

## QUASILINEAR EQUATIONS AND EQUATIONS WITH LARGE NONLINEARITIES<sup>1</sup>

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**1. Introduction.** Two kinds of equations studied in nonlinear functional analysis are: (i) quasilinear equations, i.e., equations which differ from a linear equation by a small nonlinear term, usually a nonlinear expression multiplied by a parameter  $\mu$  which is restricted to small values and (ii) nonlinear equations with well-behaved but "large" nonlinearities. These two kinds of equations are usually treated by different methods, and different kinds of results are obtained. (More extensive and detailed results can be obtained for the quasilinear equations.) Our purpose here is to study nonlinear equations with large nonlinearities by regarding them as quasilinear equations in which the parameter  $\mu$  is allowed to take "large" values, i.e., we let  $\mu \in [0, 1]$ . We will study the equations with large nonlinearities by obtaining results about the quasilinear systems which are valid if the parameter  $\mu$  is allowed to take "large" values. We study an equation in a linear space of the form

$$(E) \quad L(x) + \mu T(x, \mu) = 0,$$

where  $L$  is linear and  $T$  is a compact transformation which satisfies a uniform Lipschitz condition and a condition on the rate of growth of  $\|T(x, \mu)\|$  as  $\|x\|$  increases. In this paper, we consider only the case in which  $L$  has an inverse. (In a later paper, a different hypothesis will be used). By using Brouwer degree or Leray-Schauder degree theory, we obtain a theorem concerning solutions of equation (E). This theorem is then applied to obtain results on the existence of periodic solutions of ordinary differential equations, periodic solutions of functional differential equations, solutions of the Dirichlet problem for nonlinear elliptic equations and solutions of the Dirichlet problem for nonlinear parabolic equations.

In Section 2, we obtain the abstract theorem. In Section 3, we apply the theorem to obtain periodic solutions for nonlinear systems of

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ordinary differential equations. The systems studied are of the form  $x' = A(t)x + \mu f(t, x, \mu)$  where  $\mu \in [0, 1]$  and where  $A$  and  $f$  are periodic in  $t$ . The result obtained is related to work of Barbalat and Halanay [3], Pliss [21], Reissig [22, 23] and Gússefeldt [12], (Gússefeldt's result has also been obtained by Mawhin [18] by using a different method), and is closest to that of Reissig and Gússefeldt. The difference between our result and those of Reissig and Gússefeldt is that we impose a hypothesis on the  $f$  which implies the existence of an a priori bound on the periodic solutions whereas Reissig and Gússefeldt assume the existence of an a priori bound on the periodic solutions. Also we use finite-dimensional techniques whereas Reissig and Gússefeldt use Leray-Schauder techniques.

In Section 4, the same abstract theorem is applied to obtain periodic solutions of functional differential equations with large nonlinearities. The essential difference between finding periodic solutions of ordinary differential equations and periodic solutions of functional differential equations is that in the case of ordinary differential equations it is only necessary to study an equation in a finite-dimensional space, i.e., Euclidean  $n$ -space, whereas in the case of functional differential equations, it is necessary to study an equation in an infinite-dimensional space, i.e., a space of continuous functions. Periodic solutions of nonlinear functional differential equations have been studied in recent years by a number of writers. Periodic solutions of quasilinear equations have been studied by Halanay [13], Perelló [20], and Aizengendler and Vainberg [1]. Periodic solutions of nonlinear systems have been obtained by Jones [15] by using fixed point theorems. Periodic and almost periodic solutions have been studied by Hale [14], and Leray-Schauder degree has been used by Mawhin [18]. (For further references on periodic solutions of functional differential equations, see [19].) Only a result in [19] is related to the result here. Mawhin obtains an extension to functional differential equations of the result of Gússefeldt. He makes essentially the assumption that the degree considered is actually defined whereas we obtain an a priori estimate from the properties of the equation studied.

In Section 5, the abstract theorem is applied to solve the Dirichlet problem for a class of nonlinear elliptic equations. The result obtained is an extension of a well-known result. (Courant and Hilbert [7, pp. 369–372].)

In Section 6, the Dirichlet problem for a class of nonlinear parabolic equations is studied.

2. **An Abstract Theorem.** Let  $X$  be a normed linear space and consider the equation in  $X$ :

$$(2.1) \quad L(x) + \mu T(x, \mu) = 0,$$

where  $\mu \in [0, 1]$ ,  $x \in X$  and

(i)  $L$  is a linear continuous map from  $X$  into  $X$  with a linear continuous inverse  $L^{-1}$  from  $X$  into  $X$ ;

(ii)  $T$  is a continuous map from  $X \times [0, 1]$  into  $X$  such that  $T(x, \mu)$  is continuous in  $\mu$  uniformly on bounded sets in  $X$ , i.e., given  $\epsilon > 0$  and  $M > 0$  then there exists  $\delta > 0$  such that if  $|\mu_1 - \mu_2| < \delta$  and if  $\|x\| < M$ , then  $\|T(x, \mu_1) - T(x, \mu_2)\| < \epsilon$ .

(iii) for each fixed  $\mu \in [0, 1]$ ,  $T(\cdot, \mu)$  is a compact map from  $X$  into  $X$  (Conditions (ii) and (iii) imply that  $T$  takes bounded sets into bounded sets, i.e., given  $M > 0$  there exists  $M_1 > 0$  such that if  $\|x\| < M$ , then for all  $\mu \in [0, 1]$ ,  $\|T(x, \mu)\| < M_1$ ).

(iv) There exist positive constants  $R, A, B, C$  and  $\epsilon \in (0, 1)$  such that if  $\|x\| \geq R$ , then for all  $\mu \in [0, 1]$

$$\|T(x, \mu)\| < A(B + C\|x\|)^{1-\epsilon}.$$

**THEOREM 1.** For each  $\mu \in [0, 1]$ , equation (2.1) has a solution  $x$ .

**PROOF.** Instead of (2.1), we solve the equivalent equation

$$(2.2) \quad x + \mu L^{-1}T(x, \mu) = 0.$$

Since for each  $\mu \in [0, 1]$ ,  $T(\cdot, \mu)$  is compact, the Leray-Schauder degree of  $I + \mu L^{-1}T(\cdot, \mu)$  where  $I$  denotes the identity map, can be investigated. (If  $X$  is finite-dimensional, then instead of the Leray-Schauder degree, we consider the Brouwer degree.)

Now suppose  $x_0 \neq 0$  is a solution of (2.2) for some  $\mu \in [0, 1]$  and suppose  $\|x_0\| \geq R$ . Then

$$\|x_0\| < \mu \|L^{-1}\| A(B + C\|x_0\|)^{1-\epsilon},$$

or

$$\frac{\|x_0\|^{1/(1-\epsilon)}}{B + C\|x_0\|} < (\mu \|L^{-1}\| A)^{1/(1-\epsilon)}.$$

Then either  $\|x_0\| < B$  or

$$\frac{\|x_0\|^{1/(1-\epsilon)}}{(1 + C)\|x_0\|} \leq \frac{\|x_0\|^{1/(1-\epsilon)}}{B + C\|x_0\|} < (\mu \|L^{-1}\| A)^{1/(1-\epsilon)},$$

or

$$\|x_0\|^{d/(1-\epsilon)} < (1 + C)(\mu\|L^{-1}\|A)^{1/(1-\epsilon)}.$$

Thus, we can conclude that the solutions of (2.2) for  $\mu \in [0, 1]$  satisfy an a priori estimate, i.e., there is a closed ball  $B_0$  such that all solutions of (2.2) for  $\mu \in [0, 1]$  are in  $B_0$ . Thus

$$\deg(I + \mu L^{-1}T(\cdot, \mu), B_0, 0),$$

where  $\deg(f, S, p)$  denotes the degree (Brouwer or Leray-Schauder) of the map  $f$  at the point  $p$  and relative to the set  $S$ , is defined and has the same value for all  $\mu \in [0, 1]$ . Since  $\deg(I, B_0, 0) = +1$ , we have for all  $\mu \in [0, 1]$

$$\deg(I + \mu L^{-1}T(\cdot, \mu), B_0, 0) = +1,$$

and by a basic property of degree theory, the conclusion of the theorem follows.

**3. Application to periodic solutions of ordinary differential equations.** We study the  $n$ -dimensional system

$$(3.1) \quad x' = A(t)x + \mu f(t, x, \mu)$$

in which the following hypotheses are satisfied:

- (1)  $\mu \in [0, 1]$ ;
- (2) the elements of matrix  $A(t)$  and the components of  $f$  are defined for all real  $t$ , all  $x \in R^n$  and all  $\mu \in [0, 1]$ ;
- (3) the elements of  $A(t)$  and the components of  $f$  have continuous first derivatives in  $t$  for all real  $t$  and the components of  $f$  have continuous first derivatives with respect to  $\mu \in [0, 1]$  and continuous first derivatives with respect to the components of  $x$  for all  $x \in R^n$ ;
- (4) matrix  $A(t)$  and function  $f(t, x, \mu)$  have period  $T$  in  $t$ ;
- (5) the equation  $x' = A(t)x$  has no nontrivial solutions of period  $T$ ;
- (6) there exist positive constants  $R_1, N$  and  $\epsilon \in (0, 1)$  such that if  $\|x\| \geq R_1$ , then for all  $\mu \in [0, 1]$  and all  $t \in [0, T]$ ,  $\|f(t, x, \mu)\| < N(\|x\|)^{1-\epsilon}$ ; also the number  $N$  is chosen large enough so that  $M < N(R_1)^{1-\epsilon}$ , where

$$M = \max_{\substack{\|x\| \leq R_1 \\ 0 \leq t \leq T \\ 0 \leq \mu \leq 1}} \|f(t, x, \mu)\|;$$

- (7)  $f$  satisfies a Lipschitz condition in  $x$  uniformly in  $t$  and  $\mu$ , i.e., there is a positive constant  $L$  such that for all  $t \in [0, T]$ , all

$$\mu \in [0, 1] \quad \text{and} \quad \text{all} \quad x_1, x_2 \in R^n, \quad \|f(t, x_1, \mu) - f(t, x_2, \mu)\| < L\|x_1 - x_2\|.$$

LEMMA 1. *There exists  $\epsilon > 0$  such that for  $t \in (-\epsilon, T + \epsilon)$ ,  $\mu \in [0, 1]$  and for given  $t_0 \in (-\epsilon, T + \epsilon)$  and given  $x_0 \in R^n$  there is a unique solution  $x(t, \mu, t_0, x_0)$  of (3.1) on the interval  $(-\epsilon, T + \epsilon)$  and  $x(t, \mu, t_0, x_0)$  is a continuous function of  $(t, \mu, t_0, x_0)$ .*

(We are actually interested in the solution only on the interval  $[0, T]$ . We consider intervals of the form  $(-\epsilon, T + \epsilon)$  in order to avoid consideration of one-sided derivatives at  $t = 0$  and  $t = T$ .)

PROOF. Most of this statement is contained in standard existence theorems (see Coddington and Levinson [6, Chapter 1]). The only exceptional part lies in the fact that we are considering a “large” interval for the values  $\mu$ . However, because of the hypotheses on  $f$ , it is easy to show that the sequence

$$x_0(t, t_0, x_0), \dots, \\ x_{n+1}(t, \mu, t_0, x_0) = x_0 + \int_{t_0}^t f[s, x_n(s, \mu, t_0, x_0), \mu] ds,$$

where  $x_0(t, t_0, x_0)$  is the solution of  $x' = A(t)x$  which is equal to  $x_0$  at  $t = t_0$ , converges uniformly for  $t \in [-\epsilon, T + \epsilon]$  (where  $\epsilon$  is any fixed positive number),  $\mu \in [0, 1]$ ,  $t_0 \in [-\epsilon, T + \epsilon]$  and  $x_0$  in a bounded set in  $R^n$ . Hence the solution  $x(t, \mu, t_0, x_0)$  exists and has the desired properties.

Now by using the variation of constants formula it follows that the problem of finding periodic solutions of (3.1) becomes the problem of solving the equation in Euclidean  $n$ -space

$$(3.2) \quad [F(T) - F(0)]x_0 \\ + F(T) \int_0^T [F(s)]^{-1} \{ \mu f[s, x(s, \mu, 0, x_0), \mu] \} ds = 0,$$

where  $F(t)$  is the fundamental matrix of  $x' = A(t)x$  such that  $F(0)$  is the identity matrix, for  $x_0$  as a function of  $\mu$ . See [8, pp. 65-66]. In order to apply Theorem 1 to equation (3.2), we note first that since  $x' = A(t)x$  has no nontrivial solutions of period  $T$ , the matrix  $F(T) - F(0)$  is nonsingular (see [8, p. 66]). Next we want to obtain an estimate on

$$(3.3) \quad \|F(T) \int_0^T [F(s)]^{-1} \{ \mu f[s, x(s, \mu, 0, x_0), \mu] \} ds\|,$$

in terms of  $\|x_0\|$ . First, let

$$B_1 = \max_{0 \leq t \leq T} \|F(t)\|, \quad B_2 = \max_{0 \leq t \leq T} \|[F(t)]^{-1}\|.$$

Then (3.3) is less than

$$(3.4) \quad B_1 B_2 T \max_{\substack{0 \leq s \leq T \\ 0 \leq \mu \leq 1}} \|f[s, x(s, \mu, 0, x_0), \mu]\|.$$

Now either

$$\max_{\substack{0 \leq s \leq T \\ 0 \leq \mu \leq 1}} \|x(s, \mu, 0, x_0)\| < R_1,$$

or

$$\max_{\substack{0 \leq s \leq T \\ 0 \leq \mu \leq 1}} \|f[s, x(s, \mu, 0, x_0), \mu]\| < N \left( \max_{\substack{0 \leq s \leq T \\ 0 \leq \mu \leq 1}} \|x(s, \mu, 0, x_0)\| \right)^{1-\epsilon}.$$

If  $x_0$  is such that

$$\max_{\substack{0 \leq s \leq T \\ 0 \leq \mu \leq 1}} \|x(s, \mu, 0, x_0)\| \geq R_1,$$

then

$$(3.5) \quad \max_{\substack{0 \leq s \leq T \\ 0 \leq \mu \leq 1}} \|f[s, x(s, \mu, 0, x_0), \mu]\| < N \left( \max_{\substack{0 \leq s \leq T \\ 0 \leq \mu \leq 1}} \|x(s, \mu, 0, x_0)\| \right)^{1-\epsilon}.$$

But

$$(3.6) \quad \begin{aligned} \max_{\substack{0 \leq s \leq T \\ 0 \leq \mu \leq 1}} \|x(s, \mu, 0, x_0)\| &\leq \max_{\substack{0 \leq s \leq T \\ 0 \leq \mu \leq 1}} \|x(s, \mu, 0, 0)\| \\ &+ \max_{\substack{0 \leq s \leq T \\ 0 \leq \mu \leq 1}} \|x(s, \mu, 0, x_0) - x(s, \mu, 0, 0)\|. \end{aligned}$$

Let

$$(3.7) \quad B = \max_{\substack{0 \leq s \leq T \\ 0 \leq \mu \leq 1}} \|x(s, \mu, 0, 0)\|.$$

Since

$$(3.8) \quad \max_{\substack{0 \leq s \leq T \\ 0 \leq \mu \leq 1}} \|x(s, \mu, 0, x_0) - x(s, \mu, 0, 0)\| \leq \|x_0\| e^{TL},$$

(See Halanay [13, p. 10]), combining (3.3), (3.4), (3.5), (3.6), (3.7), and (3.8), we obtain: if  $x_0$  is such that

$$\max_{\substack{0 \leq s \leq T \\ 0 \leq \mu \leq 1}} \|x(s, \mu, 0, x_0)\| \geq R_1,$$

(in particular, if  $\|x_0\| \cong R_1$ ) then

$$\begin{aligned} & \|F(T) \int_0^T [F(s)]^{-1} \{ \mu f[s, x(s, \mu, 0, x_0), \mu] \} ds \| \\ & \cong B_1 B_2 T N (B + \|x_0\| e^{TL})^{1/\epsilon}. \end{aligned}$$

Thus, the hypotheses of Theorem 1 are satisfied by equation (3.2) and applying Theorem 1, we obtain:

**THEOREM 2.** *For each  $\mu \in [0, 1]$ , equation (3.1) has a solution of period  $T$ .*

The conclusion of Theorem 2 does not preclude the possibility that the periodic solution obtained is the trivial one, i.e., the identically zero solution.

**COROLLARY 2.** *If  $\mu_0 \in (0, 1)$  is such that*

$$\int_0^T [F(s)]^{-1} f(s, 0, \mu_0) ds \neq 0,$$

*then the periodic solution obtained in Theorem 2 for  $\mu = \mu_0$  is non-trivial.*

**PROOF.** If the condition in the corollary holds, then by the variation of constants formula,  $x(t, \mu_0, 0, x_0)$  cannot be identically zero.

Next we consider the autonomous equation

$$(3.9) \quad x' = Ax + \mu f(x, \mu),$$

and assume that the same hypotheses as for equation (3.1) are satisfied and that, in addition,  $A$  is a constant matrix and  $f$  is independent of  $t$ . Let  $T > 0$  be such that the equation  $x' = Ax$  has no nontrivial solutions of period  $T$ . We study the question of whether (3.9) has solutions of period  $T$  for  $\mu \in (0, 1]$ .

Before proceeding to this study, we make one comment concerning the status of the problem. In the study of quasilinear autonomous equations, the classical approach is to assume that as  $\mu$  is varied, the period of the sought after periodic solution also varies. That is, one seeks a solution of period  $T(\mu)$  where  $T$  is a continuous function of  $\mu$ . (See, for example, Coddington and Levinson [6, p. 352ff.] .) This is intuitively reasonable because one would expect that if the system described by the differential equation is changed, its "natural frequency" will be changed. In the main, the mathematical considerations agree with this because if one attempts to apply the classical Poincaré small parameter method to the autonomous case and if the problem is a nonresonance problem, then either no periodic solution will exist or if a periodic solution exists, it will most frequently be

identically zero. There are, however, exceptions to this. For example, if

$$f(x, \mu) = g(x, \mu) + h(\mu),$$

where  $g(0, \mu) = 0$  for all  $\mu$  and where  $h(\mu)$  is a continuous function of  $\mu$  such that  $h(0) = 0$  and, for  $|\mu|$  sufficiently small but nonzero,  $h(\mu) \neq 0$  and  $\int_0^T [F(s)]^{-1} ds$  is nonsingular, where  $F(s)$  is a fundamental matrix of  $x' = Ax$  such that  $F(0)$  is the identity matrix, then there exists a nontrivial periodic solution for each sufficiently small  $|\mu|$ . There will also be nontrivial periodic solutions in the resonance case if a further condition on  $g$  (a nonzero condition so that the topological degree of the appropriate mapping is defined and nonzero) is satisfied. (Actually in the resonance case there is a much larger class of equations for which there exist nontrivial solutions of period  $T$ . This can be demonstrated for autonomous functional differential equations as well as autonomous ordinary differential equations by using the techniques of [9] and [10].)

We obtain similar results for equation (3.9) by using the same theory that was applied to equation (3.1).

**COROLLARY 2a.** *If  $x' = Ax$  has no nontrivial solutions of period  $T$  and if  $\mu_0 \in (0, 1]$  is such that*

$$\int_0^T [F(s)]^{-1} f(0, \mu_0) ds \neq 0,$$

*then the equation (3.9) with  $\mu = \mu_0$  has a nontrivial solution of period  $T$ .*

**4. Application to periodic solutions of functional differential equations.** We study the  $n$ -dimensional functional differential equation

$$(4.1) \quad x' = L(t, x_t) + \mu f(t, x_t, \mu),$$

where

$$(4.2) \quad L(t, x_t) = \int_{-\infty}^0 x_t(s) d_s \eta(t, s),$$

where  $x_t(s)$  denotes the function  $x(t+s)$ , where  $-\infty < s \leq 0$  and  $\eta(t, s)$  is an  $n \times n$  matrix with elements  $\eta_{ij}(t, s)$  such that for  $i, j = 1, \dots, n$ ,

(1)  $\eta_{ij}(t, s)$ , is defined for  $t \geq 0$ ,  $-\infty < s < \infty$ , and  $\eta_{ij}(t, s) = 0$ , if  $s \geq 0$ .

(2) There exist positive valued functions  $\tau_{ij}(t)$ ,  $V_{ij}(t)$ , defined and bounded for  $t \geq 0$ , such that for  $s \leq -\tau_{ij}(t)$ ,

$$\eta_{ij}(t, s) = \eta_{ij}(t, -\tau_{ij}(t)),$$



and

$$V_{s=-\tau_{ij}(t)}^{s=0} \eta_{ij}(t, s) \leq V_{ij}(t),$$

where

$$V_{s=-\tau_{ij}(t)}^{s=0} \eta_{ij}(t, s)$$

denotes the total variation from  $s = -\tau_{ij}(t)$  to  $s = 0$  of  $\eta_{ij}(t, s)$  regarded as a function of  $s$ . (For later use, set

$$\tau = \sup_{\substack{i,j=1,\dots,n \\ t \geq 0}} \tau_{ij}(t).$$

Note that  $\int_{-\infty}^0 x_t(s) d_s \eta(t, s) = \int_{-\tau}^0 x_t(s) d_s \eta(t, s)$ .) Also for each fixed  $t$ , the function of bounded variation  $\eta_{ij}(t, s)$  is the sum of a jump function and an absolutely continuous function, i.e.,  $\eta_{ij}(t, s)$  has no singular part. See Kolmogorov and Fomin [16, p. 363]. (We will use results of Halanay [13, Chapter 4]. Halanay does not state this last hypothesis. However, Banks [2] has pointed out that for Halanay's results to be valid, it is necessary to assume that  $\eta_{ij}(t, s)$  has no singular part. Actually, Banks has corrected Halanay's work for the case of  $\eta_{ij}(t, s)$ , an arbitrary function of bounded variation. Since we want to use later results of Halanay, we restrict  $\eta_{ij}(t, s)$  as described above. It should also be remarked that Halanay does not give a satisfactory theory for the adjoint system (see Banks [2, p. 400]). But the representation obtained by Halanay [13, pp. 365-366, formulas (23) and (23')] is correct, provided each  $\eta_{ij}(t, s)$  has no singular part, and this representation and later results based on it are all we need.)

- (3)  $\eta_{ij}(t, s)$  is continuous in  $t$  uniformly with respect to  $s$ .
- (4) For each  $s$ ,  $\eta(t, s)$  has period  $T$  in variable  $t$  where  $T > \tau$ .
- (5) The equation  $x' = L(t, x_t)$  has no nontrivial solutions of period  $T$ .

We assume further that  $f(t, x_t, \mu)$  is a continuous map from  $I \times C[-\tau, 0] \times [0, 1]$ , where  $I$  is the real  $t$ -axis and  $C[-\tau, 0]$  is the Banach space of real continuous  $n$ -vector functions on  $[-\tau, 0]$  with the usual sup norm into real Euclidean  $n$ -space and that  $f(t, x_t, \mu)$  has period  $T$  in variable  $t$ , and that  $f(t, x_t, \mu)$  has the following additional properties:

- (6)  $f$  takes sets of the form

$$J \times C_1 \times [0, 1],$$

where  $J$  is a subset of  $I$  and  $C_1$  is a bounded subset of  $C[-\tau, 0]$ , into bounded sets in  $R^n$ .

(7) There exists  $R_2 > 0$  and  $N_1 > 0$  and  $\epsilon \in (0, 1)$  such that if  $y \in C[-\tau, 0]$  and  $\|y\| \geq R_2$ , then for  $t \in [0, T]$  and  $\mu \in [0, 1]$ ,  $\|f(t, y, \mu)\| \leq N_1(\|y\|)^{1-\epsilon}$ . Also the number  $N_1$  is chosen large enough so that  $M < N_1(R)^{1-\epsilon}$ , where

$$M = \text{lub}_{\substack{\|y\| \leq R_2 \\ t \in [0, T] \\ \mu \in [0, 1]}} \|f(t, y, \mu)\|.$$

(8)  $f$  satisfies a Lipschitz condition in  $y$  uniformly in  $t$  and  $\mu$ , i.e., there exists a positive constant  $L$  such that if  $t \in [0, T]$  and  $\mu \in [0, 1]$  and if  $y_1, y_2 \in C[-\tau, 0]$ , then

$$\|f(t, y_1, \mu) - f(t, y_2, \mu)\| \leq L\|y_1 - y_2\|,$$

where  $\|y_1 - y_2\|$  denotes the  $C[-\tau, 0]$  norm of  $y_1 - y_2$ . For later computation we will take  $L > 1$ .

(9)  $f$  is continuous in  $\mu$  uniformly in  $t$  and bounded sets in  $C[-\tau, 0]$ , i.e., given  $\epsilon > 0$  and  $B$  a bounded set in  $C[-\tau, 0]$ , then there exists  $\delta > 0$  such that if  $|\mu_1 - \mu_2| < \delta_1$  and  $y \in B$ , then for all  $t \in [0, T]$

$$\|f(t, y, \mu_1) - f(t, y, \mu_2)\| < \epsilon.$$

The first step in finding periodic solutions of (4.1) is to obtain an analog of Lemma 1 in Section 3.

**LEMMA 2.** *For each  $\phi \in C[-\tau, 0]$  and each  $\mu \in [0, 1]$ , there exists a function  $x(t, \phi, \mu)$  defined on  $[-\tau, T + \epsilon]$ , where  $\epsilon$  is any fixed positive number, such that  $x(t, \phi, \mu)$  is a solution of (4.1) for  $0 \leq t < T + \epsilon$  and  $\phi$  is the initial function for solution  $x(t, \phi, \mu)$  i.e., if  $t \in [-\tau, 0]$ , then  $x(t, \phi, \mu) = \phi(t)$ . Also  $x(t, \phi, \mu)$  is a continuous function from  $C[-\tau, 0] \times [0, 1]$  into  $C[-\tau, T + \epsilon]$ .*

**PROOF.** If  $\phi$  is fixed, then there is a unique solution  $x(t, \phi, \mu)$  of (4.1) on  $[0, T + \epsilon]$  where  $\epsilon$  is an arbitrary fixed positive number for each  $\mu \in [0, 1]$ . The function  $x(t, \phi, \mu)$  is defined for  $t \in [-\tau, T + \epsilon]$  and is the limit of the sequence  $x_0(t, \phi), x_{n+1}(t, \phi, \mu)$ , where  $x_0(t, \phi)$  is the solution of  $x' = L(t, x_t)$  which has the initial function  $\phi$  and

$$\begin{aligned} x_{n+1}(t, \phi, \mu) &= \phi(0) + \int_0^t \{L(\sigma, x_{n\sigma}(\xi, \phi, \mu)) \\ &\quad + \mu f[\sigma, x_{n\sigma}(\xi, \phi, \mu), \mu]\} d\sigma, \\ &\quad \text{for } 0 \leq t \leq T + \epsilon, \\ &= \phi(t), \text{ for } -\tau \leq t \leq 0, \end{aligned}$$

where  $x_{n\sigma}(\xi, \phi, \mu)$  denotes  $(x_n)_\sigma(\xi, \phi, \mu)$ . Because of the hypotheses imposed on  $L$  and  $f$ , it is not difficult to show, by using standard arguments that for each  $n$ ,  $x_n(t, \phi, \mu)$  is continuous in  $(t, \mu)$  and the sequence  $\{x_n(t, \phi, \mu)\}$  converges uniformly in  $(t, \mu)$  for  $(t, \mu) \in [0, T + \epsilon] \times [0, 1]$ . Hence, for fixed  $\phi$ , the solution  $x(t, \phi, \mu)$  is continuous in  $(t, \mu)$ .

Now suppose  $\mu$  is fixed. If  $\phi_1, \phi_2 \in C[-\tau, 0]$ , then according to the fundamental inequality for solutions of functional differential equations (see Halanay [13, p. 338]) we have: for each  $\mu \in [0, 1]$ ,

$$\max_{0 \leq t \leq T + \epsilon} |x(t, \phi_1, \mu) - x(t, \phi_2, \mu)| \leq \|\phi_1 - \phi_2\| e^{L(T + \epsilon)}.$$

Since

$$\begin{aligned} \max_{-\tau \leq t \leq 0} |x(t, \phi_1, \mu) - x(t, \phi_2, \mu)| &= \max_{-\tau \leq t \leq 0} |\phi_1(t) - \phi_2(t)| \\ &= \|\phi_1 - \phi_2\|, \end{aligned}$$

we can conclude that there exists a positive constant  $B_3$  such that for all  $\mu \in [0, 1]$ ,

$$(4.3) \quad \max_{-\tau \leq t \leq T + \epsilon} |x(t, \phi_1, \mu) - x(t, \phi_2, \mu)| < B_3 \|\phi_1 - \phi_2\|.$$

Now suppose  $\phi_1, \phi_2 \in C[-\tau, 0]$ ,  $\mu_1, \mu_2 \in [0, 1]$ . We will show that  $x(t, \phi, \mu)$  is continuous at the point  $(\phi_2, \mu_2)$ :

$$(4.4) \quad \begin{aligned} &\max_{-\tau \leq t \leq T + \epsilon} |x(t, \phi_1, \mu_1) - x(t, \phi_2, \mu_2)| \\ &\leq \max_{-\tau \leq t \leq T + \epsilon} |x(t, \phi_1, \mu_1) - x(t, \phi_2, \mu_1)| \\ &+ \max_{-\tau \leq t \leq T + \epsilon} |x(t, \phi_2, \mu_1) - x(t, \phi_2, \mu_2)| \leq B_3 \|\phi_1 - \phi_2\| + \epsilon. \end{aligned}$$

The first part of the inequality is from inequality (4.3). The second part of the inequality comes from the fact that  $x(t, \phi_2, \mu)$  is continuous in  $(t, \mu)$  on the compact set  $[-\tau, T + \epsilon] \times [0, 1]$ . This completes the proof of Lemma 2.

Next we recast the problem of finding periodic solutions of (4.1) as the problem of solving a functional equation that is the analog of equation (3.2) in Section 3. This analog is the following functional equation in  $C[-\tau, 0]$ .

$$(4.5) \quad \phi(s) = z(T + s, \phi) + \int_0^{T+s} \mu f[\alpha, x_\alpha(\sigma, \phi, \mu), \mu] X(T + s, \alpha) d\alpha,$$

where  $s \in [-\tau, 0]$ ,  $z(t, \phi)$  is the solution of  $x' = L(t, x_t)$  which has the

initial function  $\phi$  and  $X(\alpha, \alpha)$  is the identity matrix and  $X(t, \alpha) = 0$  for  $t < \alpha$ . Equation (4.5) is derived by Halanay [13, pp. 411–412]. In order to find a solution of period  $T$  of (4.1), it is sufficient to solve (4.5) for  $\phi$  as a function of  $\mu$ . The solution of (4.1) which has this  $\phi$  as its initial function is a solution of (4.1) with period  $T$ .

**LEMMA 3.** *There exists a positive constant  $M_1$ , such that for all  $(t, \alpha) \in [0, T + \epsilon] \times [0, T + \epsilon]$ ,  $\|X(t, \alpha)\| < M_1$ .*

**PROOF.** First  $X(t, \alpha) = Y(\alpha, t)$  where  $Y(\alpha, t)$  is the matrix solution of the system

$$-\frac{d}{d\alpha} [y(\alpha) + \int_{-\tau}^0 \eta(\alpha - \beta, \beta)y(\alpha - \beta) d\beta] = 0,$$

such that  $Y(t, t)$  is the identity matrix and  $Y(\alpha, t) = 0$  for  $\alpha > t$ . See Halanay [13, pp. 362–365]. But  $Y(\alpha, t)$  can be obtained by using successive approximations and the “step-by-step” method. See [13, p. 363]. In the process of obtaining  $Y(\alpha, t)$  in this way, it can be shown that for each fixed  $t$ ,  $Y(\alpha, t)$  is of bounded variation on finite intervals of the  $\alpha$ -axis. Also  $Y(\alpha, t)$  is continuous in  $t$  uniformly on closed finite intervals of the  $\alpha$ -axis, which consist of points  $\alpha$  such that  $\alpha \leq t$ . (The continuity in  $t$  is one-sided on the right if  $\alpha = t$ .) But from these facts it follows that there exists  $M_1 > 0$  such that  $\|Y(\alpha, t)\| < M_1$  for  $(\alpha, t) \in [0, T + \epsilon] \times [0, T + \epsilon]$ .

**LEMMA 4.** *Given  $\epsilon > 0$  then there exists  $\delta > 0$  such that if  $t_1, t_2 \in [0, T + \epsilon]$  and  $|t_1 - t_2| < \delta$  and if  $0 \leq \alpha \leq \min(t_1, t_2)$  then  $\|X(t_1, \alpha) - X(t_2, \alpha)\| < \epsilon$ .*

**PROOF.**  $\|X(t_1, \alpha) - X(t_2, \alpha)\| = \|Y(\alpha, t_1) - Y(\alpha, t_2)\|$ . Since  $Y(\alpha, t)$  is continuous in  $t$  uniformly on closed finite intervals of the  $\alpha$ -axis, there exists  $\delta > 0$  such that if  $|t_1 - t_2| < \delta$  then for all  $\alpha \in [0, T + \epsilon]$ ,  $\|Y(\alpha, t_1) - Y(\alpha, t_2)\| < \epsilon$ .

Since  $z(T + s, \phi)$  is uniquely determined by  $\phi$ , the operator

$$U : \phi(s) \rightarrow z(T + s, \phi),$$

is a well-defined map from  $C[-\tau, 0]$  into  $C[-\tau, 0]$  and we may rewrite equation (4.5) as

$$(4.6) \quad (I - U)\phi - \int_0^{T+s} \mu f[\alpha, x_a(\sigma, \phi, \mu), \mu] X(T + s, \alpha) d\alpha = 0.$$

**LEMMA 5.** *Operator  $U$  is a linear compact map from  $C[-\tau, 0]$  into  $C[-\tau, 0]$ .*

PROOF.  $U$  is additive (i.e.,  $U(ax + y) = aU(x) + U(y)$ ) because  $x' = L(t, x_t)$  is a linear equation.  $U$  is bounded (and hence continuous) by the fundamental inequality for solutions of functional differential equations [13, p. 338] referred to earlier. (To show that the inequality holds for solutions of  $x' = L(t, x_t)$ , it is only necessary to show that  $L(t, y)$  satisfies a Lipschitz condition in  $y$  which is uniform in  $t$ . But this follows at once from condition (2) on the matrix  $\eta(t, s)$  which is used to define  $L(t, y)$ .)

The fact that  $U$  is a compact mapping is indicated by Halanay [13, pp. 402-403]. The details of the proof can be filled in with the use of Lemma 3.

LEMMA 6. *The map*

$$\mathfrak{D} : (\phi(s), \mu) \rightarrow \int_0^{T+s} \mu f[\alpha, x_\alpha(\sigma, \phi, \mu), \mu] X(T + s, \alpha) d\alpha,$$

is a continuous map from  $C[-\tau, 0] \times [0, 1]$  into  $C[-\tau, 0]$  and is continuous in  $\mu$  uniformly on bounded sets in  $C[-\tau, 0]$ . Also for each fixed  $\mu \in [0, 1]$ ,  $\mathfrak{D}$  is a compact map of  $C[-\tau, 0]$  into  $C[-\tau, 0]$ .

PROOF. To prove the first statement we will show that given  $\epsilon > 0$  and  $\phi_2 \in C[-\tau, 0]$ , there exist  $\delta_1 > 0$ ,  $\delta_2 > 0$  such that if  $\|\phi_1 - \phi_2\| < \delta_1$  and  $|\mu_1 - \mu_2| < \delta_2$ , then

$$\max_{s \in [-\tau, 0]} \left\| \int_0^{T+s} \{ \mu_1 f[\alpha, x_\alpha(\sigma, \phi_1, \mu_1), \mu_1] - \mu_2 f[\alpha, x_\alpha(\sigma, \phi_2, \mu_2), \mu_2] \} X(T + s, \alpha) d\alpha \right\| < \epsilon.$$

First

$$\begin{aligned} & \left\| \int_0^{T+s} \{ \mu_1 f[\alpha, x_\alpha(\sigma, \phi_1, \mu_1), \mu_1] \right. \\ & \qquad \qquad \qquad \left. - \mu_2 f[\alpha, x_\alpha(\sigma, \phi_2, \mu_2), \mu_2] \} X(T + s, \alpha) d\alpha \right\| \\ & \leq \left\| \int_0^{T+s} \{ \mu_1 f[\alpha, x_\alpha(\sigma, \phi_1, \mu_1), \mu_1] \right. \\ & \qquad \qquad \qquad \left. - \mu_2 f[\alpha, x_\alpha(\sigma, \phi_1, \mu_1), \mu_1] \} X(T + s, \alpha) d\alpha \right\| \\ & + \left\| \int_0^{T+s} \{ \mu_2 f[\alpha, x_\alpha(\sigma, \phi_1, \mu_1), \mu_1] \right. \\ & \qquad \qquad \qquad \left. - \mu_2 f[\alpha, x_\alpha(\sigma, \phi_2, \mu_2), \mu_1] \} X(T + s, \alpha) d\alpha \right\| \\ & + \left\| \int_0^{T+s} \{ \mu_2 f[\alpha, x_\alpha(\sigma, \phi_2, \mu_2), \mu_1] \right. \\ & \qquad \qquad \qquad \left. - \mu_2 f[\alpha, x_\alpha(\sigma, \phi_2, \mu_2), \mu_2] \} X(T + s, \alpha) d\alpha \right\|. \end{aligned}$$

But

$$(4.7) \quad \begin{aligned} & \left\| \int_0^{T+s} \{ \mu_1 f[\alpha, x_\alpha(\sigma, \phi_1, \mu_1), \mu_1] \right. \\ & \quad \left. - \mu_2 f[\alpha, x_\alpha(\sigma, \phi_1, \mu_1), \mu_1] \} X(T+s, \alpha) d\alpha \right\| \\ & \leq |\mu_1 - \mu_2| M_1 \int_0^T \|f[\alpha, x_\alpha(\sigma, \phi_1, \mu_1), \mu_1]\| \alpha. \end{aligned}$$

By the definition of  $x_\alpha$  and the fundamental inequality, we have

$$\begin{aligned} \|x_\alpha(\sigma, \phi_1, \mu_1)\| & \leq \|x(t, \phi_1, \mu_1)\| \\ & \leq \|x(t, \phi_1, \mu_1) - x(t, \phi_2, \mu_1)\| \\ & \quad + \|x(t, \phi_2, \mu_1) - x(t, \phi_2, \mu_2)\| \\ & \quad + \|x(t, \phi_2, \mu_2)\| \\ & \leq B_3 \|\phi_1 - \phi_2\| + \epsilon_1 + B_4, \end{aligned}$$

where  $\|x_\alpha(\sigma, \phi_1, \mu_1)\|$  means the norm in  $C[-\tau, 0]$  and  $\|x(t, \phi_1, \mu_1)\|$  means the norm in  $C[-\tau, T + \epsilon]$  and where  $B_3$  is independent of  $\mu_1$  and  $B_4$  is independent of  $\mu_2$  and where  $\epsilon_1$  is independent of  $\mu_1$  and  $\mu_2$ , i.e., we only require that  $|\mu_1 - \mu_2|$  be small enough.

Thus, if  $\phi_1$  ranges over any bounded set in  $C[-\tau, 0]$  and  $\mu_1$  ranges over any set in  $[0, 1]$  such that  $|\mu_1 - \mu_2|$  is sufficiently small then the set of numbers  $\{\|x_\alpha(\sigma, \phi_1, \mu_1)\|\}$  is bounded. Hence, by condition (6) on  $f$ , the set of numbers  $\{\|f[\alpha, x_\alpha(\sigma, \phi_1, \mu_1), \mu_1]\|\}$  is bounded. Thus, from inequality (4.7), we obtain: if  $s \in [-\tau, 0]$

$$\begin{aligned} & \left\| \int_0^{T+s} \{ \mu_1 f[\alpha, x_\alpha(\sigma, \phi_1, \mu_1), \mu_1] \right. \\ & \quad \left. - \mu_2 f[\alpha, x_\alpha(\sigma, \phi_1, \mu_1), \mu_1] \} X(T+s, \alpha) d\alpha \right\| \\ & \leq |\mu_1 - \mu_2| B_5, \end{aligned}$$

where  $B_5$  is a positive constant.

Next for all  $s \in [-\tau, 0]$

$$\begin{aligned} & \left\| \int_0^{T+s} \mu_2 \{ f[\alpha, x_\alpha(\sigma, \phi_1, \mu_1), \mu_1] \right. \\ & \quad \left. - f[\sigma, x_\alpha(\sigma, \phi_2, \mu_2), \mu_1] \} X(T+s, \alpha) d\alpha \right\| \\ & \leq |\mu_2| M_1 \int_0^T \|f[\alpha, x_\alpha(\sigma, \phi_1, \mu_1), \mu_1] \\ & \quad - f[\alpha, x_\alpha(\sigma, \phi_2, \mu_2), \mu_1]\| d\alpha \\ & \leq |\mu_2| M_1 T L \max_{0 \leq \alpha \leq T} \|x_\alpha(\sigma, \phi_1, \mu_1) - x_\alpha(\sigma, \phi_2, \mu_2)\| \\ & \leq |\mu_2| M_1 T L \|x(t, \phi_1, \mu_1) - x(t, \phi_2, \mu_2)\| \\ & < |\mu_2| M_1 T L \epsilon, \end{aligned}$$

by Lemma 2 if  $\|\phi_1 - \phi_2\|$  and  $|\mu_1 - \mu_2|$  are sufficiently small. (Note that while  $\phi_2$  is fixed, neither  $\mu_1$  nor  $\mu_2$  need be fixed because, as follows from the proof of Lemma 2,  $x(t, \phi, \mu)$  is uniformly continuous in  $\mu$ .)

Finally for all  $s \in [-\tau, 0]$

$$\begin{aligned} & \left\| \int_0^{T+s} \{ \mu_2 f[\alpha, x_\alpha(\sigma, \phi_2, \mu_2), \mu_1] \right. \\ & \quad \left. - \mu_2 f[\alpha, x_\alpha(\sigma, \phi_2, \mu_2), \mu_2] \} X(T + s, \alpha) d\alpha \right\| \\ & \leq |\mu_2| M_1 \int_0^T \| f[\alpha, x_\alpha(\sigma, \phi_2, \mu_2), \mu_1] \\ & \quad - f[\alpha, x_\alpha(\sigma, \phi_2, \mu_2), \mu_2] \| d\alpha \\ & < |\mu_2| M_1 T \epsilon, \end{aligned}$$

if  $|\mu_1 - \mu_2|$  is sufficiently small by property (9) of function  $f$ .

The proof that  $\mathfrak{S}$  is continuous in  $\mu$  uniformly on bounded sets in  $C[-\tau, 0]$  follows by the same steps as in the preceding argument except that  $\phi_1 = \phi_2$ .

The mapping  $\mathfrak{S}$  is the composition of the maps

$$\mathcal{S} : (\phi, \mu) \rightarrow x(t, \phi, \mu), \quad (t \in [-\tau, T + \epsilon])$$

and

$$\mathcal{G} : x(t, \phi, \mu) \rightarrow \int_0^{T+s} \mu f[\alpha, x_\alpha(\sigma, \phi, \mu), \mu] X(T + s, \alpha) d\alpha.$$

By Lemma 2, the map  $\mathcal{S}$  is continuous and for fixed  $\mu$ ,  $\mathcal{S}$  takes bounded sets in  $C[-\tau, 0]$  into bounded sets in  $C[-\tau, T + \epsilon]$  by the fundamental inequality for solutions [13, p. 338]. Hence, to prove the last statement of Lemma 6, it is sufficient to show that for fixed  $\mu$ , the map  $\mathcal{G}$  is a compact map from  $C[-\tau, T + \epsilon]$  into  $C[-\tau, 0]$ . We must show:

- (i)  $\mathcal{G}$  takes  $C[-\tau, T + \epsilon]$  into  $C[-\tau, 0]$ ;
- (ii)  $\mathcal{G}$  is continuous;
- (iii)  $\mathcal{G}$  takes bounded sets into compact sets.

Proof of (i): Let  $x(t) \in C[-\tau, T + \epsilon]$ . Then

$$(\mathcal{G}x)(s) = \int_0^{T+s} \mu f[\alpha, x_\alpha(\sigma), \mu] X(T + s, \alpha) d\alpha.$$

Assuming for definiteness, that  $s_1 < s_2$

$$\begin{aligned}
& \|(\mathcal{G}x)(s_1) - (\mathcal{G}x)(s_2)\| \\
&= \left\| \int_0^{T+s_1} \mu f[\alpha, x_\alpha(\sigma), \mu] X(T+s_1, \alpha) d\alpha \right. \\
(4.8) \quad & \left. - \int_0^{T+s_2} \mu f[\alpha, x_\alpha(\sigma), \mu] X(T+s_2, \alpha) d\alpha \right\|, \\
&\leq \left\| \int_0^{T+s_1} \{ \mu f[\alpha, x_\alpha(\sigma), \mu] \} \{ X(T+s_2, \alpha) \} d\alpha \right\| \\
&+ \left\| \int_{T+s_1}^{T+s_2} \mu f[\alpha, x_\alpha(\sigma), \mu] X(T+s_2, \alpha) d\alpha \right\|.
\end{aligned}$$

Since  $\|x_\alpha(\sigma)\| \leq \|x(t)\|$ , where the first norm is for  $C[-\tau, 0]$  and the second norm is for  $C[-\tau, T + \epsilon]$ , then  $\{x_\alpha(\sigma) \mid 0 \leq \alpha \leq T\}$  is a bounded set in  $C[-\tau, 0]$  and thus by property (6) of  $f$ , there exists a positive constant  $B_6$  such that

$$\text{lub}_{0 \leq \alpha \leq T} \|f[\alpha, x_\alpha(\sigma), \mu]\| < B_6.$$

Hence, (4.8) becomes, by applying Lemmas 3 and 4,

$$\|(\mathcal{G}x)(s_1) - (\mathcal{G}x)(s_2)\| \leq TB_6\epsilon + B_6M_1|s_1 - s_2|,$$

if  $|s_1 - s_2| < \delta$ .

Proof of (ii): Let  $x_1(t), x_2(t) \in C[-\tau, T + \epsilon]$ . Then

$$\begin{aligned}
\|\mathcal{G}x_1 - \mathcal{G}x_2\| &= \max_{-\tau \leq s \leq 0} \left\| \int_0^{T+s} \mu \{ f[\alpha, x_{1\alpha}(\sigma), \mu] \right. \\
&\quad \left. - f[\alpha, x_{2\alpha}(\sigma), \mu] \} X(T+s, \alpha) d\alpha \right\|,
\end{aligned}$$

(where  $x_{1\alpha}(\sigma)$  denotes  $(x_1)_\alpha(\sigma)$ ),

$$\begin{aligned}
&\leq TM_1 \max_{0 \leq \alpha \leq T} \|f[\alpha, x_{1\alpha}(\sigma), \mu] \\
&\quad - f[\alpha, x_{2\alpha}(\sigma), \mu]\| \\
&\leq TM_1 L \max_{0 \leq \alpha \leq T} \|x_{1\alpha} - x_{2\alpha}\|,
\end{aligned}$$

by property (8) of function  $f$ . But

$$\max_{0 \leq \alpha \leq T} \|x_{1\alpha} - x_{2\alpha}\| \leq \|x_1 - x_2\|.$$

Proof of (iii): Let  $\{x_i(t)\}$  be a bounded set in  $C[-\tau, T + \epsilon]$ , say with bound  $B_7$ . We prove that  $\{\mathcal{G}x_i\}$  is a bounded equicontinuous set in  $C[-\tau, 0]$ . The compactness then follows by Ascoli's Theorem.



$$\begin{aligned} \|(\mathcal{G}x_\nu)(s)\| &= \max\left\|\int_0^{T+s} \mu f[\alpha, x_{\nu\alpha}(\sigma), \mu] X(T+s, \alpha) d\alpha\right\| \\ &\leq M_1 T \max_{0 \leq \alpha \leq T} \|f[\alpha, x_{\nu\alpha}(\sigma), \mu]\|. \end{aligned}$$

Since

$$\|x_{\nu\alpha}(\sigma)\| \leq \|x_\nu\| \leq B_7,$$

by property (6) of function  $f$ , it follows that  $\{\|\mathcal{G}x_\nu(s)\|\}$  is a bounded set.

The proof of the equicontinuity follows from an examination of the proof of (i).

This completes the proof of Lemma 6.

**LEMMA 7.** *There exist positive constants  $R_3, A, B, C$  and  $\epsilon \in (0, 1)$  such that if  $\|\phi\| \geq R_3$ , then for all  $\mu \in [0, 1]$ ,*

$$\|\mathcal{V}(\phi, \mu)\| < A(B + C\|\phi\|)^{1-\epsilon}.$$

**PROOF.** For all  $s \in [-\tau, 0]$  and all  $\mu \in [0, 1]$

$$(4.9) \quad \begin{aligned} &\left\|\int_0^{T+s} \mu f[\alpha, x_\alpha(\sigma, \phi, \mu), \mu] X(T+s, \alpha) d\alpha\right\| \\ &\leq M_1 T \max_{0 \leq \alpha \leq T} \|f[\alpha, x_\alpha(\sigma, \phi, \mu), \mu]\|. \end{aligned}$$

Now if  $\|\phi\| \geq R_2$  (the number which occurs in the description of property (7) of function  $f$ ) and if  $\|x_\alpha(\sigma, \phi, \mu)\| \geq R_2$  for some  $\alpha \in [0, T]$ , then by property (7) of  $f$

$$(4.10) \quad \max_{0 \leq \alpha \leq T} \|f[\alpha, x_\alpha(\sigma, \phi, \mu), \mu]\| \leq N \left(\max_{0 \leq \alpha \leq T} \|x_\alpha(\sigma, \phi, \mu)\|\right)^{1-\epsilon}.$$

But

$$(4.11) \quad \begin{aligned} \|x_\alpha(\sigma, \phi, \mu)\| &\leq \|x(\sigma, \phi, \mu)\| \\ &\leq \|x(\sigma, 0, \mu)\| + \|x(\sigma, \phi, \mu) - x(\sigma, 0, \mu)\|, \end{aligned}$$

and  $\|x(\sigma, 0, \mu)\| \leq B_8$  where  $B_8$  is a positive constant and  $\mu \in [0, 1]$  because  $x(\sigma, 0, \mu)$  is continuous in  $(\sigma, \mu)$ . By the fundamental inequality for  $\mu \in [0, 1]$

$$(4.12) \quad \|x(\sigma, \phi, \mu) - x(\sigma, 0, \mu)\| \leq L e^{T+\epsilon} \|\phi\|.$$

Thus, (4.10), (4.11) and (4.12) yield: if  $\|\phi\| \geq R_2$  and if  $\|x_\alpha(\sigma, \phi, \mu)\| \geq R_2$  for some  $\alpha \in [0, T]$ , then

$$(4.13) \quad \max_{0 \leq \alpha \leq T} \|f[\alpha, x_\alpha(\sigma, \phi, \mu), \mu]\| \leq N(B_8 + L e^{T+\epsilon} \|\phi\|)^{1-\epsilon},$$

and from (4.9) we have the desired result. Now suppose  $\|\phi\| \cong R_2$  and  $\|x_\alpha(\sigma, \phi, \mu)\| < R_2$  for all  $\alpha \in [0, T]$ . Then

$$\begin{aligned} \|f[\alpha, x_\alpha(\sigma, \phi, \mu), \mu]\| &\leq M \leq N(R)^{1-\epsilon} \leq N_1(\|\phi\|)^{1-\epsilon} \\ &< N_1(B_8 + Le^T\|\phi\|)^{1-\epsilon}. \end{aligned}$$

Thus, we have (4.13) and hence the desired result if  $\|\phi\| \cong R_2$ .

Thus, the hypotheses of Theorem 1 are satisfied by equation (4.6) and applying Theorem 1, we obtain

**THEOREM 3.** *For each  $\mu \in [0, 1]$ , equation (4.1) has a solution of period  $T$ .*

**COROLLARY 3.** *If for  $\mu = \mu_0$ , there exists  $s \in [-\tau, 0]$  such that*

$$\int_0^{T+s} f[\alpha, 0, \mu_0] X(T + s, \alpha) d\alpha \neq 0,$$

*then the periodic solution obtained in Theorem 3 is nontrivial.*

**PROOF.** If the condition in the corollary holds, then by the variation of constants formula (Halanay [13, p. 366, equation (23')]), the solution  $x(t, \phi, \mu_0)$  cannot be identically zero.

**REMARK.** If in equation (4.1) it is assumed that  $L$  and  $f$  are independent of  $t$ , then a result for (4.1) strictly analogous to Corollary 2a follows.

**5. Application to the Dirichlet problem for nonlinear elliptic equations.** Let  $\alpha$  be a fixed number such that  $\alpha \in (0, 1)$  and let  $\mathcal{D}$  be a bounded connected open set in the  $xy$ -plane and suppose that its boundary  $\mathcal{D}'$  is in  $C'_{2+\alpha}$  (For definitions of this term and other terms concerning Hölder spaces which will be used subsequently in this discussion, see [8, p. 157ff.]) We study the Dirichlet problem in  $\mathcal{D}$  for the nonlinear elliptic equation

$$(5.1) \quad \begin{aligned} a(x, y)z_{xx} + b(x, y)z_{xy} + c(x, y)z_{yy} + d(x, y)z_x \\ + e(x, y)z_y + f(x, y)z = G(z), \end{aligned}$$

where:

- (i)  $a, b, c, d, e, f \in C_\alpha(\bar{\mathcal{D}})$ ;
- (ii) there is a positive constant  $m$  such that for all real  $\xi, \eta$  and all  $(x, y) \in \mathcal{D}$

$$a\xi^2 + b\xi\eta + c\eta^2 \geq m(\xi^2 + \eta^2);$$

- (iii)  $f(x, y) \geq 0$  for all  $(x, y) \in \mathcal{D}$ ;

- (iv) For  $z(x, y) \in C_\alpha(\bar{\mathcal{D}})$ , the map

$$G : z \rightarrow G(z)$$

is a continuous map of  $C_\alpha(\bar{\mathcal{D}})$  into itself and there exist positive constants  $R_3, N_2$  and  $\epsilon \in (0, 1)$  such that if  $\|z\|_\alpha \geq R_3$ , then

$$\|G(z)\|_\alpha < N_2 \|z\|_\alpha^{1-\epsilon}.$$

Also  $G$  takes bounded sets in  $C_\alpha(\bar{\mathcal{D}})$  into bounded sets in  $C_\alpha(\bar{\mathcal{D}})$  and  $N_2$  is selected large enough so that  $M < N_2 R^{1-\epsilon}$  where  $M = \text{lub}_{\|z\|_\alpha \leq R} \|G(z)\|_\alpha$ .

Let  $\zeta(x, y) \in C_\alpha(\bar{\mathcal{D}})$  and consider the equation

$$(5.2) \quad \begin{aligned} a(x, y)z_{xx} + b(x, y)z_{xy} + c(x, y)z_{yy} + d(x, y)z_x \\ + e(x, y)z_y + f(x, y)z = \zeta(x, y). \end{aligned}$$

We apply the

*Schauder Existence Theorem.* Given  $\phi(s) \in C_{2+\alpha}(\mathcal{D}')$ , then (5.2) has a unique solution  $z = w(\zeta, \phi)$  where  $z \in C_{2+\alpha}(\bar{\mathcal{D}})$  and  $z|_{\mathcal{D}'} = \phi$ . Also there exists a positive number  $k$  which depends only on the  $C_\alpha(\bar{\mathcal{D}})$  norms of  $a, b, c, d, e, f$ , the constant  $m$ , and  $\bar{\mathcal{D}}$  such that

$$(5.3) \quad \|w(\zeta, \phi)\|_{2+\alpha} \leq k\{\|\phi\|_{2+\alpha} + \|\zeta\|_\alpha\}.$$

(For later computation, select  $k > 1$ .) For references to the proof of the Schauder Existence Theorem, see [8, p. 162].

Now let  $\phi \in C_{2+\alpha}(\mathcal{D}')$  be fixed. Using (5.2) and the Schauder Existence Theorem, we may rewrite (5.1) as  $\zeta = G[w(\zeta, \phi)]$ . Inequality (5.3) shows that  $w(\zeta, \phi)$  is a compact map of  $C_\alpha(\bar{\mathcal{D}})$  into itself. Since  $G$  is a continuous map of  $C_\alpha(\bar{\mathcal{D}})$  into itself, then  $G[w(\zeta, \phi)]$  is a compact map of  $C_\alpha(\bar{\mathcal{D}})$  into  $C_\alpha(\bar{\mathcal{D}})$ . Also if  $\|\zeta\|_\alpha \geq R_3$  and  $\|w(\zeta, \phi)\|_\alpha \geq R_3$ , then

$$\begin{aligned} \|G[w(\zeta, \phi)]\|_\alpha &< N_2(\|w(\zeta, \phi)\|_\alpha)^{1-\epsilon} \\ &\leq N_2[k\|\phi\|_{2+\alpha} + \|\zeta\|_\alpha]^{1-\epsilon} \\ &= N_2(k\|\phi\|_{2+\alpha} + k\|\zeta\|_\alpha)^{1-\epsilon}. \end{aligned}$$

If  $\|w(\zeta, \phi)\|_\alpha < R_3$ , then

$$\|G[w(\zeta, \phi)]\|_\alpha \leq M < N_2 R^{1-\epsilon} \leq N_2 \|\zeta\|_\alpha^{1-\epsilon} < N_2(k\|\zeta\|_\alpha)^{1-\epsilon}.$$

Thus, the hypotheses of Theorem 1 are satisfied for the functional equation:

$$(5.4) \quad \zeta + \mu G[w(\zeta, \phi)] = 0, \quad \mu \in [0, 1].$$

Applying Theorem 1, we may solve (5.4), and hence (5.1), and obtain:

**THEOREM 4.** *If  $\phi \in C_{2+\alpha}(\mathcal{D}')$  is given, then for each  $\mu \in [0, 1]$ , the equation*

$$\begin{aligned} a(x, y)z_{xx} + b(x, y)z_{xy} + c(x, y)z_{yy} \\ + d(x, y)z_x + e(x, y)z_y + f(x, y)z = \mu G(z) \end{aligned}$$

has at least one solution  $z \in C_{2+\alpha}(\bar{\mathcal{D}})$  such that  $z|_{\mathcal{D}'} = \phi$ .

(Theorem 4 also holds for elliptic equations in  $n$ -space. The proof goes through in exactly the same way except that it is necessary to use a more general version of the Schauder Existence Theorem, i.e., the version for elliptic equations in  $n$ -space).

**6. Application to the Dirichlet problem for nonlinear parabolic equations.** Let  $D$  be a bounded connected open set in the  $xy$ -plane and let  $\mathcal{D} = D \times (0, T)$  where  $T$  is a positive constant. Let  $p = (x_1, y_1, t_1)$  and  $q = (x_2, y_2, t_2)$  be a pair of points in  $\mathcal{D}$  and define

$$\bar{d}(p, q) = ((x_1 - x_2)^2 + (y_1 - y_2)^2 + |t_1 - t_2|)^{1/2}.$$

If real-valued function  $f$  has domain  $\mathcal{D}$  and  $\alpha \in (0, 1)$ , let

$$H_\alpha(f) = \text{lub}_{p, q \in \mathcal{D}} |f(p) - f(q)| / (\bar{d}(p, q))^\alpha,$$

and let

$$\|f\|_\alpha^p = \text{lub}_{p \in \mathcal{D}} |f(p)| + H_\alpha(f).$$

Let

$$\mathcal{E} = (D \times [0]) \cup (D' \times [0, T]),$$

where  $D'$  is the point set boundary of  $D$ . By standard arguments, it follows that if  $\|f\|_\alpha^p < \infty$  then  $f$  has a continuous extension on  $\bar{\mathcal{D}}$  where  $\bar{\mathcal{D}} = \mathcal{D} \cup \mathcal{E}$ , and if  $\|f\|_\alpha^p < \infty$ , we shall understand  $f$  to be this extension. If  $f$  has second derivatives, define

$$\begin{aligned} \|f\|_{2+\alpha}^p &= \|f\|_\alpha^p + \|f_x\|_{(1+\alpha)/2}^p + \|f_y\|_{(1+\alpha)/2}^p \\ &\quad + \|f_t\|_\alpha^p + \|f_{xx}\|_\alpha^p + \|f_{xy}\|_\alpha^p + \|f_{yy}\|_\alpha^p. \end{aligned}$$

Let  $\phi$  be a real-valued function whose domain is the set  $\mathcal{E}$  and define  $\|\phi\|_{2+\alpha}^p$  analogously.

We study the Dirichlet problem for the nonlinear parabolic equation:

$$(6.1) \quad \begin{aligned} a_{11}(x, y, t)u_{xx} + a_{12}(x, y, t)u_{xy} + a_{22}(x, y, t)u_{yy} - u_t + b_1(x, y, t)u_x \\ + b_2(x, y, t)u_y + cu = G(u), \end{aligned}$$

where there exist constants  $\alpha$  and  $\epsilon$  such that  $0 < \alpha < \alpha + \epsilon < 1$  and the set  $\mathcal{D}$  is of class  $B_{2+\alpha+\epsilon}$ , i.e., the set  $D$  is such that for each point  $(x, y) \in D'$  there is a neighborhood  $\mathcal{M}$  of  $(x, y)$  and a neighborhood  $\mathcal{N}$  of the origin in another plane (whose points have coordinates  $s_1, s_2$ ) and a 1-1 mapping from  $\mathcal{N}$  onto  $\mathcal{M}$  described by a pair of real-valued functions  $x(s_1, s_2)$  and  $y(s_1, s_2)$  such that  $x(s_1, s_2), y(s_1, s_2) \in C_{2+\alpha}(\bar{\mathcal{N}})$  where  $C_{2+\alpha}(\bar{\mathcal{N}})$  is the Banach space of real-valued functions with  $\alpha$ -Hölder continuous second derivatives on  $N$ . It is also assumed that there is positive number  $M$  such that the numbers  $\|a_{ij}\|_{\alpha+\epsilon}^p$  ( $i, j = 1, 2$ ),  $\|b_i\|_{\alpha}^p$  ( $i = 1, 2$ ) and  $\|c\|_{\alpha}^p$  are all less than  $M$  and that the matrix  $(\alpha_{ij}(x, y, t))$ , where  $\alpha_{12} = \alpha_{21} = \frac{1}{2}a_{12}$  and  $\alpha_{11} = a_{11}, \alpha_{22} = a_{22}$ , is such that its quadratic form is positive definite on  $\mathcal{D}$  (which implies there is a positive constant  $N$  such that  $\det(\alpha_{ij}(x, y, t)) \geq N$ , for all  $(x, y, t) \in \mathcal{D}$ ).

Finally, we assume that  $G(u)$  satisfies the following conditions:

(1) If  $\bar{C}_{\alpha}^0(\bar{\mathcal{D}})$  is the linear normed space of functions  $f(x, y, t)$  such that  $\|f\|_{\alpha}^p < \infty$  and such that  $f(D' \times [0]) \cup (D' \times [T])$  is identically zero, then if  $z \in \bar{C}_{\alpha}^0(\bar{\mathcal{D}})$ , the mapping  $G: z \rightarrow G(z)$  is a continuous mapping from  $\bar{C}_{\alpha}^0(\bar{\mathcal{D}})$  into  $\bar{C}_{\alpha}^0(\bar{\mathcal{D}})$ .

(2) There exist positive constants  $R_4, N_3$  and  $\epsilon_1 \in (0, 1)$  such that if  $f \in \bar{C}_{\alpha}^0(\bar{\mathcal{D}})$  and if  $\|f\|_{\alpha}^p \geq R_4$ , then  $\|G(f)\|_{\alpha}^p < N_3(\|f\|_{\alpha}^p)^{1-\epsilon_1}$ . Also  $G$  takes bounded sets in  $\bar{C}_{\alpha}^0(\bar{\mathcal{D}})$  into bounded sets in  $\bar{C}_{\alpha}^0(\bar{\mathcal{D}})$  and  $N_3$  is selected large enough so that  $M_0 < N_3(R_4)^{1-\epsilon_1}$  where  $M_0 = \text{lub} \|G(f)\|_{\alpha}^p$ , where  $f \in \bar{C}_{\alpha}^0(\bar{\mathcal{D}})$  and  $\|f\|_{\alpha}^p \leq R_4$ .

Now let  $\zeta(x, y, t) \in \bar{C}_{\alpha}^0(\bar{\mathcal{D}})$ , the linear normed space of functions  $f(x, y, t)$  such that  $\|f\|_{\alpha} < \infty$ , and consider the equation

$$(6.2) \quad a_{11}u_{xx} + a_{12}u_{xy} + a_{22}u_{yy} - u_t + b_1u_x + b_2u_y + cu = \zeta.$$

We apply:

*Barrar-Friedman Existence Theorem.* Suppose the function  $\phi$  with domain  $\mathcal{E}'$  such that  $\|\phi\|_{2+\alpha}^p < \infty$  is given and suppose  $\phi$  is compatible with (6.2), i.e.,  $\phi$  satisfies (6.2) on  $(D' \times [0]) \cup (D' \times [T])$ . Then (6.2) has a unique solution  $w(\zeta, \phi)$  and

$$(6.3) \quad \|w(\zeta, \phi)\|_{2+\alpha}^p < C(M, N, \bar{\mathcal{D}}) (\|\zeta\|_{\alpha}^p + \|\phi\|_{2+\alpha}^p)$$

where  $C(M, N, \bar{\mathcal{D}})$  is a positive constant which depends only on  $M, N$ , and  $\bar{\mathcal{D}}$ . (For later computation, select  $C(M, N, \bar{\mathcal{D}}) > 1$ ).

For proof of the Barrar-Friedman Existence Theorem, see Barrar [4, 5] and Friedman [11].

Let  $\phi$  be identically zero and let  $\zeta \in \bar{C}_\alpha^0(\bar{\mathcal{D}})$ . Using (6.2) and the Barrar-Friedman Existence Theorem, we may rewrite (6.1) as  $\zeta = G[w(\zeta, 0)]$ . From inequality (6.3), it follows by standard arguments that  $w(\zeta, \phi)$ , where  $\phi$  is fixed, is a compact map from  $\bar{C}_\alpha(\bar{\mathcal{D}})$  into  $\bar{C}_\alpha(\bar{\mathcal{D}})$  and hence by condition (1) on  $G$ , it follows that  $G[w(\zeta, 0)]$  is a compact mapping from  $\bar{C}_\alpha^0(\bar{\mathcal{D}})$  into itself. Also if  $\|\zeta\|_\alpha^p \cong R_4$  and  $\|w(\zeta, \phi)\|_\alpha^p \cong R_4$ , then

$$\begin{aligned} \|G[w(\zeta, 0)]\|_\alpha^p &< N_3(\|w(\zeta, 0)\|_\alpha^p)^{1-\epsilon_1} \\ &\cong N_3[C(M, N, \bar{\mathcal{D}})]^{1-\epsilon_1} [\|\zeta\|_\alpha^p]^{1-\epsilon_1}. \end{aligned}$$

If  $\|w(\zeta, \phi)\|_\alpha^p < R_4$ , then

$$\begin{aligned} \|G[w(\zeta, \phi)]\|_\alpha^p &\cong M_0 < N_3(R_4)^{1-\epsilon_1} \cong N_3[\|\zeta\|_\alpha^p]^{1-\epsilon_1} \\ &< N_3[C(M, N, \bar{\mathcal{D}})] [\|\zeta\|_\alpha^p]^{1-\epsilon_1}. \end{aligned}$$

Thus, the hypotheses of Theorem 1 are satisfied for the functional equation

$$\zeta + \mu G[w(\zeta, 0)] = 0,$$

and we obtain:

**THEOREM 5.** *For each  $\mu \in [0, 1]$  the equation (6.1) with  $G(u)$  replaced by  $\mu G(u)$  has at least one solution  $u \in \bar{C}_{2+\alpha}(\bar{\mathcal{D}})$  such that  $u|_{\mathcal{E}}$  is identically zero.*

**REMARKS.** 1. Theorem 5 also holds in the  $n$ -dimensional case and for more general regions because the Barrar-Friedman Theorem holds in these more general cases.

2. Note that the arguments above cannot be employed if  $\phi$  is not identically zero because if  $\phi$ , not identically zero, is given, then the set of functions  $\zeta$  such that  $\phi$  is compatible with (6.2) will not in general be a linear space and so the domain of  $w(\zeta, \phi)$  and  $G[w(\zeta, \phi)]$  is not a linear space. So there is no possibility of applying Theorem 1.

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