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UPPER AND LOWER BOUNDS FOR THE NUMBER OF SOLUTIONS OF FUNCTIONAL EQUATIONS INVOLVING k-SET CONTRACTIONS

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1. **Introduction.** In [2] J. Scanlon proved theorems yielding upper and lower bounds on the number of solutions to a functional equation where the function was a compact perturbation of the identity operator. In [7] the author generalized some of the results to get a local version for certain noncompact mappings. The purpose of this paper is to show that these results can be generalized much further. We shall see that the results are true for the k-set contractions studied by F. E. Browder and R. D. Nussbaum in [1] and [5]. Since the work on this paper was done, R. D. Nussbaum has published a similar result. See Appl. Anal. 1 (1971).

2. **Preliminaries.** Let X be a real Banach space. For any subset of X, Ω, define the *measure of compactness* of Ω to be $\gamma(\Omega) = \inf\{d \mid \Omega$ can be covered by a finite number of sets of diameter less than or equal to $d\}$ (See [4] p. 413). We shall say that g is a *k-set contraction* if given any bounded set $A \subset X$, $g(A)$ is a bounded subset of X and $\gamma(g(A)) \leqq k\gamma(A)$ (See [3] and [5]). We shall consider only k-set contractions for which $k < 1$. An example of a k-set contraction is the sum of a compact map and a contraction.

Now suppose that G is an open bounded subset of X and that $f : \bar{G} \to X$ is a k-set contraction for which $a \notin (I - f)(\partial G)$. It is then possible to define the topological degree of $I - f$ at a with respect to G, $d(I - f, G, a)$, as is shown by Nussbaum in [5]. This definition of topological degree has all of the usual properties of topological degree and contains the usual definition of degree.

We next state a lemma giving some properties of k-set contractions that we shall need. Proofs of these properties can be found in [5].

LEMMA 1. *Suppose that f is a differentiable k-set contraction. Then*

(1) *$f'(x)$ is also a k-set contraction,*

(2) *the spectrum of $f'(x)$ is finite for $|\lambda| > k$, and*

(3) *$I - f$ is a Fredholm map of index zero.*

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The above lemma will be seen to be the key to the proofs given in this paper. These three properties are also satisfied by compact mappings and are the properties that make possible much of the work done with compact mappings. It is for this reason that the class of k-set contractions seems to be a convenient setting in which to do much of the analysis and applied mathematics.

We might add that Lemma 1 is true for complex Banach spaces also. We next state our last preliminary lemma which can be found on page 95 of [5].

LEMMA 2. *Suppose that G is a bounded open subset of the complex Banach space X_c. Suppose also that f is a differentiable k-set contraction mapping $\bar{G}$ into X_c for which $\theta \notin (I-f)(\partial G)$. Then $(I-f)^{-1}(\theta)$ consists of a finite number of points, say $x_1, \cdots, x_r$, and $d(I-f, G, \theta) \geqq r$.*

3. **A Lower Bound and An Implicit Function Theorem.** We are now able to state and prove a theorem which will yield a lower bound on the number of solutions to a functional equation. This theorem is a generalization of Lemma 3 of [2] and Lemma 3 of [7].

THEOREM 3. *Let G be a bounded open subset of the real Banach space X. Suppose that f is a differentiable k-set contraction mapping $\bar{G}$ into X and $y \in X$ is such that $y \notin (I-f)(\partial G)$. If N is a sufficiently small neighborhood of y, then there exists a set $K \subset N$ of first category for which*

(1) if $y_1 \in N-K$, then $(I-f)^{-1}(y_1) \cap G$ is a finite set,

(2) if $y_1 \in N-K$ and $x \in (I-f)^{-1}(y_1) \cap G$, then $I-f'(x)$ is invertible, and

(3) the number of points in $(I-f)^{-1}(y_1) \cap G$ is $\geqq |d(I-f, G, y)|$.

PROOF. Let N be a neighborhood of y such that $y_1 \in N$ implies that $d(I-f, G, y_1) = d(I-f, G, y)$, and let $A = \{x \in G \mid I-f'(x)$ does not have an inverse$\}$. Properties (1) and (3) of Lemma 1 allow us to use Smale's infinite dimensional Sard's Theorem ([6] Theorem 1.3) to get that $(I-f)(A)$ is of first category. Let $K = N \cap (I-f)(A)$ and choose $y_1 \in N-K$.

If we let $M = (I-f)^{-1}(y_1)$, then it is clear that $M \subset f(M) + y_1$. Thus $\gamma(M) \leqq \gamma[f(M)] \leqq k\gamma(M) < \gamma(M)$. Therefore M is compact and hence a lemma due to Browder (Lemma 3 [5]) implies that $(I-f)^{-1}(y_1) \cap G$ is a finite set, say $\{x_1, \cdots, x_m\}$.

Since $y_1 \in N-K$ implies that $I-f'(x_i)$ is injective, $i = 1, \cdots, m$, we get that $I-f$ is injective in some neighborhood of each x_i

and that $d(I - f, G_i, y_1) = \pm 1$ where G_i is an open neighborhood of x_i such that $G_i \cap G_j = \phi$ for $i \neq j$. Then since

$$d(I - f, G, y) = \sum_{i=1}^{m} d(I - f, G_i, y_1),$$

we have that $m \geqq |d(I - f, G, y)|$.

As a corollary to Theorem 3 (with the aid of the usual implicit function theorem) we then have the following result.

COROLLARY 4. *Let the function $M(x, y) = I(x, y) - f(x, y)$ be defined on $G \times U$ where G is a bounded open subset of the real Banach space X and U is an open neighborhood of θ in X. Suppose that M is continuously differentiable in $G \times U$, $f(\circ, \theta)$ is a k-set contraction, $I(\circ, \theta)$ is the identity mapping on G, and that $\theta \notin M(\circ, \theta)(\partial G)$. We then have that if*

$$|d(I(\circ, \theta) - f(\circ, \theta), G, \theta)| = m > 0,$$

then there exists a neighborhood of θ, N, and a set of first category $K \subset N$ such that $w \in N - K$ implies that the equation

$$I(x, y) - f(x, y) = w$$

has a finite set of families of solutions $x_i(y)$, $i = 1, \cdots, q$ where $q \geqq m$ and each family $x_i(y)$ is continuously differentiable in some neighborhood, N_i, of θ.

4. **An Upper Bound.** In this section we shall prove a theorem which yields an upper bound to the number of solutions. As in the previous section we shall investigate solutions of $(I - f)(x) = y$ where f is a continuously differentiable mapping from the closure of an open bounded set G into X.

Let X_c denote the complexification of the space X; i.e., let $X_c = X \times X$, define addition by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, define scalar multiplication by $(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x)$, and the norm by $\|(x, y)\| = \sup_{0 \leqq \theta \leqq 2\pi} \|(\sin \theta)x + (\cos \theta)y\|$.

We must next assume that the mapping f has a complexification F. That is, we assume that there exists a mapping $F : G_c \to X_c (G_c = G \times G)$ such that:

(1) $(I - F)(x, \theta) = ((I - f)(x), \theta)$ for all $x \in G$,

(2) F is continuously differentiable at each point of G_c, and

(3) $(I - f)'(x)$ injective implies that $(I - F)'(x, \theta)$ is injective.

It is easy to see that if f is linear or a polynomial, then f has the above type of complexification. Since we shall be considering mappings in both X and X_c, we shall use $d_c(I - F, G_c, \theta)$ to denote degree in X_c. We then have the following theorem.

THEOREM 5. *Suppose that f, G, X and X_c are as above. Then if $d_c(I - F, G_c, \theta)$ is defined, there exists a neighborhood N of θ in X and a set $K \subset N$ of first category such that $y \in N - K$ implies that the number of solutions in G to the equation $(I - f)(x) = y$ is finite and $\leqq d_c(I - F, G_c, \theta)$.*

PROOF. Suppose $(x, \theta) \in \partial G_c$. Since $d_c(I - F, G_c, \theta)$ is defined, $(I - F)(x, \theta) \neq (\theta, \theta)$. Condition (1) on F then implies that $(I - f)(x) \neq \theta$ and hence that $d(I - f, G, \theta)$ is defined.

We can then apply Theorem 3 to get a neighborhood of θ in X, N, and a set $K \subset N$ of first category such that if $y \in N - K$, then $(I - f)^{-1}(y) \cap G$ is a finite set, say $\{x_1, \cdots, x_m\}$ and $(I - f)'(x_i)$ is injective for $i = 1, \cdots, m$. Condition (3) above implies that the points (x_i, θ) are isolated points in $(I - F)^{-1}(\theta, \theta) \cap G_c$.

Since $(I - F)'(x_i, \theta)$ is injective, we know that 1 is not an eigenvalue of $(I - F)'(x_i, \theta)$. By property (2) of Lemma 1 we see that there is a smooth path P, say $z(t) = g_1(t) + ig_2(t)$ for $t \in [0, 1]$, from 0 to 1 such that $I - z(t)F'(x_i, \theta)$ is invertible for all $t \in [0, 1]$. Let S be an open ball centered at the origin. By the homotopy property we have that

$$\begin{aligned} d_c((I - F)'(x_i, \theta), S, \theta) &= d_c(I - z(t)F'(x_i, \theta), S, \theta) \\ &= d_c(I, S, \theta) \\ &= 1. \end{aligned}$$

The above statement is true for all i, $i = 1, \cdots, m$.

We then approximate $I - F$ near the points (x_i, θ) by $(I - F)'(x_i, \theta)$ and get the fact that there exists pairwise disjoint open balls $B_i \subset X_c$ about (x_i, θ) such that for $i = 1, \cdots, m$, $B_i \cap (I - F)^{-1}(\theta) = \{(x_i, \theta)\}$ and

$$d_c(I - F, B_i, \theta) = d_c((I - F)'(x_i, \theta), S, \theta) = 1.$$

We then let W be an open set contained in G_c that contains $(I - F)^{-1}(\theta) \cap G_c - \{(x_1, \theta), \cdots, (x_m, \theta)\}$. Then

$$\begin{aligned} d_c(I - F, G_c, \theta) &= \sum_{j=1}^{m} d_c(I - F, B_j, \theta) + d_c(I - F, W, \theta) \\ &= m + d_c(I - F, W, \theta). \end{aligned}$$

But by Lemma 2 we have that $d_c(I - F, W, \theta) \geqq \theta$. Thus

$$d_c(I - F, G_c, \theta) \geqq m,$$

which is what we were to prove.

It is sometimes convenient to consider mappings of the form $h - f$ where h is a homeomorphism and f is a k-set contraction. If we add the conditions that (1) h is such that $f \circ h^{-1}$ is a k'-set contraction for $k' < 1$ and (2) $h'(x)$ exists and is such that $h'(x)$ is a homeomorphism, then Theorem 3 and 5 will hold true for maps of the form $h - f$.

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