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THE PADÉ TABLE OF MEROMORPHIC FUNCTIONS OF SMALL ORDER WITH NEGATIVE ZEROS AND POSITIVE POLES

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Introduction. The Padé table of a power series

$$a_0 + a_1 z + a_2 z^2 + \cdots = f(z) \qquad (a_0 \neq 0) \tag{1}$$

is an infinite array of rational functions

$$\begin{array}{cccc} R_{00} & R_{10} & R_{20} & \cdots \\ R_{01} & R_{11} & R_{21} & \cdots \\ R_{02} & R_{12} & R_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array} \tag{2}$$

defined as follows:

$$R_{mn} = \frac{P_{mn}}{Q_{mn}},$$

where

(i) P_{mn} is a polynomial of degree not greater than m;
(ii) Q_{mn} is a polynomial of degree not greater than n;
(iii) the polynomials P_{mn} and Q_{mn} are chosen so as to maximize the multiplicity of the zero, at $z = 0$, of

$$f(z) - R_{mn}(z).$$

The obvious questions concerning the existence and uniqueness of the Padé table have well known answers which will be taken for granted [3; pp. 235–244].

The study of the general convergence problem, stated below, has made little progress since Perron's masterful account of the theory (in the first edition (1914) of [3]).

Convergence Problem. *Find relations between the analytic character of $f(z)$ and the convergence of suitable sequences of approximants of the associated Padé table.*

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The present note continues a recently published paper of Arms and Edrei [2] and is devoted to the convergence problem posed by the following class of meromorphic functions.

Definition of the class $\mathcal{S}$. An analytic function $f(z)$ belongs to the class $\mathcal{S}$ if it is representable in the form

$$f(z) = a_0 e^{\gamma z} \frac{\prod_{j=1}^{\infty} (1 + \alpha_j z)}{\prod_{j=1}^{\infty} (1 - \beta_j z)}, \tag{3}$$

where

$$\begin{cases} a_0 > 0, \gamma \geqq 0, \\ \alpha_j \geqq 0, \beta_j \geqq 0 \text{ (for all } j \geqq 1), \sum_j (\alpha_j + \beta_j) < +\infty. \end{cases}$$

We shall always assume that

$$\alpha_1 \geqq \alpha_2 \geqq \alpha_3 \geqq \cdots, \quad \beta_1 \geqq \beta_2 \geqq \beta_3 \geqq \cdots,$$

and do not exclude the possibility that some or all of the quantities α and β be zero.

I prove here a single

Theorem. *Let* (1) *be the expansion of a nonrational member f of the class $\mathcal{S}$ and let* (2) *be the associated Padé table. Let $\{m_\lambda\}_{\lambda=1}^{\infty}$ and $\{n_\lambda\}_{\lambda=1}^{\infty}$ be two sequences of positive integers such that $m_\lambda \to \infty$, $n_\lambda \to \infty$. Assume that $\mathfrak{D}$ is a compact subset of the complex plane and that it contains no poles of $f(x)$. Then, as $\lambda \to \infty, R_{m_\lambda, n_\lambda}(z) \to f(z)$, uniformly on $\mathfrak{D}$.*

If $\gamma = 0$ in (3), the above Theorem is an obvious consequence of Theorem 1 of Arms and Edrei. In the general case $\gamma > 0$, treated here, I obtain the Theorem by combining results of Arms and Edrei with well known properties of normal families.

Is it possible to enlarge the class of functions considered in the Theorem without seriously affecting the conclusion?

Consideration of the class of real meromorphic functions of order less than two, with real zeros and poles, suggests itself naturally at this stage and may warrant further study.

1. Notation and references. This note continues the paper of Arms and Edrei [2].

In order to avoid the repetition of published material

(i) I assume that the reader has access to [2];

(ii) I adopt without modification all the notations and assumptions used in the statement and proof of Theorem 1 of [2];

(iii) I refer, without additional explanations to specific formulae of [2] and use, with their obvious meaning references such as [2; p. 15, (4.13)];

(iv) [2; p. 4, (13)] contains a misprint which has been corrected in the corresponding formula (3) of the present note.

2. NORMALITY OF THE PADÉ APPROXIMANTS. From this point on, $\{m_\lambda\}_{\lambda=1}^{\infty}$ and $\{n_\lambda\}_{\lambda=1}^{\infty}$ denote two sequences of positive integers such that

$$m_\lambda \to \infty, \quad n_\lambda \to \infty. \tag{2.1}$$

We first simplify our notation by writing P_λ, Q_λ, instead of P_{m_λ, n_λ}, Q_{m_λ, n_λ}, and focus our attention on the sequence of approximants

$$R_\lambda(z) = \frac{P_\lambda(z)}{Q_\lambda(z)} \quad (\lambda = 1, 2, 3, \cdots). \tag{2.2}$$

Let Ω be a bounded region and $\bar{\Omega}$ its closure [the word region is used in the sense of nonempty open connected subset of the complex plane]. We assume

$$\frac{1}{\beta_j} \notin \bar{\Omega} \quad (j = 1, 2, 3, \cdots). \tag{2.3}$$

LEMMA 1. *Let the sequences of integers* $\{m_\lambda\}$, $\{n_\lambda\}$, *satisfy the conditions* (2.1), *let the approximants* $R_\lambda(z)$ *be defined by* (2.2), *and let* Ω *be a bounded region satisfying the condition* (2.3). *Then,*

(i) *for* $\lambda > \lambda_0(\Omega)$, *the approximants* $R_\lambda(z)$ *have no poles in* $\bar{\Omega}$;

(ii) *the functions* $\{R_\lambda(z)\}$ $(\lambda > \lambda_0(\Omega))$ *form a family normal in* Ω.

PROOF OF ASSERTION (i). Assume that (i) does not hold. It is then possible to find a sequence

$$\lambda_1, \lambda_2, \lambda_3, \cdots, \tag{2.4}$$

such that each $Q_{\lambda_j}(z)$ has some zero in $\bar{\Omega}$.

From (2.4) extract a subsequence $\{\mu_k\}_{k=1}^{\infty}$ chosen so that

$$\lim_{k\to\infty} \frac{m_{\mu_k}}{n_{\mu_k}} = \omega.$$

[We do not exclude the occurrence of $\omega = +\infty$.]

By Theorem 1 of [2; p. 4]

$$Q_{\mu_k}(z) \to \exp\left(\frac{-\gamma z}{1+\omega}\right)\Pi(1-\beta_j z)\ (k\to\infty) \tag{2.5}$$

uniformly in $\bar{\Omega}$. [For $\omega = +\infty$, the exponential term is absent from (2.5).]

Now the limit function of the polynomials Q_{μ_k} does not vanish on the compact set $\bar{\Omega}$. Hence, for k large enough, these polynomials have no zeros in $\bar{\Omega}$. This contradicts the construction of (2.4) and hence proves assertion (i) of the lemma.

Proof of assertion (ii). By assertion (i) all the functions of the family

$$\{R_\lambda(z)\} \quad (\lambda > \lambda_0(\Omega)) \tag{2.6}$$

are regular in Ω. This enables us to use the simplest definition of normality (see, for instance, [**1**; p. 168]) and to avoid the slightly more cumbersome consideration of normal families of meromorphic functions.

To establish the normality of the family (2.6) we start from some given infinite sequence of strictly increasing integers and select one of its subsequences, say $\{\nu_k\}_{k=1}^{\infty}$, such that

$$\lim_{k\to\infty} \frac{m_{\nu_k}}{n_{\nu_k}} = \omega \quad (0 \leqq \omega \leqq +\infty).$$

Hence, by Theorem 1 of [2]

$$P_{\nu_k}(z) \to a_0 \exp\left(\frac{\omega\gamma z}{1+\omega}\right) \prod_{j\geqq 1} (1+\alpha_j z), \tag{2.7}$$

$$Q_{\nu_k}(z) \to \exp\left(-\frac{\gamma z}{1+\omega}\right) \prod_{j\geqq 1} (1-\beta_j z), \tag{2.8}$$

uniformly in $\bar{\Omega}$. [For $\omega = +\infty$, the exponential term is absent from (2.8) and equal to $\exp(\gamma z)$ in (2.7).]

We thus conclude that

$$\frac{P_{\nu_k}(z)}{Q_{\nu_k}(z)} \to f(z),$$

uniformly on $\bar{\Omega}$. This proves the normality of (2.6) in Ω.

3. Proof of the Theorem. By assumption, $\mathfrak{D}$ contains no point of the set

$$\mathcal{P} = \{1/\beta_j\}_{j\geqq 1}.$$

Since the set $\mathcal{P}$ has no finite limit point and $\mathfrak{D}$ is compact, the distance between $\mathcal{P}$ and $\mathfrak{D}$ is positive and we may select η smaller than this distance and satisfying the additional conditions

$$0 < \eta < \frac{1}{\beta_1}. \tag{3.1}$$

Consider now the set

$$\Omega = \left\{ z : |z| < L (0 < L < +\infty), \left| z - \frac{1}{\beta_j} \right| > \eta \; (j = 1, 2, 3, \cdots) \right\},$$

where L is large enough to imply

$$\mathfrak{D} \subset \{z : |z| < L\}.$$

It is clear that Ω is a bounded region which contains the interval

$$-L < x \leqq 0, \quad y = 0 \quad (z = x + iy),$$

satisfies (2.3) and the additional condition

$$\mathfrak{D} \subset \Omega. \tag{3.2}$$

In view of Lemma 1, the sequence

$$\{R_\lambda(z)\} \quad (\lambda \geqq \lambda_0(\Omega))$$

forms a family normal in Ω.

Put

$$z = -x \left(0 \leqq x < \frac{1}{\beta_1} \right),$$

in [2; p. 14, (4.3)]. Using the notation of (2.2), we conclude that

$$|f(-x) - R_\lambda(-x)| = \left| f(-x) - \frac{P_\lambda(-x)}{Q_\lambda(-x)} \right| \leqq \sum_{k=m_\lambda+1}^{\infty} |a_k| x^k.$$

Hence, in view of (2.1)

$$R_\lambda(z) \to f(z) \quad (\lambda \to \infty), \tag{3.3}$$

at all points of the interval

$$\max \left(-\frac{1}{\beta_1}, -L \right) < x \leqq 0, y = 0,$$

which is contained in Ω. From a fundamental property of normal families [1; p. 172, exercise 2] we now deduce that (3.3) holds uniformly on every compact subset of Ω. By (3.2), $\mathfrak{D}$ is such a subset; this remark completes the proof of the Theorem.

References

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3. O. Perron, *Die Lehre von den Kettenbrüchen*, 3rd ed., vol. 2, B. G. Teubner, Stuttgart, 1957.

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