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THE INTERPOLATION OF PICK FUNCTIONS

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Before stating our version of the Cauchy Interpolation Problem it is desirable to recall the definition of the degree of a rational function. Let $f(z)$ be a rational function; then, in a known way, $f(z)$ may be regarded as a continuous map of the Riemann sphere into itself. This mapping has a Brouwer degree, d, which we take to be the degree of the rational $f(z)$. Equivalently, if $f(z)$ is presented as the quotient of two relatively prime polynomials $p(z)$ and $q(z)$, where d' is the algebraic degree of p and d'' is the algebraic degree of q then the degree of $f(z)$ is given by $d = \max(d', d'')$. Finally, we should note that for all but finitely many values of λ the function $f(z) - \lambda$ has exactly d distinct and finite zeros, and these are simple. If it is known that the rational $f(z)$ has degree at most N and that it has at least $N + 1$ zeros, multiplicitly included, then $f(z)$ vanishes identically.

Cauchy Interpolation Problem. Let there be given k distinct interpolation points on the real axis $x_1, x_2, x_3, \cdots, x_k$ and equally many non-negative integers $\nu_1, \nu_2, \nu_3, \cdots, \nu_k$ as well as $N = \sum_{i=1}^{k} (\nu_i + 1)$ real numbers f_{ij} where $1 \leqq i \leqq k$ and $0 \leqq j \leqq \nu_i$. It is required to find a rational function $f(z)$ of degree at most $N/2$ satisfying the N conditions $f^{(j)}(x_i) = f_{ij}$. In any case that we study, the problem will in fact be an interpolation problem: there will be a function $F(z)$, usually not rational, so that the data f_{ij} are obtained from $F^{(j)}(x_i)$.

In the special case when $N = k$, where no derivatives were considered in the problem, the Cauchy Interpolation Problem was exhaustively studied by Löwner in a famous paper [2]. The other extreme case, where $k = 1$, corresponds to the determination of certain Padé approximations of a function, these approximations being on the diagonal or adjacent to the diagonal in the Padé table.

It is important to note that if $f(z)$ is a solution to the Cauchy Interpolation Problem for which the degree of $f(z)$ is strictly smaller than $N/2$ then the solution is unique. Were there another solution $g(z)$, the rational function $f(z) - g(z)$ would have degree at most $N - 1$, but would have at least N zeros, since at each interpolation point x_i there would be a zero of degree $\nu_i + 1$. Thus the difference would vanish identically. We emphasize that this will always be the case when N is odd. It is therefore clear that the Interpolation Problem depends significantly on the parity of N.

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In order to state the useful theorems concerning the Cauchy Interpolation Problem it is convenient to introduce the concept of the divided differences of a function. In view of the fact that we shall be concerned almost exclusively with analytic functions, we give a somewhat unorthodox definition. Let $F(z)$ be analytic in some region $\mathfrak{D}$ and suppose that $\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_\ell$ is a system of $\ell + 1$ not necessarily distinct points in $\mathfrak{D}$. Let C be a rectifiable curve in $\mathfrak{D}$ which surrounds these points. We then set

$$[\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_\ell] = \frac{1}{2\pi i}\int_C \frac{F(z)}{(z-\lambda_0)(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_\ell)}\,dz.$$

It is plain to see that this "divided difference" can be evaluated by residues and that it will be an algebraic expression in the points λ_i and the values of the function F at those points as well as the values of some of the derivatives. Indeed, if we take for the λ the points of the Cauchy Interpolation Problem, each x_i being taken $\nu_i + 1$ times, then the resulting difference will involve exactly the data of the Cauchy Interpolation Problem. It becomes clear that we could have stated that problem in an equivalent way: all possible values of the differences $[\lambda_0, \lambda_1, \cdots, \lambda_\ell]$ are prescribed, whenever the set of λ's is a subset of the N numbers

$$[x_1, x_1, \cdots, x_2, x_2, \cdots, x_3, x_3, \cdots, x_k, x_k]$$

where each x_i is taken $\nu_i + 1$ times. Then it is necessary to find a rational function $f(z)$ of degree at most $N/2$ realizing these differences.

Let us suppose that N is even and write $N = 2n$. Take the set of N numbers corresponding to the Cauchy problem and divide these numbers into two sets of n elements each in any way that is convenient. Call the elements of the first set $\xi_1, \xi_2, \cdots, \xi_n$ and those of the second $\eta_1, \eta_2, \cdots, \eta_n$. Now form the matrix of order n: $L =$

$$\begin{bmatrix} [\xi_1, \eta_1] & [\xi_1, \eta_1, \eta_2] & [\xi_1, \eta_1, \eta_2, \cdots, \eta_n] \\ [\xi_1, \xi_2, \eta_1] & [\xi_1, \xi_2, \eta_1, \eta_2] & [\xi_1, \xi_2, \eta_1, \eta_2, \cdots, \eta_n] \\ [\xi_1, \xi_2, \cdots, \xi_n, \eta_1] & & [\xi_1, \xi_2, \cdots, \xi_n, \eta_1, \eta_2, \cdots, \eta_n] \end{bmatrix}$$

Note that the ij-th element of this matrix is $[\xi_1, \xi_2, \cdots, \xi_i, \eta_1, \eta_2, \cdots, \eta_j]$.

THEOREM 1. *L will be singular or non-singular in a way that depends only on the data of the interpolation problem, and independently of the assignment of the names ξ_i and η_j in that problem. When L is non-singular there exist two pairs of real polynomials*

$[\sigma_0, \tau_0]$ and $[\sigma_\infty, \tau_\infty]$ of degree at most n such that all solutions to the Cauchy problem occur in the family

$$f_t(z) = \frac{\sigma_0(z) + t\sigma_\infty(z)}{\tau_0(z) + t\tau_\infty(z)} .$$

All functions in this family are solutions to the Interpolation Problem save for at most k exceptional solutions where numerator and denominator have a common zero at an interpolation point. The corresponding rational function is then of degree strictly smaller than n and is not a solution to the Interpolation Problem.

Our next theorem does not require that N be even.

THEOREM 2. *A solution to the Cauchy Interpolation Problem exists if there exists an integer n with $2n \geqq \max \nu_i$ such that all n by n matrices L of the type above formed from data of the problem are non-singular, while all such matrices of higher order are singular. The solution then is exactly of degree n, but it need not be unique.*

Pick Functions.

A function $\varphi(\zeta) = U(\zeta) + iV(\zeta)$ is called a Pick function if it is analytic in the upper half-plane and has positive imaginary part. Such functions admit a canonical representation, easily derived from the Poisson integral representation of the harmonic and positive $V(\zeta)$. We will have

$$\varphi(\zeta) = \alpha\zeta + \beta + \int \left[\frac{1}{\lambda - \zeta} - \frac{\lambda}{\lambda^2 + 1} \right] d\mu(\lambda)$$

where $\alpha \geqq 0$, β is real and μ a positive Radon measure on the real λ-axis for which $\int(\lambda^2 + 1)^{-1} \, d\mu(\lambda)$ is finite. The representation is unique, in fact, putting $\zeta = \xi + i\eta$ we have $\alpha = \lim_{\eta \to \infty} V(i\eta)/i\eta$ and $\beta = \text{Re}[\varphi(i)]$ while the measure μ may be determined from the function in the following way. We consider a monotone increasing function $\mu(\lambda)$ corresponding to the measure and normalize it so that $\mu(\lambda) = (\mu(\lambda + 0) + \mu(\lambda - 0))/2$. Under these circumstances, then for every interval (a, b)

$$\mu(b) - \mu(a) = \lim_{\eta \to 0} \frac{1}{\pi} \int_a^b V(x + i\eta) \, dx.$$

Associated with the open interval (a, b) we have the subclass of Pick functions denoted $P(a, b)$: these are the Pick functions which admit an analytic continuation from the upper half-plane across the interval (a, b) into the lower half-plane such that the continuation is by reflection with respect to the real axis. It is not difficult to show that $\varphi(\zeta)$ is in $P(a, b)$ if and only if the corresponding measure μ puts no mass in the interval (a, b). A function $F(z)$ is a series of Stieltjes if and only if $-F(z)$ belongs to $P(0, \infty)$. If a rational function $\varphi(\zeta)$ belongs to $P(a, b)$ then its poles are simple, fall on the real axis outside the interval (a, b) and have negative residues.

In this paper we will suppose that a Pick function $\varphi(\zeta)$ is given and that it belongs to the class $P(a, b)$; we seek to approximate the function with solutions to the Cauchy Interpolation Problem associated with the given function where the interpolation points x_i are all within the open interval. It is easier to describe the situation if we suppose, in addition, that φ is not a rational function.

THEOREM 3. *The square matrices L associated with the interpolation are non-singular.*

While we avoid giving the proofs of our theorems, it is worthwhile to indicate its nature here. Consider the functions $f_i(\lambda)$ and $g_j(\lambda)$ where $f_i(\lambda)^{-1} = (\lambda - \xi_1)(\lambda - \xi_2) \cdots (\lambda - \xi_i)$ and $g_j(\lambda)^{-1} = (\lambda - \eta_1)(\lambda - \eta_2) \cdots (\lambda - \eta_j)$. Using the canonical representation of φ, at least under the additional hypothesis that $\alpha = 0$, we find

$$L_{ij} = [\xi_1, \xi_2, \cdots, \xi_1, \eta_1, \eta_2, \cdots, \eta_j] = \int f_i(\lambda) g_j(\lambda)\, d\mu(\lambda)$$
$$= (f_i, g_j)$$

the inner product being taken in the L^2-space associated with the measure μ. Thus the matrix L_{ij} is a sort of Gram's matrix, and as in [1], it is easy to show that it is non-singular when the support of the measure is not a finite set.

The important theorem is essentially due to Loewner [1, 2].

THEOREM 4. *Let N be odd and φ_N the corresponding (unique) interpolation for φ; then φ_N is also in P(a, b). When N is even many of the interpolating functions are in the Pick class, but not all.*

In view of certain well-known compactness properties for the Pick class and the class $P(a, b)$ and because of the form of the approximating functions when N is even, the successive approximations to φ display the usual limit-point and limit circle behavior associated with the moment problem as well as with the Sturm-Liouville problem. However, in almost every case, only the limit-point case occurs and the approximating functions converge to $\varphi(\zeta)$ uniformly on compact subsets of the union of the upper half-plane, the lower half-plane and the interval (a, b). The convergence is generally extremely rapid and the approximations oscillate about the limiting function in certain intervals. We refer to the paper of M. F. Barnsley in these proceedings for further detail.

References

1. William F. Donoghue, Jr., *The theorems of Loewner and Pick,* Israel Journal of Mathematics 4 (1966), 153–170.

2. K. Löwner, *Über Monotone Matrixfunktionen,* Mathematische Zeitschrift 38 (1934), 177–216.

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