ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 4, Number 1, Winter 1974

THE NIELSEN FIXED POINT THEORY FOR NONCOMPACT SPACES

U. KURT SCHOLZ[1]

Introduction. The Nielsen fixed point theory provides a lower bound for the number of fixed points of a map $f: X \to X$ on a compact metric ANR (absolute neighbourhood retract) ([5], [11], [15]). This lower bound, denoted by $N(f)$ (the Nielsen number of f) is a non-negative integer and is known to be homotopy invariant ([11], [15]). Recently, Brown [7], has shown that one can obtain a Nielsen theory on noncompact ANR's by requiring only that the maps and homotopies be compact (i.e., their images have compact closure).

The purpose of this paper is to establish a Nielsen theory for very general classes of self maps (§1) and to give a proof of the homotopy invariance of the Nielsen number. The motivation for this type of generality becomes clear in §3 where several interesting and analytically important examples of such classes are produced.

§1. **Preliminaries.** Given a self map $f: X \to X$, let $\Phi(f)$ represent the set of fixed points of f. We let I denote the closed unit interval. For a given homotopy $H: X \times I \to X$, we shall make use of the following maps: $\mathbf{H}: X \times I \to X \times I$ given by $\mathbf{H}(x, t) = (H(x, t), t)$; for $t \in I, h_1 : X \to X$ defined by $h_t(x) = H(x, t)$; finally, for $r, s \in I$, $H^{r,s}: X \times I \to X$ defined by $H^{r,s}(x, t) = H(x, (1-t)r + ts)$.

Let $\mathcal{G}$ be a class of self maps. An $\mathcal{G}$-homotopy is a map $H: X \times I \to X$ such that $H^{r,s} \in \mathcal{G}$ and $h_t \in \mathcal{G}$ for all $r, s, t \in I$. If H is an $\mathcal{G}$-homotopy, we say that h_0 and h_1 are $\mathcal{G}$-homotopic.

For a given class of self maps $\mathcal{G}$, we let $\mathcal{C}_{\mathcal{G}}$ be the class of all triples (X, f, U) where $f: X \to X$ is a map in $\mathcal{G}$, and U is an open subset of X which has no fixed points of f on its boundary.

Let $f: X \to X$ be a map and $H_* = \{H_p, \partial_p\}$ a rational homology theory defined for a category of spaces including X. Let $f_{*,p}: H_p(X) \to H_p(X)$ denote the induced homomorphism. We say that f has a

Received by the editors February 2, 1972.
AMS 1970 *subject classifications*. Primary 54H25; Secondary 47H10, 55C20.
It is a pleasure to acknowledge the encouragement given me by Professor Robert F. Brown during this writing. His advice and criticism have been invaluable.
[1]This work was supported by NSF Grant GU 1534.

generalized trace if for each integer p there is a finite dimensional subspace $E_p \subseteq H_p(X)$ so that:

(1) $f_{*,p}(E_p) = 0$ for all but a finite number of p.

(2) $f_{*,p}(E_p) \subseteq E_p$.

(3) For each $\alpha \in H_p(X)$, there is an n so that $f^n_{*,p}(\alpha) \in E_p$ where $f^n_{*,p}$ denotes the nth iterate of $f_{*,p}$.

In this case one defines trace $(f_{*,p}) =$ trace $(f_{*,p} \mid E_p)$. It is easily verified that this definition is independent of the choice of E_p satisfying condition (2) and (3); (see Browder [3], Leray [**10**]).

If $f : X \to X$ is a map having a generalized trace, define $L(f) = \sum_{p=0}^{\infty} (-1)^p$ trace $(f_{*,p})$, the generalized Lefschetz number.

A class of self maps $\mathcal{G}$ will be called admissible if for each map $f : X \to X$ in $\mathcal{G}$ the following three conditions are satisfied:

(1) f has a generalized Lefschetz number.

(2) $\Phi(f)$ is compact.

(3) X is a metric ANR.

We will call $\mathcal{C}_{\mathcal{G}}$ admissible if $\mathcal{G}$ is admissible.

An index on a class of admissible triples $\mathcal{C}_{\mathcal{G}}$ is a function $i : \mathcal{C}_{\mathcal{G}} \to$ integers satisfying the following five axioms:

(1) (Excision). If $(X, f, U) \in \mathcal{C}_{\mathcal{G}}$ and $g \in \mathcal{G}$, $g : X \to X$ is a map such that $g|\text{cl}(U) = f|\text{cl}(U)$ ($\text{cl}(U) =$ closure of U) then $i(X, f, U) = i(X, g, U)$.

(2) (Homotopy). If $H : X \times I \to X$ is a homotopy and $H \in \mathcal{G}$ and $(X, h_t, U) \in \mathcal{C}_{\mathcal{G}}$ for all $t \in I$ then $i(X, h_0, U) = i(X, h_1, U)$.

(3) (Additivity). If $(X, f, U) \in \mathcal{C}_{\mathcal{G}}$ and $\{U_1 \cdots U_n\}$ is a collection of mutually disjoint open subsets of U so that $\Phi(f) \cap (U - \bigcup_{i=1}^{n} U_i) = \emptyset$, then $i(X, f, U) = \sum_{i=1}^{n} i(X, f, U_i)$.

(4) (Normalization). If $f \in \mathcal{G}$ then $i(X, f, X) = L(f)$.

(5) (Commutativity). If $f : X \to Y$ and $g : Y \to X$ are maps such that fg and gf are in $\mathcal{G}$ then if $(x, gf, U) \in \mathcal{C}_{\mathcal{G}}$ we have $i(X, gf, U) = i(Y, fg, g^{-1}(U))$.

For the existence of indexes in the compact case, see e.g., Leray [**10**], O'Neill [13], Dold [8]. Numerous indexes exist for non-compact spaces and varieties of self maps.

§2 **The Nielsen Number.** In this section, let $\mathcal{G}$ be an admissible class such that $\mathcal{C}_{\mathcal{G}}$ admits an index.

Given $f \in \mathcal{G}$ and $x_0, x_1 \in \Phi(f)$, then x_0 and x_1 are called f-equivalent if there exists a path $C : I \to X$ so that $C(0) = x_0$, $C(1) = x_1$ and C and fC are homotopic keeping endpoints fixed (Wecken [**15**]). Using the property that an ANR can be embedded as a closed subspace of a normed linear space, it is not difficult to show that for each $x \in \Phi(f)$, there exists a $\delta = \delta(x) > 0$ so that whenever $x' \in \Phi(f)$ and $d(x, x') < \delta$ then x and x' are f-equivalent (Brown [7]). This implies that each fixed point class is open in $\Phi(f)$ and hence by the compactness of the latter set, the number of f-equivalence classes is finite. An equivalence class will be called a *fixed point class* of f.

For each fixed point class F of f, we can now produce an open set U containing F so that $\mathrm{cl}(U) \cap \Phi(f) = F$ (e.g., let $U = \bigcup_{x \in F} N(x, \delta(x)/2)$ where $N(x, \delta(x)/2)$ is the $\delta(x)/2$ neighborhood of x and $\delta(x)$ is from the preceding paragraph). We define the index $i(F)$ of the fixed point class F by $i(F) = i(X, f, U)$. One easily verifies that this is independent of the particular U chosen. If $i(F) \neq 0$, we shall call F an *essential fixed point class*.

The *Nielsen number* $N(f)$ of a map $f \in \mathcal{G}$ is defined to be the number of essential fixed point classes of f. The following is then obvious.

THEOREM 1. *If $f : X \to X$ is a map in $\mathcal{G}$ then f has at least $N(f)$ fixed points.*

The rest of this section is devoted to the proof that the Nielsen number is homotopy invariant i.e., if $f, g : X \to X$ are maps in $\mathcal{G}$ which are $\mathcal{G}$-homotopic, then $N(f) = N(g)$.

Let X be a space and $A \subseteq X \times I$ a subset of $X \times I$; for each $t \in I$, the t-slice of A, written A_t, is defined by $A_t = \{x : (x, t) \in A\}$.

LEMMA 1. *Let $H : X \times I \to X$ be an $\mathcal{G}$-homotopy and let F be a fixed point class of $\boldsymbol{H}$. Then for each $t \in I$, either $F_t = \emptyset$ or F_t is a single fixed point class of h_t.*

PROOF. For notational convenience, we shall write $C \simeq D$ to indicate that C and D are fixed end-point homotopic paths in X.

Let $t \in I$. To prove the lemma, it suffices to show that two points (x, t) and (y, t) in $X \times I$ are $\boldsymbol{H}$-equivalent if and only if x and y are h_t-equivalent.

If (x, t) and (y, t) are $\boldsymbol{H}$-equivalent, then there is a path $\boldsymbol{C}' : I \to X \times I$ so that $\boldsymbol{C}'(0) = (x, t)$, $\boldsymbol{C}'(1) = (y, t)$ and $\boldsymbol{HC}' \simeq \boldsymbol{C}'$. Write $\boldsymbol{C}'(\tau) = (C_1(\tau), C_2(\tau))$ and let $\boldsymbol{C}$ be the path defined by $\boldsymbol{C}(\tau) = (C_1(\tau), t)$. Clearly $\boldsymbol{C}' \simeq \boldsymbol{C}$ so that $\boldsymbol{C} \simeq \boldsymbol{HC}$. It now follows that $C_1 \simeq h_t C_1$ and consequently since $C_1(0) = x$, $C_1(1) = y$, we conclude that x and y are h_t-equivalent.

Conversely, if x and y are h_t-equivalent, then choose a path $D : I \to X$ so that $D(0) = x$, $D(1) = y$ and $D \simeq h_t D$. We now define $\boldsymbol{D} : I \to X \times I$ by $\boldsymbol{D}(\tau) = (D(\tau), t)$. Clearly $\boldsymbol{D} \simeq \boldsymbol{HD}$, showing that (x, t) and (y, t) are $\boldsymbol{H}$-equivalent.

Lemma 2. *Let H be an $\mathcal{G}$-homotopy and for a given $t \in I$, let F' be a fixed point class of h_t. Then there is a unique fixed point class F of $\boldsymbol{H}$ so that $F' = F_t$.*

Proof. If $x \in F'$, then $(x, t) \in \Phi(\boldsymbol{H})$, so that if F is the unique fixed point class containing (x, t) then $F' = F_t$ by Lemma 1.

Lemma 3. (Compare Brooks [1, Theorem 24, page 81]). *Let $H : X \times I \to X$ be an $\mathcal{G}$-homotopy and let F be a fixed point class of $\boldsymbol{H}$. Then $i(F_0) = i(F_1)$ (where we define $i(F_t) = 0$ if $F_t = \emptyset$).*

Proof. It suffices to show that for each $r \in I$, there is an $\epsilon > 0$ so that if $s \in I$ and $|r - s| < \epsilon$ then $i(F_r) = i(F_s)$. Then using the connectivity of the set I we obtain the desired result.

Let d be the metric defined on X and let $p : X \times I \to X$ be the projection map. Choose an open set $U \subseteq X \times I$ so that $F \subseteq U$ and $\mathrm{cl}(U) \cap \Phi(\boldsymbol{H}) = F$. If $r \in I$, then we easily have $F_r \subseteq U_r$ and $\mathrm{cl}(U_r) \cap \Phi(h_r) = F_r$. Thus $i(F_r) = i(X, h_r, U_r)$. Also, the compact set $K = p(F) - U_r$ clearly contains no fixed points of h_r and so we can find a $\delta > 0$ so that $d(h_r(x), x > \delta$ for all $x \in K$. Now since K is compact, H is uniformly continuous on $K \times I$; consequently there exists an $\epsilon > 0$ so that if $|r - s| < \epsilon$ then $d(H(x, r), H(x, s)) < \delta$ for all $x \in K$. Thus if $|r - s| < \epsilon$ then h_s can have no fixed points on K. It then follows that since $F_s \subseteq p(F)$, we must have $F_s \subseteq U_r$. Clearly, $\mathrm{cl}(U_r) \cap \Phi(h_s) = F_s$ since $\mathrm{cl}(U)$ contains only the fixed point class F. Thus $i(F_s) = i(X, h_s, U_r)$.

Finally, using the fact that $\boldsymbol{H}_{r,s} \in \mathcal{G}$, and for each $t \in I$, $h_t^{r,s}$ has no fixed points on the boundary of U_r, we conclude that $i(X, h_r, U_r) = i(X, h_s, U_r)$ by the homotopy axiom. Thus $i(F_r) = i(F_s)$ as asserted.

We are now prepared to prove our main theorem.

Theorem 2. *Let $H : X \times I \to X$ be an $\mathcal{G}$-homotopy. Then $N(h_0) = N(h_1)$.*

Proof. A given fixed point class of h_0 is the 0-slice of a fixed point class F of H (Lemma 2). Then by Lemma 3, $i(F_0) = i(F_1)$; hence if F_0 is essential, so is F_1. Thus $N(h_0) \leqq N(h_1)$.

Using the homotopy $H^{-1} = H^{1,0}$ and the same argument as above we obtain $N(h_1) \leqq N(h_0)$.

Remarks. We note that the definition of the Nielsen number and the conclusion of Theorem 2 require only axioms (1) and (2) of the fixed point index and the property that if $i(X, f, U) \neq 0$ then f has a fixed point in U (this is implicit in axiom 3). In view of this fact, Theorem 2 can be generalized to include those classes $\mathcal{G}$ of self maps on metric ANR having compact fixed point sets and such that $\mathcal{C}_{\mathcal{F}}$ admits an "index" satisfying the three properties above.

As a final remark, we note that it is possible to generalize the Nielsen theory to maps of the type $f : \mathrm{cl}(U) \to X$ where U is an open subset of X (metric ANR), $\Phi(f)$ is compact and f is fixed point free on the boundary of U. Using the fact that an open subset of an ANR is an ANR, one could obtain a "local Nielsen number," $N(f, U)$. We omit the obvious details of its definition and proof of its homotopy invariance.

§3. EXAMPLES. The condition of being an $\mathcal{F}$-homotopy is quite awkward and in practice it is usually satisfied by a homotopy H whenever $H \in \mathcal{F}$. This is the case for the three examples which follow.

1. *Palais maps.* A map $f : X \to X$ on a metric ANR is called a Palais map if the following two conditions are satisfied: (a) f is core compact; i.e., $\mathrm{cl}(\bigcap f^n(X))$ is compact. (b) f is locally compact; i.e., for each $x \in X$, there exists an open set U containing x so that $f(U) \subseteq U$ and $\mathrm{cl}(f(U))$ is compact.

Let $\mathcal{F}$ be the class of Palais maps and let $H : X \times I \to X$ be a homotopy so that $H \in \mathcal{F}$. To show H is a Palais homotopy, suppose that $U \subseteq X \times I$ is an open set such that $H(U) \subseteq U$ and $\mathrm{cl}(H(U))$ is compact. For $r, s \in I$ define $M_{r,s} : X \times I \to X \times I$ by $M_{r,s}(x, t) = (x, (t - r)/(s - r))$ and let $U^{r,s} = M_{r,s}(U)$. We then have $H^{r,s}(U^{r,s}) = M_{r,s}(H(U))$. From this it follows easily that $H^{r,s}$ is locally compact. The remaining conditions for being an $\mathcal{F}$-homotopy are straight forward.

The following is due to R. S. Palais (unpublished).

THEOREM 3. *Let $\mathcal{F}$ be the class of Palais maps on metric ANR. Then $\mathcal{F}$ is admissible and $\mathcal{C}_{\mathcal{F}}$ admits an index.*

Consequently by the above discussion and Theorem 2 we conclude that if $H \in \mathcal{F}$ then $N(h_0) = N(h_1)$.

2. *k-set contractions.* Let A be a bounded subset of a Banach space B. Define $\gamma(A)$ to be the infimum of all numbers $d > 0$ so that A can be covered by a finite number of sets in B of diameter less than or equal to d. If $X \subseteq B$ and $f : X \to X$ is a map such that for each bounded set $A \subseteq X$, $\gamma(f(A)) \leqq k\gamma(A)$ then f is called a k-set contraction.

Let $\mathcal{F}$ be the class of all self maps $f : X \to X$ such that X is a locally finite union of closed convex subsets of a Banach space, and f is a k-set contraction with $k < 1$ for which $f^n(X)$ is bounded for some n.

THEOREM 4. (Nussbaum [12]). *The class $\mathcal{C}_{\mathcal{G}}$ is admissible and admits an index.*

If H is a homotopy with $H \in \mathcal{G}$, it is easily verified that H is an $\mathcal{G}$-homotopy.

3. Let X be a subspace of a Banach space B and assume that there exist an increasing sequence of ANR subspaces $X_n \subseteq X$ contained in finite dimensional subspaces of B, and retractions $r_n : X \to X_n$ such that (a) $\mathrm{cl}(\bigcup X_n) = X$ and (b) $\lim r_n(x) = x$ for each $x \in X$. Call $\{X_n, r_n\}$ an approximation scheme for X. For $f : X \to X$, define $f_n : X_n \to X_n$ by $f_n = r_n \circ (f \mid X_n)$.

Note that if X admits an approximation scheme $\{X_n, r_n\}$ then so does $X \times I$, given by $\{X_n \times I, r_n = r_n \times \mathrm{id}\}$ where id is the identity map on I.

Let $\mathcal{G}$ be the class of all maps $f : X \to X$ such that $\Phi(f)$ is compact and X possesses an approximation scheme with the following property: for each sequence $\{x_{n_j}\}$ (with $x_{n_j} \in X_{n_j}$ and $n_j \to \infty$) such that $d(x_{n_j}, f_{n_j}(x_{n_j})) \to 0$, there exists a subsequence $x_{nj(k)}$ converging to some $x \in X$ and $f(x) = x$.

We remark that such maps are entirely analogous to and modelled upon the concept of A-proper maps [4].

We define an "index" as in [4]:

Let U be a bounded open subset of X so that (X, f, U) is admissible and set $U_n = U \cap X_n$. By an easy argument, it is easily seen that there exists some integer n_0 so that (X_n, f_n, U_n) is admissible whenever $n \geqq n_0$. By Dold [8], an index is defined for (X_n, f_n, U_n).

Let Z be the set of integers and let $Z' = Z \cup \{-\infty, \infty\}$. Define $i(X, f, U)$ to equal the set of limit points of $\{i(X_n, f_n, U_n) \mid (X_n, f_n, U_n)$ is admissible$\}$. Then $i(X, f, U)$ is a non-empty subset of Z'. Now if U is unbounded, choose a bounded open subset $V \subset U$ so that (X, f, V) is admissible and $(U - V) \cap \Phi(f) = \emptyset$; define $i(X, f, U) = i(X, f, V)$.

The following properties are easily verified:

(i) If $i(X, f, U) \neq \{0\}$ then f has a fixed point in U.

(ii) If H is a homotopy such that $H \in \mathcal{G}$ and (X, h_t, U) is admissible for each $t \in I$, then $i(X, h_0, U) = i(X, h_1, U)$.

(iii) If $(X, f, U) \in \mathcal{C}_{\mathcal{G}}$ and $U_1, \cdots, U_n$ are disjoint open subsets of U so that f is fixed point free on $U - (\bigcup_{j=1}^{n} U_j)$ then $i(X, f, U) \subseteq \sum_{j=1}^{n} i(X, f, U_j)$ (where $\infty - \infty = Z'$ by convention).

(iv) If $H : X \times I \to X$ is a map so that $H \in \mathcal{G}$ then H is an $\mathcal{G}$-homotopy.

Finally, to define the Nielsen number, we further require that for each map $f : X \to X$ in $\mathcal{G}$, X is also an ANR.

Together with the remarks following Theorem 2, we obtain a homotopy invariant Nielsen number for this class $\mathcal{G}$.

References

1. R. Brooks, *Coincidences, roots and fixed points*, Doct. Diss., U.C.L.A., 1967.

2. F. Browder, *On the fixed point index for continuous mappings of locally connected spaces*, Summa Brasil. Math. 4, (1960), 253–293.

3. ———, *Asymptotic fixed point theorems*, Math. Ann. **185** (1970), 38–61.

4. F. E. Browder and W. V. Petryshyn, *Noncompact operators in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 641–646.

5. R. F. Brown, *The Lefschetz fixed point Theorem*, Scott, Foresman and Company (1971).

6. ———, *An elementary proof of the uniqueness of the fixed point index*, Pac. J. Math. **35** (1971), 549–559.

7. ———, *On the Nielsen fixed point theorem for compact maps*, Duke Math. J. **36** (1969), 699–708.

8. A. Dold, *Fixed point index and fixed point theorem for Euclidean neighbourhood retracts*, Topology **4** (1965), 1–8.

9. A. Gramas, *Generalizing the Hopf-Lefschetz fixed point theorem for non-compact ANR-s*, Symposium on Infinite Dimensional Topology, Bâton-Rouge, 1967.

10. J. Leray, *Théorie des pointes fixés: indice total et nombre de Lefschetz*, Bull. Soc. Math. France, **87** (1959), 221–233.

11. J. Nielsen, *Untersuchung zur Topologie der geschlossenen zweiseitigen Flächen*, Acta Math. **50** (1927), 189–358.

12. R. Nussbaum, *The fixed point index for k-set contractions*, Doct. Diss., Univ. of Chicago, 1969.

13. B. O'Neill, *Essential sets and fixed points*, Amer. J. Math. **75** (1953), 497–509.

14. U. K. Scholz, *The Nielsen Theory on fixed point classes for Palais maps*, Doct. Diss., U.C.L.A., 1970.

15. R. Wecken, *Fixpunktklassen* I, II, Math. Ann. **117** (1941), 659–671; Math. Ann. **118** (1941), 216–234.

University of Arizona, Tucson, Arizona 85721

College of St. Thomas, St. Paul, Minnesota 55105