[f_\tau]_r \neq 0$ $(r \equiv I)$, the series $\sum f_\nu z^\nu$ generates a nonterminating associated continued fraction ([4–11], [1] Ch. 3, [2] Ch. 11) of the form $\{f_0 : 1 + w_1 z +; v_\nu z^2 : 1 + w_\nu z +\}$;

Received by the Editors February 8, 1973.

if $H[f_\tau]_r \neq 0$ $(r \equiv I_0^{r'})$, $H[f_\tau]_r = 0$ $(r \equiv I_{r'+1})$ for some $r' \in I$, this expansion terminates; in either case we denote the continued fraction in question by $\mathcal{A}\{\sum f_\nu z^\nu\}$ (this expansion is characterized by the property that, in the nonterminating case, if for sufficiently small values of z, $C[\mathcal{A}\{\sum f_\nu z^\nu\}]_r = \sum f_\nu^{(r)} z^\nu$, then $f_\nu^{(r)} = f_\nu$ $(\nu \equiv I_0^{2r-1}$, $r \equiv I_1)$ with analogous properties holding in the terminating case). If $H[f_\tau]_r$, $H[f_{\tau+1}]_r \neq 0$ $(r \equiv I)$, the series $\sum f_\nu z^\nu$ generates a nonterminating corresponding continued fraction ([4–11], [1] Ch. 3, [2] Ch. 11) of the form $\{f_0 : 1 +; u_\nu z : 1+\}$; if for some $r' \in I$, either $H[f_\tau]_r$, $H[f_{\tau+1}]_r \neq 0$ $(r \equiv I_0^{r'})$, $H[f_\tau]_r$, $H[f_{\tau+1}]_r = 0$ $(r \equiv I_{r'+1})$ or $H[f_\tau]_r \neq 0$ $(r \equiv I_0^{r'+1})$, $H[f_{\tau+1}]_r \neq 0$ $(r \equiv I_0^{r'})$, $H[f_\tau]_r = 0$ $(r \equiv I_{r'+2})$, $H[f_{\tau+1}]_r = 0$ $(r \equiv I_{r'+1})$, this expansion terminates; in either case we denote the continued fraction in question by $\mathcal{C}\{\sum f_\nu z^\nu\}$ (we now have $C[\mathcal{C}\{\sum f_\nu z^\nu\}]_r = \sum \hat{f}_\nu^{(r)} z^\nu$ $\hat{f}_\nu^{(r)} = f_\nu$ $(\nu \equiv I_0^{r-1}$, $r \equiv I_1)$ in the nonterminating case, and similar relationships in the terminating cases). $P\{\sum f_\nu z^\nu\}$ denotes the ensemble of Padé quotients generated by the series $\sum f_\nu z^\nu$; $P_{i,j}(z) \in P\{\sum f_\nu z^\nu\}$ means that for fixed $i, j \in I$, $P_{i,j}(z)$ is the Padé quotient of order i, j derived from $\sum f_\nu z^\nu$; $P\{\sum f_\nu z^\nu\} \equiv \{P_{i,j}(z)\}$ means that the Padé quotients generated by $\sum f_\nu z^\nu$ are $P_{i,j}(z)$ $(i, j \equiv I)$. $\xi \in \hat{\boldsymbol{B}}_\alpha^\beta$ means that $\xi(\varsigma)$ is a bounded and nondecreasing real valued function for $\alpha \leqq \varsigma \leqq \beta$, where α, β are real and prescribed; if $\xi \in \hat{\boldsymbol{B}}_\alpha^\beta$ is not a simple step function with a finite number of salti over the prescribed range, then we write $\xi \in \boldsymbol{B}_\alpha^\beta$. $\{f_\nu\} \in \boldsymbol{M}[\xi]_\alpha^\beta$ means that $f_\nu = \int_\alpha^\beta \varsigma^\nu d\xi(\varsigma)$ $(\nu \equiv I)$ where $\xi \in \boldsymbol{B}_\alpha^\beta$; $\{f_\nu\} \in \boldsymbol{M}_\alpha^\beta$ means that there exists a function ξ such that $\{f_\nu\} \in \boldsymbol{M}[\xi]_\alpha^\beta$. We set $\int_\alpha^\beta d\xi(\varsigma)/(1 - z\varsigma) = \mathfrak{s}t[z; \xi]_\alpha^\beta$.

2. **Interpolatory rational functions.** For $m \in I$, let $f_\nu^{(m)}$ be the νth order divided difference of the function $f(x)$ with respect to the arguments $x_{m+\tau}(\tau \equiv I_0^\nu)$ for $\nu \equiv I$. The rth partial sum $\Phi_r^{(m)}(x)$ of the Newton interpolation series [12] $\sum f_\nu^{(m)} \prod_0^{\nu-1} (x - x_{m+\tau})$ (taking $\Phi_0^{(m)}(x) = f_0^{(m)}$, and so on) is a polynomial of degree r whose value agrees with that of $f(x)$ when $x = x_{m+\tau}(\tau \equiv I_0^r)$. The partial sums $\{\Phi_r^{(m)}(x)\}$ may be computed by means of a well known recursive algorithm — the Aitken-Neville process [13, 14].

Let $m, i, j \in I$, let $N_{i,j}^{(m)}(x)$, $D_{i,j}^{(m)}(x)$ be two polynomials of degree j and i respectively, and set $R_{i,j}^{(m)}(x) = N_{i,j}^{(m)}(x)/D_{i,j}^{(m)}(x)$. The equations $R_{i,j}^{(m)}(x) = f(x)$ $(x = x_\tau,\ \tau \equiv I_m^{m+i+j})$ may be expressed as a system of $i + j + 1$ homogeneous linear equations involving the coefficients of $N_{i,j}^{(m)}(x)$ and $D_{i,j}^{(m)}(x)$, which under certain conditions determine $R_{i,j}^{(m)}(x)$ as the quotient, first obtained by Jacobi [15], of two determinants. The derivation of this quotient may be regarded as being a transformation of the interpolatory series referred to above.

The rational functions $R_{r,r}^{(m)}(x)$, $R_{r,r+1}^{(m)}(x)$ $(r \equiv I)$ may be constructed by means of Thiele's process ([16], [17] Ch. 15): if numbers $\{\rho_r^{(m)}\}$ can be computed from the initial values $\rho_{-1}^{(m)} = 0$ $(m \equiv I_1)$, $\rho_0^{(m)} = f(x_m)$ $(m \equiv I)$ by means of the recursion $\rho_{r+1}^{(m)} = \rho_{r-1}^{(m+1)} + (x_{m+r+1} - x_m)/(\rho_r^{(m+1)} - \rho_r^{(m)})$ $(r,\ m \equiv I)$ then $R_{r,r}^{(m)}(x)$ $[R_{r,r+1}^{(m)}(x)]$ is the $2r^{\text{th}}$ $[(2r+1)^{\text{th}}]$ convergent $(r \equiv I)$ of the continued fraction

$$(1) \qquad \{f(x_m) + ; x - x_m : \rho_1^{(m)} + ; x - x_{m+\nu-1} : \rho_\nu^{(m)} - \rho_{\nu-2}^{(m)} + \}.$$

An entirely different and somewhat more efficient process for determining the values of the $\{R_{r,r}^{(m)}(x)\}$, $\{R_{r,r+1}^{(m)}(x)\}$ was devised by the author [18], this approach was considerably developed by Stoer [19] and the theory has been further refined by Larkin [20]. A special case of Larkin's algorithm is as follows: setting $R_{r,r}^{(m)}(x) = R_{2r}^{(m)}(x)$, $R_{r,r+1}^{(m)}(x) = R_{2r+1}^{(m)}(x)$ $(m, r \equiv I)$, $\hat{x}_m = x - x_m$, $(m \equiv I)$, we have $R_0^{(m)}(x) = f(x_m)$, $R_1^{(m)}(x) = \{\hat{x}_m R_0^{(m+1)}(x) - \hat{x}_{m+1} R_0^{(m)}(x)\}/(\hat{x}_m - \hat{x}_{m+1})$ $(m \equiv I)$;

$$(2) \qquad R_{r+1}^{(m)}(x) = R_{r-1}^{(m+1)}(x) + (\hat{x}_m - \hat{x}_{m+r+1}) \Big/ \left(\frac{\hat{x}_m}{R_r^{(m+1)}(x) - R_{r-1}^{(m+1)}(x)} + \frac{\hat{x}_{m+r+1}}{R_{r-1}^{(m+1)}(x) - R_r^{(m)}(x)} \right). \quad (r \equiv I_1,\ m \equiv I)$$

3. **Rational functions derived from power series.** As the arguments in Newton's interpolation series tend to the same value, μ say, this series becomes Taylor's series $\sum f_\nu z^\nu$, where $f_\nu = \mathfrak{D}_\mu{}^\nu f(\mu)/\nu!$ $(\nu \equiv I)$ and $z = x - \mu$; Jacobi's determinantal quotient becomes the *Näherungsbruch* [15] or approximating fraction or what is now called the Padé quotient $P_{i,j}(z)$ derived from this power series. The derivation of the quotient $P_{i,j}(z)$ expressed in the (possibly reducible) form $P_{i,j}(z) = N_{i,j}(z)/D_{i,j}(z)$ where $N_{i,j}(z) = \sum_0^j n_\nu^{(i,j)} z^\nu$, $D_{i,j}(z) = \sum_0^i d_\nu^{(i,j)} z^\nu$ from the equations

$$\sum_0^r d_\nu^{(i,j)} f_{r-\nu} = n_r^{(i,j)} \ (r \equiv I_0^j), \quad \sum_0^r d_\nu^{(i,j)} f_{r-\nu} = 0 \ (r \equiv I_{j+1}^{j+i})(d_\nu^{(i,j)} = 0, \nu > i)$$

was placed on a rigorous basis by Frobenius [9]. A systematic study of the whole ensemble of rational functions $\{P_{i,j}(z)\}$ was carried out by Padé [21] who considered [22] the exponential series in detail (for results on the approximation of the exponential function by Padé quotients, see [23]); the Padé quotients derived from the exponential series are identical with rational approximations to the exponential function derived earlier by Darboux [24] (for a recent use of the method of Darboux in obtaining error estimates for Padé approximants, see [25]).

In his general inquiry, Padé derived a number of structural theorems. In particular, that if $P\{\sum f_\nu z^\nu\} \equiv \{P_{i,j}(z)\}$, $P\{\sum f_{m+\nu} z^\nu\} \equiv \{P_{i,j}^{(m)}(z)\}$ $(m \in I)$ then $P_{i,j+m}(z) = \sum_0^{m-1} f_\nu z^\nu + z^m P_{i,j}^{(m)}(z)$ $(j \equiv I, \quad i \equiv I_{j+1})$; furthermore, if $\{\sum f_\nu z^\nu\}\{\sum \tilde{f}_\nu z^\nu\} = \sum i_\nu z^\nu$ where $i_0 = 1$, $i_\nu = 0$ $(\nu \equiv I_1)$, and $P\{\sum \tilde{f}_\nu z^\nu\} \equiv \{\tilde{P}_{i,j}(z)\}$, then $\tilde{P}_{i,j}(z) = P_{j,i}(z)$ $(i, j \equiv I)$ (these results may, of course, be combined). He also established a connection between Padé quotients and continued fractions derived from power series: if for some $m \in I_1$, $\mathcal{A}\{\sum f_{m+\nu} z^\nu\}$ exists, then $P_{r,m+r-1}(z) = \sum_0^{m-1} f_\nu z^\nu + z^m C[\mathcal{A}\{\sum f_{m+\nu} z^\nu\}]_r$ and if $\mathcal{C}\{\sum f_{m+\nu} z^\nu\}$ exists then $P_{r,m+r-1}(z) = \sum_0^{m-1} f_\nu z^\nu + z^m C[\mathcal{C}\{\sum f_{m+\nu} z^\nu\}]_{2r}$, $P_{r,m+r}(z) = \sum_0^{m-1} f_\nu z^\nu + z^m C[\mathcal{C}\{\sum f_{m+\nu} z^\nu\}]_{2r+1}$, in each case for all $r \in I$ for which the convergent referred to exists (for convenience in exposition, we append the quotient $P_{0,-1}(z) = 0$ to the Padé table). The structure of Padé quotients derived from quotients of power series has recently been considered by Householder [**26**].

The arguments $\{x_\nu\}$ occurring in recursion (2) may also be allowed to tend to the same value μ; the resulting algorithm motivated the discovery of a difference-differential recursion [27] relating three Padé quotients $P_{i,j}(\mu; z) \in P\{\sum\{\mathcal{D}_\mu{}^\nu f(\mu)/\nu!\} z^\nu\}$ $(z = z' - \mu)$ of which two are assumed distinct: setting

$$\theta_{i,j}\{a(\mu), b(\mu)\} =$$

$$a(\mu) + (i+j)\{b(\mu) - a(\mu)\}^2/[(i+j)\{b(\mu) - a(\mu)\} - z\mathcal{D}_\mu b(\mu)],$$

$$P_{i,j}(\mu; z) = \theta_{i,j}\{P_{i-1,j-1}(\mu; z), P_{i-1,j}(\mu; z)\},$$

$$P_{i,j}(\mu; z) = \theta_{i,j}\{P_{i-1,j-1}(\mu; z), P_{i,j-1}(\mu; z)\}. \; (i, j \in I_1)$$

The above difference-differential recursion is certainly not the most straightforward method for constructing Padé quotients. The simplest such method, presented in terms of a series $\sum f_\nu z^\nu$ for which $H[f_{\tau+m}]_r \neq 0$ $(r, \quad m \equiv I)$ and for those quotients $P_{i,j}(z) \in P\{\sum f_\nu z^\nu\}$ for which $j \geqq i - 1$, is the $q - d$ algorithm [**10**, **28**, **29**]: set $a_1^{(m)} = 0$ $(m \equiv I_1)$, $a_2^{(m)} = -f_{m+1} f_m{}^{-1} (m \equiv I)$ and compute recursively

$$a_{2r+1}^{(m)} = a_{2r}^{(m+1)} + a_{2r-1}^{(m+1)} - a_{2r}^{(m)}, \quad (r \equiv I_1, m \equiv I)$$

$$a_{2r+2}^{(m)} = a_{2r+1}^{(m+1)} a_{2r}^{(m+1)} a_{2r+1}^{(m)^{-1}};$$

then set $N_{0,m}(z) = \sum_0^m f_\nu z^\nu$, $D_{0,m}(z) = 1$ $(m \equiv I_{-1})$; for $r \equiv I_1$, $m \equiv I$

$$N_{r,m+r-1}(z) = N_{r-1,m+r-1}(z) + a_{2r}^{(m)} z N_{r-1,\, m+r-2}(z),$$

$$D_{r,m+r-1}(z) = D_{r-1,\, m+r-1}(z) + a_{2r}^{(m)} z D_{r-1,\, m+r-2}(z).$$

Certain cases are known in which the coefficients $\{f_\nu\}$ and the numbers $\{a_r^{(m)}\}$ derived from them can be expressed in simple closed form; they are all subsumed within the following general result [30]: if $f_m = \prod_0^{m-1}(\psi_\tau/\phi_\tau)$ $(m \equiv I)$, where $\psi_\tau = A - q^{\alpha+\tau} \neq 0$, $\phi_\tau = C - q^{\gamma+\tau} \neq 0$ $(\tau \equiv I)$, then for $r \equiv I_1$, $m \equiv I$

$$a_{2r}^{(m)} = q^{r-1}\psi_{m+r-1}\phi_{m+r-2}/\phi_{m+2r-3}\phi_{m+2r-2},$$
$$a_{2r+1}^{(m)} = q^{m+r-1}(1-q^r)(Cq^\alpha - Aq^{\gamma+r-1})/\phi_{m+2r-2}\phi_{m+2r-1},$$

and closed expressions may also be given for the $\{n_r^{(m)}\}$, $\{d_r^{(m)}\}$. Special cases, relating to the coefficients $f_m = \prod_0^{m-1}(1-q^{\alpha+\tau})$, $\prod_0^{m-1}(1-q^{\gamma+\tau})^{-1}$, q^{m^2}, $\prod_0^{m-1}\{(\alpha+\tau)/(\gamma+\tau)\}$, $\prod_0^{m-1}(\alpha+\tau)$, $\prod_0^{m-1}(\gamma+\tau)^{-1}$ may be derived as limiting forms of this special result.

Numerical values of Padé quotients are most economically determined by use of the ϵ-algorithm [31, 32]: set $\epsilon_{-1}^{(m)} = \epsilon_{2m}^{(-m)} = 0$ $(m \equiv I_1)$, $\epsilon_0^{(m)} = \sum_0^{m-1} f_\nu z^\nu (m \equiv I)$; if numbers $\{\epsilon_r^{(m)}\}$ can be computed by use of the recursion

$$\epsilon_{r+1}^{(m)} = \epsilon_{r-1}^{(m+1)} + (\epsilon_r^{(m+1)} - \epsilon_r^{(m)})^{-1} (r \equiv I, r' = -[r/2], m \equiv I_{r'})$$

then $\epsilon_{2r}^{(m)} = R_{r,m+r-1}(z)$ $(r \equiv I,\ m \equiv I_{-r+1})$. The numbers $\{\epsilon_r^{(m)}\}$ with odd suffix may be eliminated from the above formulae, and for distinct Padé quotients we obtain [33] the recursion

$$\{P_{i+1,j}(z) - P_{i,j}(z)\}^{-1} + \{P_{i-1,j}(z) - P_{i,j}(z)\}^{-1}$$
$$= \{P_{i,j+1}(z) - P_{i,j}(z)\}^{-1} + \{P_{i,j-1}(z) - P_{i,j}(z)\}^{-1}$$

and by rearrangement, a similar formula in which each Padé quotient is replaced by its reciprocal.

The theory of the ϵ-algorithm may be used to derive a second difference-differential recursion [27] for Padé quotients $\hat{P}_{i,j}(\mu; z) \in P\{\sum \mathfrak{D}_\mu^\nu \phi(\mu)\} z^\nu\}$ which, assuming two of the three quotients concerned to be distinct, is as follows: setting

$$\psi\{c(\mu), d(\mu)\} =$$
$$[d(\mu)^2 - c(\mu)\{\phi(\mu) + z\mathfrak{D}_\mu d(\mu)\}]/[2d(\mu) - c(\mu) - \phi(\mu) - z\mathfrak{D}_\mu d(\mu)],$$
$$\left.\begin{aligned}\hat{P}_{i,j}(\mu; z) &= \psi\{\hat{P}_{i-1,j-1}(\mu; z), \hat{P}_{i-1,j}(\mu; z)\},\\ \hat{P}_{i,j}(\mu; z) &= \psi\{\hat{P}_{i-1,j-1}(\mu; z), \hat{P}_{i,j-1}(\mu; z)\}.\end{aligned}\right. \quad (i,j \in I_1).$$

4. **Discrete extrapolation algorithms.** The interpolation processes described in § 2 may be used to estimate the limit or formal limit S of a sequence of numbers $\{S_\nu\}$. As an example, we consider extra-

polation to the limit by the use of polynomials. We take $x = 1/(\nu + \sigma)$ as the variable in the interpolatory polynomials where σ is a fixed constant, and select a subsequence $\{\nu_m\}$ $(m \equiv I)$ of the integers $\nu \equiv I$. The estimate $\phi_r^{(m)}$ of S derived from the numbers S_{ν_τ} $(\tau \equiv I_m^{m+r})$ is the value of the interpolating polynomial, when $x = 1/(\nu + \sigma) = 0$, which assumes the value S_{ν_τ} when $x = 1/(\nu_\tau + \sigma)$ $(\tau \equiv I_m^{m+r})$. The various estimates of S for $r, m \equiv I$ may be constructed by use of the Aitken-Neville scheme: we set $\phi_0^{(m)} = S_{\nu_m}$ $(m \equiv I)$ and have

$$\phi_{r+1}^{(m)} = \{(\nu_{m+r+1} + \sigma)\phi_r^{(m+1)} - (\nu_m + \sigma)\phi_r^{(m)}\}/(\nu_{m+r+1} - \nu_m). \; (r, m \equiv I)$$

Taking $\nu_m = m$ $(m \equiv I)$, $\sigma = 1$, we obtain a somewhat unstable process equivalent in principle to a method of Salzer [34] (for a related algorithm, see [35]). Taking $\nu_m = 2^m$ $(m \equiv I)$, $\sigma = 0$, we obtain Romberg's algorithm [36] (see also [37]) used for numerical integration by successive doubling of the number of subintervals of integration.

The successive convergents of the continued fraction (1) are interpolating rational functions. It is easily shown that $\lim_{x=\infty} R_{2r}^{(m)}(x) = \rho_{2r}^{(m)}$ $(r, m \in I)$. Setting $\rho_{-1}^{(m)} = 0$ $(m \equiv I_1)$, $\rho_0^{(m)} = S_{\nu_m}$ $(m \equiv I)$ and computing

$$\rho_{r+1}^{(m)} = \rho_{r-1}^{(m+1)} + (\nu_{m+r+1} - \nu_m)/(\rho_r^{(m+1)} - \rho_r^{(m)}) \; (r, m \equiv I)$$

the numbers $\{\rho_{2r}^{(m)}\}$ provide estimates, derived from extrapolation of rational functions, of S. Setting $\nu_m = m$ $(m \equiv I)$ we obtain the ρ-algorithm [39]; setting $\nu_m = 2^m$ $(m \equiv I)$ we obtain an algorithm which can be used for numerical integration [40].

The values of the Padé quotients $P_{r,m+r-1}(z) \in P\{\sum f_\nu z^\nu\}$ $(r, m \equiv I)$, in particular, provide estimates of the sum or formal sum S of the series whose partial sums are $S_m = \sum_0^{m-1} f_\nu z^\nu$ $(m \equiv I)$; the ϵ-algorithm, which may be used to construct these quotients, is thus a process for extrapolating the partial sums $\{S_m\}$ to a limit. To provide another motivation, we remark that if $\sum f_\nu z^\nu$ is the series expansion of the rational function $S = \sum_1^{r'} \sum_1^{\tau_\nu} d_{\nu,\nu'} (1 - z\lambda_\nu)^{-\nu'}$, then $S_m = \sum_1^{r'} \lambda_\nu{}^m \sum_0^{\tau_\nu - 1} A_{\nu,\nu'} m^{\nu'}$ $(m \equiv I)$ (where the $\{A_{\nu,\nu'}\}$ depend upon z). If $\sum_1^{r'} \tau_\nu = r$, then $P_{r,m+r-1}(z) = S$ $(m \equiv I)$ (the determinantal formulae involved, which are of course identical to those obtained by Jacobi [15], were also derived by Schmidt [40] and Shanks [41]). Thus the ϵ-algorithm can be regarded as being an extrapolation procedure using exponential cum polynomial sums of the above form.

It is possible to repeat application of the ϵ-algorithm by selecting a sequence of one set of numbers $\{{}_\tau\epsilon_{2r}^{(m)}\}$ to be the initial sequence for the construction of the next $\{{}_{\tau+1}\epsilon_{2r}^{(m)}\}$. In one mode of repetition, called corresponding repeated application, [43] we set ${}_0\epsilon_0^{(m)} = S_m$ $(m \equiv I)$ and thereafter ${}_{\tau+1}\epsilon_0^{(2m)} = {}_\tau\epsilon_{2m}^{(0)}$, ${}_{\tau+1}\epsilon_0^{(2m+1)} = {}_\tau\epsilon_{2m}^{(1)}$ $(m \equiv I)$. For an $m' \in I_2$, the numbers ${}_0\epsilon_0^{(1)}$ and ${}_0\epsilon_0^{(m')}$ taken together indicate the rate of convergence of the sequence $\{S_m\}$ under transformation, the number ${}_1\epsilon_0^{(m')}$ indicates the effect of one application of the ϵ-algorithm, the number ${}_2\epsilon_0^{(m')}$ that of two, and so on. Taking $S_m = \sum_0^{m-1} \binom{-1/2}{\nu} (\nu+1)^{-1}$ $(m \equiv I)$, we find that ${}_0\epsilon_0^{(1)} = 1.0$, ${}_0\epsilon_0^{(6)} = 0.81$ $\cdots$, ${}_1\epsilon_0^{(6)} = 0.82840$ $\cdots$; ${}_2\epsilon_0^{(6)} = 0.82842\ 71247\ 49$. Since $\lim_{m=\infty} S_m = 2(2^{1/2}-1) = 0.82842\ 71247\ 43$, it will be seen that we have extracted from the first six terms of the series under transformation information otherwise to be obtained by direct summation of more than ten thousand million terms. (This is perhaps an appropriate juncture at which to remark that details of the numerical behaviour of continued fractions are given in [43]; Algol procedures relating to continued fractions and the ϵ-algorithm for complex numbers are given in [44] and for real numbers in [42, 45]; programs in Fortran are given in [46]).

Brezinski [47] and Gekeler [48] have recently investigated application of the ϵ-algorithm to the sequence $\{S_m\}$ produced by means of the scheme $S_{m+1} = F(S_m)(m \equiv I)$, and in this way have constructed high order iterative processes not involving the use of derivatives for finding a fixed point of the equation $S = F(S)$.

The ϵ-algorithm possesses a number of invariant and other properties: If the ϵ-algorithm can be applied to the numbers $\{S_m\}$ satisfying an irreducible recursion of the form $\sum_0^h d_\nu S_{m+\nu} = G$ $(m \equiv I)$ to produce numbers $\{\epsilon_{2h}^{(m)}\}$, then $\epsilon_{2h}^{(m)} = G/D$ $(m \equiv I)$ where $D = \sum_0^h d_\nu$ [31]; furthermore, if in this recursion $D \neq 0$ then, assuming again that all numbers concerned can be produced, $\sum_0^{2h-1} (-1)^\nu \epsilon_\nu^{(m)} \epsilon_{\nu+1}^{(m)} = -D'/D$ $(m \equiv I)$, where $D' = \sum_1^h \nu d_\nu$ [49]. The numbers $\{\epsilon_r^{(m)}\}$ produced from a prescribed sequence $\{S_m\}$ are simply related to those produced from the sequences $\{S_{m+1} - S_m\}$ and $\{\sum_0^m S_\nu\}$ [50]. Further algebraic results are given in [51]. Auxiliary numerical transformations are described in [52, 53]. Algorithms related to the ϵ-algorithm have been investigated by Brezinski [54].

5. **Convergence theory and quantitative behaviour.** In a celebrated thesis [55] Hadamard related the behaviour of a function $f(z)$ analytic at the origin to that of ratios of the form $H[f_{\tau+j+1}]_i/H[f_{\tau+j}]_i$ of the Hankel determinants formed from the coefficients $\{f_\nu\}$ of the Taylor series expansion of $f(z)$, and of certain polynomials also formed from these coefficients. Very astutely, de Montessus de Ballore noted [56] that just these determinantal ratios and polynomials occur in the formula for the difference $P_{i,j+1}(z) - P_{i,j}(z)$, and in this

way derived his convergence result: if $f(z) = \sum f_\nu z^\nu$ has $i \in I_1$ poles (counted according to their multiplicity) and no other singularities within the circle $C_\zeta : |z| = \zeta$, then the sequence $R_{i,j}(z) \in P\{\sum f_\nu z^\nu\}$ $(j \equiv I)$ converges uniformly to $f(z)$ except in the neighborhoods of the included poles within C_ζ; if $f(z)$ has a singularity upon C_ζ, this sequence diverges for $|z| > \zeta$. (This result says nothing concerning the further sequences $\{R_{i+\tau,j}(z)\}$ $(\tau \in I_1$ and j increasing); Perron ([1] § 45) gives an example in which $\{R_{0,j}(z)\}$ converges in any bounded domain of the z-plane, whilst $\{R_{1,j}(z)\}$ diverges on a point set everywhere dense in the domain). Dienes [**57**] extended the work of Hadamard by investigating the behaviour of a Taylor series upon its circle of convergence; Wilson [**58**, **59**] continued the theory by investigating the behaviour of the sequence $\{R_{i,j}(z)\}$ $(j \equiv I)$ upon its circle of convergence and at the included poles. Pólya [**60**], Wilson [**61**] and Edrei [**62**] have related the behaviour of $f(z)$ to that of expressions of the form $\{H[f_{\tau+\nu}]_i\}^{1/\nu}$ (see, also [**64**]).

The first general investigation of the convergence of continued fractions derived from power series (i.e., concerning classes of functions, and not special functions generating expansions whose coefficients are expressible in closed form) was conducted by Markoff [**64**] in continuation of earlier work by Tschebyscheff [**65**]. He was concerned with series $\sum f_\nu z^\nu$ for which $\{f_\nu\} \in M[\xi]_\alpha^\beta$ $(-\infty < \alpha < \beta < \infty)$; his result is that $\mathcal{A}\{\sum f_\nu z^\nu\}$ converges to $f(z) = \mathfrak{st}\,[z;\xi]_\alpha^\beta$ uniformly in D_α^β, any bounded open domain not containing any point z such that $z^{-1} \in [\alpha, \beta]$. (If, in the preceding, $\xi \in \hat{B}_\alpha^\beta$ and is a simple step function with a finite number of salti, then $f(z)$ is a rational function, $\mathcal{A}\{\sum f_\nu z^\nu\}$ terminates and reproduces $f(z)$). Stieltjes [**10**] extended this investigation to the expansion $\mathcal{C}\{\sum f_\nu z^\nu\}$ where $\{f_\nu\} \in M[\xi]_\alpha^\beta$ and either $0 \leqq \alpha < \beta \leqq \infty$ or $-\infty \leqq \alpha < \beta \leqq 0$, and it was subsequently shown by Carleman [**66**] that convergence of $\mathcal{C}\{\sum f_\nu z^\nu\}$ to $f(z)$ in D_α^β is ensured if the series $\sum_1^\infty f_\nu^{-1/2\nu}$ diverges. The theory was extended further by Hamburger [**67**] and Nevanlinna [**68**] who considered coefficient sequences $\{f_\nu\} \in M[\xi]_\alpha^\beta$ for general real intervals $-\infty \leqq \alpha < \beta \leqq \infty$ (for an application of this theory to orthogonal polynomials, see [**70**]); Carleman's result in this case is that $\mathcal{A}\{\sum f_\nu z^\nu\}$ converges to $f(z)$ uniformly in D_α^β if the series $\sum_1^\infty f_{2\nu}^{-1/2\nu}$ diverges.

Stieltjes' theory was extended to the Padé table generated by the series $\sum t_\nu z^\nu$ in question by Van Vleck [**70**] who showed, in particular, that if $\{f_\nu\} \in M_0^\infty$, then $\{f_{m+\nu}\} \in M_0^\infty$ also $(m \equiv I)$, and that if $\sum f_\nu z^\nu = f_0\{1 - z\sum \tilde{f}_\nu z^\nu\}^{-1}$ (in the sense of formal power series) then $\{\tilde{f}_{m+\nu}\} \in M_0^\infty$ $(m \equiv I)$. Using these results Wall, in his doctoral dissertation [72], gave a complete analysis of the convergence behaviour of the forward diagonal sequences of the Padé table derived from a Stieltjes series. He later gave a similar treatment [72] relating to series $\sum f_\nu z^\nu$ for which $\{f_\nu\} \in M_\alpha^\beta$ and $-\infty \leqq \alpha < \beta \leqq \infty$, and also completely investigated the convergence of all forward diagonal sequences of the Padé table relating to the case in which $\{f_\nu\} \in M_\alpha^\beta$ and $-\infty < \alpha < \beta < \infty$ [73].

An important *motif* occurring in the theory of Hamburger and Nevanlinna is that for $\{f_\nu\} \in M[\xi]_\alpha^\beta$ and nonreal values of z, $\mathcal{A}\{\sum t_\nu z^\nu\}$ yields a sequence of nested inclusion domains for the value of $f(z) = \mathfrak{st}\,[z; \xi]_\alpha^\beta$. This theory can be extended to the Padé table generated by $\sum f_\nu z^\nu$ and *a fortiori* to the case in which $\sum f_\nu z^\nu$ is a Stieltjes series, with $\{f_\nu\} \in M[\xi]_0^\infty$. We derive the result [74] that for quotients $R_{i,j}(z)$ for which $j \geqq i-1$ derived from this Stieltjes series with nonreal argument, the circle through the values of $P_{i,j}(z)$, $P_{i,j+1}(z)$, and $P_{i+1,j+1}(z)$ and the circle through the values of $P_{i,j}(z)$, $P_{i+1,j}(z)$ and $P_{i+1,j+1}(z)$ both include the value of $f(z) = \mathfrak{st}\,[z; \xi]_0^\infty$. Such triads taken from the sequences $P_{r,m+r-1}(z)$, $P_{r,m+r}(z)$ $(m \in I,\ r \equiv I)$ intersect to provide nested convex inclusion domains for $f(z)$; in particular, those deriving from the case $m = 0$ yield the inclusion domains of Henrici and Pfluger [75]. The reciprocals of the values of similar triads of quotients for which $i \geqq j$ yield inclusion domains for the value of $f(z)^{-1}$.

It was shown by Stieltjes [10] that when $\{f_\nu\} \in M[\xi]_0^\infty$ and $-\infty < z < 0$, the successive convergents of $\mathcal{C}\{\sum f_\nu z^\nu\}$ yield approximations to $f(z) = \mathfrak{st}\,[z; \xi]_0^\infty$ which are in a certain sense optimal. This theory can be extended to the Padé table generated by the series $\sum f_\nu z^\nu$, and we obtain the result [76] that for the series in question and $-\infty < z < 0$,

$$z^{-m}\{f(z) - P_{r,m+r-1}(z)\} = \tag{4}$$

$$\min_{X_1, X_2, \cdots, X_r} \int_0^\infty \left\{1 + \sum_1^r X_\nu (1 - z\varsigma)^\nu\right\}^2 \frac{\varsigma^m d\xi(\varsigma)}{1 - z\varsigma} \quad (r, m \equiv I)$$

the $\{X_\nu\}$ being real numbers. The partial index m determines the weight function $(\varsigma^m d\xi(\varsigma)/(1 - z\varsigma))$ with respect to which each quotient is a best approximation; the partial index r determines the number of disposable parameters.

It is clear that the expression upon the right hand side of relationship (4) with r replaced by $r+1$ has one more disposable parameter; since neighbouring distinct Padé quotients have, when $z \neq 0$, distinct values, its value is therefore less than that of the expression as it stands. Indeed a quantitative study of the forward and backward diagonal sequences of the Padé table in question can be based upon an examination of relationships (4). If the terminating or nonterminating sequence $a_r (r \equiv I_0^{r'})$ is monotonically increasing [decreasing] with upper [lower] bound A, we write $a_r |_{r=0}^{r'} \in MI\{A\}$ $[a_r |_{r=0}^{r'} \in MD\{A\}]$. Using this notation we find [76] that for $-\infty < z < 0$ and the Padé quotients in question and $m \in I$, $P_{r,2m+r-1}(z) |_{r=0}^{\infty} \in MI\{f(z)\}$, $P_{r,2m+r}(z) |_{r=0}^{\infty} \in MD\{f(z)\}$, $P_{r,2m-r}(z) |_{r=0}^{m} \in MD\{P_{m,m}(z)\}$, $P_{r,2m-r+1}(z) |_{r=0}^{m+1} \in MI\{P_{m+1,m}(z)\}$.

The analysis of the preceding two paragraphs has been extended [77] to the Padé quotients $\{P_{i,j}(z)\}$ for which $i > j$ and also so as to concern series $\sum f_\nu z^\nu$ for which $\{f_\nu\} \in M_\alpha^\beta$ where $[\alpha, \beta]$ is a general interval of the real axis.

Nearly all of the theory of this section was used to derive the following result [74]: let M_ν, b_ν $(\nu \equiv I_1^h)$ be two sets of positive real numbers with $b_h < \cdots < b_2 < b_1 < \infty$ and let $\xi \in B_a^b$ where $0 \leqq a \leqq b < b_h$, let D be the open disc $|z| < b^{-1}$ cut along the real segment $(-b^{-1}, -b_1^{-1})$, let

$$f(z) = \mathfrak{s}t\,[-z; \xi]_a^b + \sum_1^h M_\nu/(1 + b_\nu z),$$

let $P\{\sum t_\nu z^\nu\} \equiv \{P_{i,j}(z)\}$ where $f(z) = \sum f_\nu z^\nu$ for sufficiently small z, and define a progressive sequence of such quotients to be one in which the successor $P_{i'',j''}(z)$ to $P_{i',j'}(z)$ is such that either $i'' > i'$, $j'' \geqq j'$ or $i'' \geqq i'$, $j'' > j'$. Then any progressive sequence of quotients $\{P_{i,j}(z)\}$ for which $i, j \geqq h$ converges uniformly to $f(z)$ for $z \in D$. This result has also been extended to more general functions $f(z)$ [77].

6. **Access to the convergence theory.** It is of great help when investigating practical problems to know whether the theory of the preceding paragraphs can be applied — to know, in particular, whether the coefficients of a given series $\sum f_\nu z^\nu$ can be expressed in the form $f_\nu = \int_\alpha^\beta \varsigma^\nu d\xi(\varsigma)$ $(\nu \equiv I)$ where $0 \leqq \alpha < \beta \leqq \infty$ and $\xi \in B_\alpha^\beta$; they are unlikely to be given in this form. For this reason we give some results [78] which can be used to resolve this problem. We present the theory in terms of the function $F(\lambda) = \lambda^{-1} f(-\lambda^{-1})$, where $f(z) = \mathfrak{s}t\,[z; \xi]_\alpha^\beta$ is the function generating the series $\sum f_\nu z^\nu$ in question. We have ([79] Ch. 8)

$$(5) \qquad \text{(i)}\ F(\lambda) = \int_0^\infty e^{-\lambda\zeta} g(\zeta)\, d\zeta, \qquad \text{(ii)}\ g(\zeta) = \int_0^\infty e^{-\zeta\varsigma}\, d\xi(\varsigma).$$

It suffices for our requirements, therefore, to show that the function $g(\zeta)$ in formula (5i) has the form given by (5ii) in which $\xi \in \hat{B}_0^\infty$. The function $g(\zeta)$ is said to be completely monotone (we write $g(\zeta) \in CM$) if $(-\mathfrak{D}_\zeta)^\nu g(\zeta) \geqq 0$ $(0 \leqq \zeta \leqq \infty,\ \nu \equiv I)$; for example $(1 + \alpha\zeta)^{-\beta}$ $(0 \leqq \alpha \leqq \infty,\ 0 < \beta < \infty)$ and $e^{-\alpha\zeta}(0 \leqq \alpha \leqq \infty)$ are CM-functions. According to a theorem of Bernstein [80] and Widder [81], $g(\zeta) \in CM$ if and only if $g(\zeta)$ is expressible in the form (5ii) with $\xi \in \hat{B}_0^\infty$. Thus we have shown that $\sum f_\nu z^\nu$ is a Stieltjes series if the related function $F(\lambda) = \lambda^{-1} f(-\lambda^{-1})$ is expressible in the form (5i) where $g(\zeta) \in CM$. It may be possible to show that $g(\zeta) \in CM$ directly. There are, however, a number of results which can be used to construct CM functions from others of the same type; in favorable cases they may be used in reverse to show that a given function $g(\zeta)$ is composed of CM constituents and is hence itself CM: if $g_1(\zeta), g_2(\zeta) \in CM$, then $g_1(\zeta) + g_2(\zeta)$, $g_1(\zeta)g_2(\zeta) \in CM$; if $\{\mathfrak{D}_\zeta\, g(\zeta)\}/g(\zeta) \in CM$ then [82] $1/g(\zeta) \in CM$ (this result is of use in the investigation of infinite products: if $g(\zeta)$ is an infinite product, $\{\mathfrak{D}_\zeta\, g(\zeta)\}/g(\zeta)$ is an infinite sum and as such somewhat easier to deal with than $g(\zeta)$ itself); if $-\mathfrak{D}_\zeta\, g(\zeta) \in CM$, then [82] $e^{-g(\zeta)} \in CM$. We must still consider convergence: having shown that the function $g(\zeta)$ in formula (5i) is CM, it follows from Carleman's criterion that if $g(\zeta)$ is regular at the origin of the ζ-plane, then all diagonal sequences of the Padé table converge to $f(z)$ for all finite $z \notin [0, \infty]$.

Naturally the series $\sum f_\nu z^\nu$ is of Stieltjes type if $\{f_\nu\} \in B_0^\beta$, where $0 < \beta < \infty$. By changing the variable from z to $z\beta$, we need only consider the case $\{f_\nu\} \in B_0^1$. The sequence $\{f_\nu\}$ is said to be totally monotone (we write $\{t_\nu\} \in TM$) if $\Delta^\tau f_\nu \geqq 0$ $(\tau, \nu \equiv I)$ where $\Delta^0 f_\nu = f_\nu$ $(\nu \equiv I)$, $\Delta^{\tau+1} f_\nu = \Delta^\tau f_\nu - \Delta^\tau f_{\nu+1}$ $(\tau, \nu \equiv I)$; for example, $\{(\alpha + \beta\nu)^{-1}\}$ $(0 < \alpha, \beta < \infty)$ and $\{\alpha^\nu\}$ $(0 < \alpha \leqq 1)$ are TM sequences. By a theorem of Hausdorff [83, 84], $\{f_\nu\} \in TM$ if and only if $\{f_\nu\} \in M_0^1$. Again there are a number of results which may be used to show that $\{f_\nu\} \in TM$. If $\{f_\nu^{(1)}\}, \{f_\nu^{(2)}\} \in TM$, then $\{f_\nu^{(1)} + f_\nu^{(2)}\}$, $\{f_\nu^{(1)} f_\nu^{(2)}\} \in TM$. If $\{f_\nu\} \in TM$ then, subject to the attached conditions the following sequences are also TM [85]: (i) $\{(1 - f_\nu)^{-1}\}$ $(f_0 < 1)$; (ii) $\{\prod_0^{\nu-1} f_\tau^{-1}\}$ $(\lim_{\nu=\infty} f_\nu \geqq 1)$; (iii) $\{\prod_0^{\nu-1}(1 - f_\tau)\}$ $(f_0 \leqq 1)$; (iv) $\{\eta^{-\Delta^h f_\nu}\}$ $(0 < \eta \leqq 1; h \in I)$; (v) $\{\eta^{\sum_{\tau=0}^{\nu-1} f_\tau}\}$ $(0 < \eta \leqq 1)$. If $g(\zeta) \in CM$ and $0 \leqq \alpha, \beta < \infty$ then $\{g(\alpha + \nu\beta)\} \in TM$. Lastly, if $\{f_\nu\} \in TM$ and $\{\sum f_\nu z^\nu\}/\{1 - z \sum \tilde{f}_\nu z^\nu\} = f_0$ for sufficiently small z, then $\{\tilde{f}_\nu\} \in TM$ [10, 86]. Having shown that $\{t_\nu\} \in TM$, it follows from Markoff's theorem that all forward diagonal sequences of quotients $P_{i,j}(z) \in P\{\sum f_\nu z^\nu\}$ converge for all finite $z \notin [1, \infty]$ to the function defined by analytic continuation of the sum of the series $\sum f_\nu z^\nu$. We remark that it has been shown that for $z \in [-1, 0)$ at least, application of the ϵ-algorithm to the sequence $\{\sum_0^{m-1} f_\nu z^\nu\}$ ($\{f_\nu\} \in \boldsymbol{TM}$) is a stable numerical process [87].

It should be pointed out that even when the above results are available, a great deal is still left to the ingenuity of the investigator.

The above theory has been used in an investigation of the continued fraction transformation of Newton's interpolation series, Newton's series for the derivative, Gregory's integration series, and the Euler-Maclaurin series [88]. It may also be used to investigate the continued fraction transformation of Fourier series [89].

7. **Classes of functions connected with the stability of operators and with smoothing operations.** Consider the equation

$$\frac{\partial \phi(t)}{\partial t} = L\phi(t) \tag{6}$$

where $\phi(t)$ is a function of variables $x, y, \cdots$ as well as t, and L is a bounded linear operator operating in the domain of $x, y, \cdots$ but not of t; (6) may, for example, be a partial differential equation. The solution to (6) satisfies the relationship $\phi(t + \Delta t) = e^{L\Delta t}\phi(t)$. If the eigenspectrum of L is confined to the open left half-plane $\mathrm{Re}(\lambda) < 0$ we have, using a suitable norm, $\|\phi(t + \Delta t)\| < \|\phi(t)\|$ for all $\Delta t \geqq 0$, and $\lim_{t=\infty}\|\phi(t)\| = 0$. In practice, equation (6) is solved by using an approximation to $e^{L\Delta t}$, for example $C_3(z) = (1 + \frac{1}{2}z)/(1 - \frac{1}{2}z)$, where $z = L\,\Delta t$, derived from the corresponding continued fraction for e^z; either of the explicit or implicit schemes

$$(7)\qquad \begin{aligned} &\text{(i) } \hat{\phi}(t + \Delta t) = (1 + \tfrac{1}{2}L\,\Delta t)(1 - \tfrac{1}{2}L\,\Delta t)^{-1}\hat{\phi}(t),\\ &\text{(ii) } (1 - \tfrac{1}{2}L\,\Delta t)\hat{\phi}(t + \Delta t) = (1 + \tfrac{1}{2}L\,\Delta t)\hat{\phi}(t) \end{aligned}$$

may be used to derive an approximate solution $\hat{\phi}(t)$ to equation (6). The zeros of the denominator of $C_3(z)$ lie in the open right half-plane $\mathrm{Re}(z) > 0$ and the function $C_3(z)$ maps the left half-plane $\mathrm{Re}(z) \leqq 0$ onto the unit disc $|C_3(z)| \leqq 1$. Thus, if the eigenspectrum of L is confined to the open left half-plane $\mathrm{Re}(\lambda) < 0$, the operator upon the right hand side of equation (7i) may be constructed, and equation (7ii) may be solved. Furthermore $\|\hat{\phi}(t + \Delta t)\| < \|\hat{\phi}(t)\|$ and $\lim_{t=\infty}\|\hat{\phi}(t)\| = 0$, in analogy with the corresponding relationships for the exact solution. Similar use may be made of any convergent $C_{2r+1}(z)$ ($r \in I_1$) of the corresponding continued fraction for e^z; the properties of $C_3(z)$ just referred to are also possessed by $C_{2r+1}(z)$ ([90], see also [91]).

The properties of e^z described above are shared by functions of a general class [92], whose members may be represented by the formula

$$f(z) = 1 + \omega z / \{1 - \tfrac{1}{2}\omega z + z^2 \mathit{s}t[-z^2; \xi]\,{}_0^\infty\}. \tag{8}$$
$$(0 < \omega < \infty, \xi \in B_0^\infty)$$

Such a function is real for real z; in the sectors $-\frac{1}{2}\pi < \arg(z) < \frac{1}{2}\pi$ and $\frac{1}{2}\pi < \arg(z) < \frac{3}{2}\pi$ it generates an asymptotic series $\sum f_\nu z^\nu$; $\mathcal{C}\{\sum t_\nu z^\nu\}$ exists; the zeros of the denominators of $C[\mathcal{C}\{\sum t_\nu z^\nu\}]_{2r+1}$ $(r \in I_1)$ lie in the half-plane $\operatorname{Re}(z) > 0$ and this convergent maps the left half plane $\operatorname{Re}(z) \leqq 0$ onto the unit disc. e^z is one such function, $z + ((1 + z^2))^{1/2}$ is another, and ${}_1F_1(\alpha + 1; 2\alpha + 1; z)/{}_1F_1(\alpha; 2\alpha + 1; z)$ $(-\frac{1}{2} < \alpha < \infty)$ yet another. Using the general properties of functions of the above class we deduce [93], for example, that if ${}_1F_1(\alpha; 2\alpha + 1; z) = 0$ $(-\frac{1}{2} < \alpha < \infty)$ then $\operatorname{Re}(z) > 0$. For a subclass of the above functions, it may be shown that the roots of the equation $f(z) = C[\mathcal{C}\{\sum t_\nu z^\nu\}]_{2r+1}$ $(r \in I)$ are pure imaginary, symmetrically distributed about the origin and, if $f(z)$ is nonrational, unbounded in number; furthermore they interlace in the sense that if the roots are $z = \pm i y_\nu^{(r)}$, then subject to a suitable ordering $y_0^{(r)} = y_0^{(r+1)} = 0$, $y_\nu^{(r)} < y_\nu^{(r+1)} < u_{\nu+1}^{(r)}$ $(\nu \equiv I_1^{r'})$, where $r' = \infty$ if $f(z)$ is nonrational. This result is a generalization of the formula $\mathrm{E}^{\pm 2\nu\pi i} = 1$ $(\nu \equiv I)$.

Functions having a representation of the form

$$g(z) = a e^{\gamma z} \left\{ \prod_1^j (1 + \alpha_\tau z) \right\} \Big/ \left\{ \prod_1^i (1 - \beta_\tau z) \right\}$$

$$(0 \leqq i, j \leqq \infty, 0 < a < \infty, 0 \leqq \gamma < \infty, \alpha_\tau > 0\ (\tau \equiv I_1^j), \tag{9}$$

$$\beta_\tau > 0\ (\tau \equiv I_1^i);\ \sum_1^j \alpha_\nu, \sum_1^i \beta_\nu < \infty)$$

were introduced by Schoenberg [94] in connection with the study of smoothing operations, and studied extensively by Edrei and others [95–110]. Arms and Edrei [111] have, by using recursions based upon known formulae relating to the function e^z, derived the salient properties of the Padé table generated by functions of the form (9). The two classes of functions having representations of the forms (8) and (9) intersect: $g(z)$ also has a representation of the form (8) if and only if $a = 1$, $i = j$ and $\alpha_\tau = \beta_\tau$ $(\tau \equiv I_1^i)$. Furthermore, the functions $g(z)$ of this intersection also belong to the subclass described at the end of the preceding paragraph.

8. **Confluent prediction algorithms and integration methods.** In

§ 4 we described algorithms for estimating the limit or formal limit of the sequence $\{S_m\}$ in terms of the numbers $S_m, S_{m+1}, \cdots$ or, equivalently, in terms of the numbers $S_m, \Delta S_m, \Delta^2 S_m, \cdots$. If we introduce the substitution $\mu = \mu' + m\Delta\mu$ where μ' and $\Delta\mu$ are finite constants (so that μ becomes $\mu + \Delta\mu$ when m is increased to $m+1$) and an auxiliary substitution in the discrete algorithm in question, we derive [112, 113] a confluent form of the algorithm which can be used to estimate $\lim_{\mu=\infty} S(\mu)$ in terms of $S(\mu)$ and its successive derivatives evaluated at a finite point μ. If yet another change of variable is introduced, replacing μ by μ'' and setting $S(\mu'') = \int_\mu^{\mu''} \psi(\mu')\, d\mu'$, the confluent form evolves to a second confluent form which may be used to estimate $\int_\mu^\infty \psi(\mu')\, d\mu'$ in terms of $\psi(\mu)$ and its successive derivatives.

The successive functions produced by the last algorithms concerned are [**113**], in the case of extrapolation by polynomials

$$\hat{\phi}_r(\sigma; \mu) = \sum_1^r \binom{r}{\nu} \frac{(\mu-\sigma)^\nu}{\nu!} \mathfrak{D}_\mu^{\nu-1}\psi(\mu), \quad (r \equiv I) \tag{10}$$

in the case of extrapolation by rational functions

$$\hat{\rho}_{2r}(\mu) = H[\mathfrak{D}_\mu^{\tau-1}\psi(\mu)/\tau!]_r / H[\mathfrak{D}_\mu^{\tau+1}\psi(\mu)/(\tau+2)!]_{r-1} \quad (r \equiv I) \tag{11}$$

and in the case of extrapolation by exponential cum polynomial sums

$$\hat{\epsilon}_{2r}(\mu) = H[\mathfrak{D}_\mu^{\tau-1}\psi(\mu)]_r / H[\mathfrak{D}_\mu^{\tau+1}\psi(\mu)]_{r-1} \quad (r \equiv I) \tag{12}$$

(taking $\mathfrak{D}_\mu^{\tau-1}\psi(\mu) = 0$ when $\tau = 0$ in both (11) and (12)).

The functions of (10)–(12) may also be produced by the use of discrete algorithms: set $\zeta_0^{(0)} = 0$, $\zeta_0^{(m)} = (\mu-\sigma)^m \mathfrak{D}_\mu^{m-1}\psi(\mu)/m!$ $(m \equiv I_1)$ and compute $\zeta_{r+1} = \zeta_r^{(m)} + \zeta_r^{(m+1)}$ $(r, m \equiv I)$, then $\zeta_r^{(0)} = \hat{\phi}_r(\sigma; \mu)(r \equiv I)$; set $\omega_{-1}^{(m)\prime} = 0$ $(m \equiv I_1)$, $\omega_0^{(0)\prime} = 0$, $\omega_0^{(m)\prime} = \mathfrak{D}^{m-1}\psi(\mu)/m!$ $(m \equiv I_1)$ and compute

$$\begin{aligned} \omega_{2r+1}^{(m)\prime} &= \omega_{2r-1}^{(m+1)\prime} + \omega_{2r}^{(m)\prime} / \omega_{2r}^{(m+1)\prime}, \\ \omega_{2r+2}^{(m)\prime} &= \omega_{2r}^{(m+1)\prime} (\omega_{2r+1}^{(m)\prime} - \omega_{2r+1}^{(m+1)\prime}), \end{aligned} \quad (r, m \equiv I) \tag{13}$$

then $\omega_{2r}^{(0)\prime} = \hat{\rho}_{2r}(\mu)$; set $\omega_{-1}^{(m)} = 0$ $(m \equiv I_1)$, $\omega_0^{(0)} = 0$, $\omega_0^{(m)} = \mathfrak{D}_\mu^{m-1}\psi(\mu)$ $(m \equiv I_1)$ and compute further numbers $\{\omega_r^{(m)}\}$ by means of recursions similar to (13), then $\omega_{2r}^{(0)} = \hat{\epsilon}_{2r}(\mu)$ $(r \equiv I_1)$. (The last two algorithms can break down).

If $\psi(\mu)$ is the polynomial in $(\mu-\sigma)^{-1}$

$$\psi(\mu) = \sum_1^{r'} A_\nu (\mu-\sigma)^{-\nu-1} \tag{14}$$

then $\mathfrak{D}_\mu \hat{\phi}_r(\sigma;\mu) = -\psi(\mu)$ $(r \equiv I_{r'})$, and if σ does not lie on the path of integration, $\hat{\phi}_r(\sigma;\mu) = \int_\mu^\infty \psi(\mu')\,d\mu'$ $(r \equiv I_{r'})$. If σ lies on the path of integration, and the coefficients of even powers of $(\mu-\sigma)^{-1}$ in formula (14) are zero, then $\hat{\phi}_r(\sigma;\mu) = \oint_\mu^\infty \psi(\mu')\,d\mu'$ $(r \equiv I_{r'})$, whilst if certain of these coefficients are nonzero the functions $\hat{\phi}_r(\sigma;\mu)$ serve to define a divergent integral, written as $(\phi(\sigma))\int_\mu^\infty \psi(\mu')\,d\mu'$. Similar considerations relate to functions of the form $\hat{\rho}_{2r'}(\mu)$ and $\hat{\epsilon}_{2r'}(\mu)$ produced from appropriate integrands. It has been shown [114], and extensive numerical experimentation has confirmed the finding, that in certain cases these methods of integration may be used to evaluate integrals whose integrands have an infinite number of poles which together with their limit point lie on the path of integration.

The functions of formula (12) occur in the theory of the continued fraction integral [115]. Subject to certain restrictions $\lim_{\Delta\mu=0}\sum \psi(\mu+\nu\,\Delta\mu)\,\Delta\mu = \int_\mu^\infty \psi(\mu')\,d\mu'$ where this integral is defined in the extended Riemann sense. The continued fraction integral $(CF)\int_\mu^\infty \psi(\mu')\,d\mu'$ is defined as follows: the successive convergents of $\mathcal{A}\{\sum \psi(\mu+\nu\,\Delta\mu)\,\Delta\mu z^\nu\}$ are functions of the form $C_r(\mu;\Delta\mu;z)$ $(r \equiv I)$. It is found that if the derivatives $\mathfrak{D}_\mu^\nu \psi(\mu)$ $(\nu \equiv I)$ exist, $\lim_{\Delta\mu=0}\lim_{z=1} C_r(\mu;\Delta\mu;z) = \hat{\epsilon}_{2r}(\mu)$ $(r \equiv I)$. (CF) $\int_\mu^\infty \psi(\mu')\,d\mu'$ is defined to be the last member of the sequence $\{\hat{\epsilon}_{2r}(\mu)\}$ if the succeeding functions of the form (12) are indeterminate, and otherwise to be the limit, if it exists and is finite, of this sequence.

9. **The partial differential equation of the Padé surface.** Two typical relationships between numbers $\{\epsilon_r^{(m)}\}$ of the ϵ-algorithm are

$$(15)\qquad \begin{aligned} (\epsilon_{2r}^{(m)} - \epsilon_{2r-2}^{(m+1)})(\epsilon_{2r-1}^{(m+1)} - \epsilon_{2r-1}^{(m)}) &= 1,\\ (\epsilon_{2r+1}^{(m)} - \epsilon_{2r-1}^{(m+1)})(\epsilon_{2r}^{(m+1)} - \epsilon_{2r}^{(m)}) &= 1. \end{aligned}$$

We introduce a change of coordinates, setting $x = x' + 2kr$, $y = y' + 2k(r+m)$ where x', y' and k are constants, and a change of dependent variable, setting $\hat{\epsilon}(x,y) = \epsilon_{2r}^{(m)}$, $\tilde{\epsilon}(x+k,\ y+k) = 4k^2\epsilon_{2r+1}^{(m)}$. Relationships (15) become

$$\left\{\frac{\hat{\epsilon}(x,y)-\hat{\epsilon}(x-2k,y)}{2k}\right\}\left\{\frac{\tilde{\epsilon}(x-k,y+k)-\tilde{\epsilon}(x-k,y-k)}{2k}\right\} = 1$$

$$\left\{\frac{\epsilon(x+k,y+k)-\tilde{\epsilon}(x-h,y+k)}{2k}\right\}\left\{\frac{\hat{\epsilon}(x,y+2k)-\epsilon(x,y)}{2k}\right\} = 1.$$

Letting k tend to zero we obtain the pair of partial differential equations

$$\hat{\epsilon}_x \tilde{\epsilon}_y = 1, \qquad \tilde{\epsilon}_x \hat{\epsilon}_y = 1 \tag{16}$$

where the suffix now denotes partial differentiation. Eliminating one of the functions $\hat{\epsilon}(x, y)$ and $\tilde{\epsilon}(x, y)$ from equation (16), we obtain a relationship for the other which may be expressed in one of the two forms

$$\{1/\epsilon_x\}_x = \{1/\epsilon_y\}_y, \quad \epsilon_{xx}\epsilon_y^{\,2} = \epsilon_{yy}\epsilon_x^{\,2}. \tag{17}$$

The numbers $\{\epsilon_{2r}^{(m)}\}$ produced from appropriate initial values are Padé quotients: relationships (17) taken together constitute the partial differential equation of the Padé surface [**116**]. If either of them is satisfied by the function $\epsilon(x, y)$, we write $\epsilon(x, y) \in PD$. Thus, $1/(x - y) \in PD$ for all x and y such that this function is defined, i.e. for all x and y except $x = y = \infty$; again, the finite constant $c \in PD$ for all x and y.

The partial differential equation of the Padé surface has a number of remarkable properties [**117**]. If $f(\xi)$ is a twice differentiable function of ξ, and α, β, γ and δ are finite constants, then $f(\alpha x + \beta x + \gamma xy + \delta) \in PD$. For example, if $f(z)$ is an analytic function of the complex variable $z = x + iy$, then $f(x + iy)$, $f(x - iy) \in PD$; again $f(x + y)$, $f(x - y) \in PD$. With the same function $f(\xi)$, relationships (17) and their first order finite difference approximations, of which the first is

$$\begin{aligned} &\{\epsilon(x + k, y) - \epsilon(x, y)\}^{-1} + \{\epsilon(x - k, y) - \epsilon(x, y)\}^{-1} \\ &= \{\epsilon(x, y + k) - \epsilon(x, y)\}^{-1} + \{\epsilon(x, y - k) - \epsilon(x, y)\}^{-1} \end{aligned}$$

and the second may be obtained by rearrangement of this equation, have the same solution of the form $\epsilon(x, y) = f(x + y)$ or $f(x - y)$, independent of the mesh length k. If $\epsilon(x, y) \in PD$ and $\zeta(\xi)$ is twice differentiable in ξ, then $\zeta\{\epsilon(x, y)\} \in PD$ also. If $\hat{\epsilon}(x, y)$, $\tilde{\epsilon}(x, y)$ satisfy equations (16), and $\psi(x, y)$ is twice differentiable in x and y, then $\psi\{\hat{\epsilon}(x, y), \tilde{\epsilon}(x, y)\} \in PD$; this result includes, of course, the case in which $\psi(x, y) \in PD$. (For the sake of conciseness we have not described the domains of the $x - y$ plane over which the above results hold.) The partial differential equation of the Padé surface is the Euler equation of a certain variational problem: if the integral $\iint_G \ln(\epsilon_x/\epsilon_y)\, dxdy$ is to have an extremum, where G is a prescribed region of the $x - y$ plane whose boundary curve has a tangent which turns piecewise continuously, then it is necessary that $\epsilon(x, y) \in PD$ at all points of G.

10. **Continued fractions with coefficients over a ring.** Noncommutative continued fractions were first introduced in connection with

the solution of quadratic equations in quaternions by Hamilton [118], who derived the fundamental recursions for the determination of successive convergents of such continued fractions. Turnbull [119] derived these recursions for the case in which the coefficients are square matrices, and Wedderburn [120] did likewise for the case in which the coefficients are mixtures of a variable x and a differential operator $\mathfrak{D}_x$ (the continued fractions concerned occur in the solution of homogeneous second order linear differential equations). An extensive structural theory of such continued fractions was given by the author [121] who considered, in particular, the derivation of such continued fractions from power series.

Let R be a ring with unit element I, a set of invertible elements R_I, and a centre $C\{R\}$ (e.g. R is the system of $p \times p$ $(p \in I_1)$ matrices, R_I is the set of invertible $p \times p$ matrices, $C\{R\}$ is the set of scalar multiples of the $p \times p$ unit matrix). The convergents $\{C_r'\}$ of the *pre*-continued fraction $pre\{B_0 +; A_\nu : B_\nu + \}$ may be defined by setting $N_{-1}' = I$, $N_0' = B_0$, $D_{-1}' = 0$, $D_1' = I$ and determining $D_r' = B_r D_{r-1}' + A_r D_{r-2}'$, $N_r' = B_r N_{r-1}' + A_r N_{r-2}'$ $(r \equiv I_1)$ when $C_r' = D_r'^{-1} N_r'$ $(r \equiv I)$. The convergents $\{C_r''\}$ of the *post*-continued fraction $post\{B_0 +; A_\nu : B_\nu + \}$ may be defined by setting $N_{-1}'' = D_0'' = I$, $D_{-1}'' = 0$, $N_0'' = B_0$ and determining $D_r'' = D_{r-1}'' B_r + D_{r-2}'' A_r$, $N_r'' = N_{r-1}'' B_r + N_{r-2}'' A_r$ $(r \equiv I_1)$, when $C_r'' = N_r'' D_r''^{-1} (r \equiv I)$. In general, the convergents of these two systems of continued fractions derived from the same coefficients are unequal; for example, the convergents of order unity are $B_0 + B_1^{-1} A_1$ and $B_0 + A_1 B_1^{-1}$ respectively. However, if in particular a *pre*-continued fraction is regular in the sense that $B_r, D_r' \in R_I$ $(r \equiv I_1)$ then it may be thrown by means of an equivalence transformation into a form in which all $A_\nu \in C\{R\}$ For such a continued fraction the *pre*- and *post*-convergents are equal.

If $F_0 \in R_I$, the formal power series $F(z) = \sum F_\nu z^\nu (F_\nu \in R$, $\nu \equiv I_1)$ has a two-sided inverse $\tilde{F}(z) = \sum \tilde{F}_\nu z^\nu$ for which $F(z)\tilde{F}(z) = \tilde{F}(z)F(z) = I(z)$, where $I(z)$ is the series $\sum I_\nu z^\nu$ for which $I_0 = I$, $I_\nu = 0$ $(\nu \equiv I_1)$ [122]. Euclid's algorithm applied to $F(z)$ is defined by setting $F^{(0)}(z) = F(z)$ and determining $F^{(r)}(z)^{-1} = \sum \tilde{F}_\nu^{(r)} z^\nu$, $B_{r+1} = \tilde{F}_0^{(r)}$, $F^{(r+1)}(z) = \sum \tilde{F}_{\nu+1} z^\nu (r \equiv I)$ [123]. If $B_r \in R_I$, $(r \equiv I)$ then $F(z)$ generates the continued fraction $\{I : B_1 +; z : B_\nu + \}$ whose *pre*- and *post*-convergents are equal: setting $C[pre\{I : B_1 +; z : B_\nu + \}]_r = D_r'(z)^{-1} N_r'(z)$, $C[post\{I : B_1 + ; z : B_\nu + \}]_r = N_r''(z) D_r''(z)^{-1}$ $(r \equiv I)$, we have $D_r'(z)^{-1} N_r'(z) = N_r''(z) D_r''(z)^{-1}$ in the sense that $D_r'(z) N_r''(z)$

$= N_r'(z)D_r''(z)$ $(r \equiv I)$ for all $z \in C\{R\}$ (this relationship may be presented in terms of the noncommutative orthogonal polynomials derived from the moment sequence $\{F_\nu\}$, and then yields a fundamental result in the theory of such polynomials). The series expansion $\hat{F}(z) \equiv \sum \hat{F}_\nu z^\nu$ of the rational function $\hat{D}(z)^{-1}\hat{N}(z)$, where $\hat{D}(z)$ and $\hat{N}(z)$ are polynomials and the constant term in $\hat{D}(z)$ belongs to R_I, may be determined by equating coefficients of corresponding powers of z in the relationship $\hat{D}(z)\hat{F}(z) \equiv \hat{N}(z)$; similarly for the series expansion of $\tilde{N}(z)\tilde{D}(z)^{-1}$. Reverting to the continued fraction $\{I : B_1 + ; z : B_\nu + \}$, the series expansion of $D_r'(z)^{-1}N_r'(z)$ (or $N_r''(z)D_r''(z)^{-1}$) agrees with $F(z)$ as far as the term $F_{r-1}z^{r-1}$. If the numbers $\{B_r^{(m)}\}$ so produced all belong to R_I, Euclid's algorithm may be applied to each of the series $\sum F_{m+\nu}z^\nu$ to produce continued fractions $\{I : B_1^{(m)} + ; z : B_\nu^{(m)} + \}$. In this way we derive two sided Padé quotients of the form $P_{r,m+r-1}(z) = \sum_0^{m-1} F_\nu z^\nu + z^m C$ $[pre,\ post\,\{I : B_1^{(m)} +;\ z : B_\nu^{(m)} + \}]_{2r}$. The continued fraction $\{I : \mathbf{B}_1^{(m)} +;\ z : B_\nu^{(m)} + \}$ may be thrown into a one-sided form $pre\{F_m : I +;\ A_\nu^{(m)} z : I + \}$ and, with the symbol a replaced by A, the *pre-q-d* algorithm relationships are (3). The Padé quotients of the form $P_{m+r,r}(z)$ $(r, m \equiv I)$ may be determined as above by use of the series reciprocal to $F(z)$. The whole ensemble of Padé quotients $\{P_{i,j}(z)\}$ may be constructed by use of Euclid's algorithm as described above if and only if they can also be constructed by applying the ϵ-algorithm to the initial values $E_{-1}^{(m)} = E_{2m}^{(-m)} = 0$ $(m \equiv I_1)$, $E_0^{(m)} = \sum_0^{m-1} F_\nu z^\nu$ $(m \equiv I)$ as described in § 3.

The above theory may be used to derive the following result: if the $p \times p$ $(p \in I_1)$ matrices $\{S_m\}$ satisfy the recursion $\sum_0^r D_\nu S_{m+\nu} = S$ $(m \equiv I)$, where the $\{D_\nu\}$ and S are also $p \times p$ matrices and $D = \sum_0^r D_\nu$ is invertible, and the ϵ-algorithm can be applied to the sequence $\{S_m\}$ to produce matrices $\{E_{2r}^{(m)}\}$ then $E_{2r}^{(m)} = D^{-1}S$ $(m \equiv I)$. The matrices $\{S_m\}$ then also satisfy a similar recursion of the form $\sum_0^r S_{m+\nu}D_\nu' = S$ $(m \equiv I)$ with $D' = \sum_0^r D_\nu'$ being invertible and $D^{-1}S = SD'^{-1}$.

A very modest convergence theory of noncommutative continued fractions exists [**118, 124, 125**].

11. **Vector continued fractions.** Addition and subtraction of vectors are defined in terms of the components of the vectors concerned; the inverse of the vector $z = (z_1, z_2, \cdots, z_p)$, where the $\{z_\nu\}$ are complex numbers, is defined by $z^{-1} = (\sum_1^p z_\nu \bar{z}_\nu)^{-1}\bar{z}$ where the bar denotes the complex conjugate. The vector sequence $\{s_m\}$ may be transformed by means of the ϵ-algorithm, since the arithmetic operations

required are defined ([**126–131**]; for application to sequences obtained by iterated projections, see [**132**], and for a connection with linear programming, see [**133**]). At first, it was difficult to establish a direct theory of the vector ϵ-algorithm, but an indirect theory was derived by using an isomorphism due to J. B. McLeod [**134**]. He constructed a set of $2^{p'} \times 2^{p'}$ $(p' = 2p+1)$ matrices $\{\Gamma_\nu\}$ satisfying the relationships $\Gamma_\nu{}^2 = I$ $(\nu \equiv I_1^{2p+1})$, $\Gamma_\nu\Gamma_{\nu'} + \Gamma_{\nu'}\Gamma_\nu = 0$ $(\nu \equiv I_1^{2p}$, $\nu' \equiv I_{\nu+1}^{2p+1})$ (these matrices are examples of a system due to Hurwitz [**135**]; they are, of course, special cases of Clifford numbers [**136, 137, 138**]; for applications of Clifford numbers in physics, see [**139–144**]). The matrix Z isomorphic to the vector z with $z_\nu = x_\nu + iy_\nu$ $(\nu \equiv I_1^p)$ is $Z = \sum_1^p x_\nu \Gamma_{2\nu} + \sum_1^p y_\nu \Gamma_1 \Gamma_{2\nu+1}$. Denoting the matrix isomorphic to $\bar{z}$ by $\tilde{Z}$, we find that $Z\tilde{Z} = (\sum_1^p z_\nu \bar{z}_\nu)I$, i.e. $Z^{-1} = (\sum_1^p z_\nu\bar{z}_\nu)^{-1}\tilde{Z}$: the vector-matrix isomorphism is preserved during inversion (and, of course, during addition and subtraction). If vectors $\{\epsilon_r{}^{(m)}\}$ [matrices $\{E_r{}^{(m)}\}$] are produced by applying the vector [matrix] ϵ-algorithm to the sequence $\{s_m\}$ $[\{S_m\}]$ and $s_m \leftrightarrow S_m$ $(m \equiv I)$, then $\epsilon_r{}^{(m)} \leftrightarrow E_r{}^{(m)}$ for all vectors produced. Using this result McLeod proved a result conjectured [**130**] by the author: if vectors $\epsilon_{2r'}^{(m)}$ $(m \equiv I)$ can be produced from the initial sequence $\{s_m\}$, where $\sum_0^{r'} d_\nu s_{m+\nu} = s$ $(m \equiv I)$, the $\{d_\nu\}$ being real and $d = \sum_0^{r'} d_\nu \neq 0$ then $\epsilon_{2r'}^{(m)} = d^{-1}s$ $(m \equiv I)$. This result was used by Brezinski [**145**] and Gekeler [**146**] to show that the scheme $s_{m+1}^{(i)} = F(s_m^{(i)})$ $(m \equiv I_0^{2p+1})$, $s_0^{(i+1)} = \epsilon_{2p}^{(i,0)}$ $(i \equiv I)$, where $\{\epsilon_r^{(i,m)}\}$ denote the p-dimensional vectors produced from the initial sequence $\{s_m^{(i)}\}$, provides, under certain quite benevolent conditions not involving convergence of the scheme $s_{m+1} = F(s_m)$ $(m \equiv I)$, a quadratically convergent process for determining the fixed point of the equation $s = F(s)$. The isomorphism described above has been extended by the author [**147**], and it has been shown that the matrices $ZZ'Z$ and $ZZ'Z'' + Z''Z'Z$ are isomorphic to vectors if Z, Z' and Z, Z', Z'' are. Furthermore, the isomorphism is an isometry in the sense that if $\|z\| = (\sum_1^p z_\nu\bar{z}_\nu)^{1/2}$ and $\|Z\| = ((\textit{maximum eigenvalue of } ZZ^*))^{1/2}$, where the asterisk denotes the complex conjugate transpose, then for companion vectors and matrices $\|z\| = \|Z\|$.

The generalized inverse A^+ of the $p \times q(p, q \in I_1)$ matrix A is uniquely determined [**148–151**] by the conditions $AA^+A = A$, $A^+AA^+ = A^+$, $(AA^+)^* = AA^+$, $(A^+A)^* = A^+A$. In this sense z^{-1} is the transpose of z^+. The generalized inverse $\tilde{f}(z) = \sum \tilde{f}_\nu z^\nu$ of the formal power series $f(z) = \sum f_\nu z^\nu$ with either $1 \times p$ or $p \times 1$ $(p \in I_1)$ vector coefficients is uniquely determined [**152**] by the four equations $f(z)\tilde{f}(z)f(z) = f(z)$, $\tilde{f}(z)f(z)\tilde{f}(z) = \tilde{f}(z)$, $f(z)\tilde{f}(z)$

$= \sum G_\nu z^\nu$, $G_\nu{}^* = G_\nu$ $(\nu \equiv I)$, $\tilde{f}(z)f(z) = \sum H_\nu z^\nu$, $H_\nu{}^* = H_\nu$ $(\nu \equiv I)$. In the $1 \times p$ vector case the formulae determining the $\{\tilde{f}_\nu\}$ are, if $f_0 \neq 0$, $z_0 = 1/(f_0 f_0{}^*)$, $\tilde{f}_0 = f_0{}^* z_0$,

$$\zeta_r = \sum_0^r (f_\nu f^*_{r-\nu})\, z_0, \quad z_r = -\sum_0^{r-1} z_\nu \zeta_{r-\nu}, \; \tilde{f}_r = \sum_0^r z_\nu \; f^*_{r-\nu} (r \equiv I_1)$$

(the vector-matrix isomorphism described above is preserved during this process of inversion: if $\sum \tilde{F}_\nu z^\nu$ is the inverse of $\sum F_\nu z^\nu$ as described in § 10, and $F_\nu \leftrightarrow f_\nu$ $(\nu \equiv I)$ then $\tilde{F}_\nu \leftrightarrow \tilde{f}_\nu$ $(\nu \equiv I)$ also [**123**]). Using this inverse, it has been possible to establish a direct theory of vector continued fractions. Euclid's algorithm for formal power series with vector valued coefficients is defined in analogy with the case described in § 10. There is a vector form of the $q - d$ algorithm which may be used to determine the coefficients in the continued fraction of the form $\{1 : b_1^{(m)} + \, ; z : b_\nu^{(m)} + \}$ derived by means of Euclid's algorithm from the series $\sum f_{m+\nu} z^\nu$ $(m \equiv I)$. We set

$$\begin{aligned} &b_1^{(m)}\, b_3^{(m)^{-1}}\, b_5^{(m)} \cdots b_{4r+1}^{(m)} = \eta_{2r}^{(m)}, \\ &b_1^{(m)^{-1}}\, b_3^{(m)}\, b_5^{(m)^{-1}} \cdots b_{4r+3}^{(m)} = \eta_{2r+1}^{(m)}, \\ &b_2^{(m)}\, b_4^{(m)^{-1}}\, b_6^{(m)} \cdots b_{4r+2}^{(m)} = \sigma_{2r}^{(m)}, \\ &b_2^{(m)^{-1}}\, b_4^{(m)}\, b_6^{(m)^{-1}} \cdots b_{4r+4}^{(m)} = \sigma_{2r+1}^{(m)}, \end{aligned} \qquad (r \equiv I)$$

define numbers $\{\hat{\eta}_r^{(m)}\}$, $\{\hat{\sigma}_r^{(m)}\}$ be reversing the order of these products (so that $b_5^{(m)}\, b_3^{(m)^{-1}}\, b_1^{(m)} = \hat{\eta}_2^{(m)}$, and so on) and further numbers $\{\eta_r{}^{(m)^{-1}}\}$, $\cdots$ by $b_{4r+1}^{(m)^{-1}} \cdots b_5^{(m)^{-1}} b_3^{(m)}\, b_1^{(m)^{-1}} = \eta_{2r}^{(m)^{-1}}$ and so on, and set $\sigma_{-2}^{(m)^{-1}} = 0$, $\sigma_{-1}^{(m)} = \eta_{-1}^{(m)} = 1$. We then have $b_1^{(m)} = f_m^{-1}$ $(m \equiv I)$ and, taking $aba = (a\bar{b}^* + \bar{b}a^*)a - (aa^*)\bar{b}$,

$$\begin{aligned} b_{2r+2}^{(m)} &= \eta_r^{(m)^{-1}}\, \sigma_{r-1}^{(m+1)}\, b_{2r+1}^{(m+1)}\, \hat{\sigma}_{r-1}^{(m+1)}\, \hat{\eta}_r{}^{(m)^{-1}} \\ b_{2r+3}^{(m)} &= \hat{\eta}_r^{(m)}\, (\sigma_{r-1}^{(m+1)}\, b_{2r+2}^{(m+1)^{-1}}\, \hat{\sigma}_{r-1}^{(m+1)} \qquad (r \equiv I) \\ &\quad + \hat{\sigma}_{r-2}^{(m+1)}\, b_{2r}^{(m+1)}\, \sigma_{2r-2}^{(m+1)} - \hat{\eta}_{r-1}^{(m)^{-1}}\, b_{2r+1}^{(m)}\, \eta_{r-1}^{(m)^{-1}})^{-1} \eta_r{}^{(m)} \end{aligned}$$

(the products occurring in these formulae are nested products of the form aba).

12. **Extensions to nonassociative number systems.** Much of the theory of § 10 can be extended to the case in which the continued fractions concerned have coefficients over a nonassociative ring (for the theories of various important nonassociative number systems, see [**153, 154**], and for applications of such systems in geometry,

see [**155**], and in physics, see [**156**, **157**]; an extensive treatment of formal power series with coefficients over nonassociative rings of various types is given in [**122**]). In particular, we mention that if s, a, b, c are Cayley numbers [**158**, **159–161**] over a field, with the norms of a, b, c nonzero, then the ϵ-algorithm can be applied to the sequence $s_m = s + a(b^m c)$ (or to $s_m = s + (ab^m)c$) $(m \equiv I)$ to produce numbers $\epsilon_2^{(m)} = s$ $(m \equiv I)$. If the s, a, b, c were complex numbers, the numbers $\{s_m\}$ would lie on a spiral with centre s in the complex plane. Thus, in the nonassociative case, the ϵ-algorithm can be used to find the centre of a spiral in a non Desarguian plane [**162**].

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