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RADICAL AND ANTIRADICAL GROUPS

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1. **Preliminaries.** To gain a better understanding of radical rings, it is important to ask which abelian groups are the additive groups of proper radical rings, that is, of rings $\mathfrak{S}$ that are not zero rings (i.e., for some s_1, $s_2 \in \mathfrak{S}$, $s_1 s_2 \neq 0$) where $\mathfrak{S} = J(\mathfrak{S})$, the Jacobson radical of $\mathfrak{S}$. A related question asks which subgroups B of an abelian group A support radicals of rings on A (that is, $\mathfrak{A}^+ = A$ and $J(\mathfrak{A})^+ = B$). It is often more convenient to state these questions from within: (1) Given an abelian group A, what radical rings does it support? (2) Given a subgroup B of an abelian group A, in how many ways can A be turned into a ring $\mathfrak{A}$ in such a way that $J(\mathfrak{A})^+ = B$? We shall give answers to (1) for some types of groups and touch briefly upon (2).

A nontrivial abelian group A is called a *radical group* if it supports at least one proper radical ring; otherwise, it is called an *antiradical group*. All groups here will be abelian, and all rings are to be associative. Some of our results will be formulated in terms of a pair of maps on the group called a bimultiplier, a sort of pre-bimultiplication [**6**]. We shall show (Theorem 1) that if a radical group supports a radical ring as the kernel of a ring extension then it supports the kernel of a related radical ring extension. Necessary and sufficient conditions are found (Theorem 2) in terms of a locus in Euclidean n-space for a given bimultiplier on a torsion-free divisible group of rank n to produce a radical ring on that group. We prove (Theorem 3) that, for rings on torsion-free divisible groups, the radical-supporting subgroups are precisely the $\mathfrak{Q}$-submodules (where $\mathfrak{Q}$ is the ring of rationals). The torsion-free groups of rank 1 that are radical groups are completely classified in terms of type (Theorem 4). For $A \oplus A$ to be antiradical it is necessary and sufficient (Theorem 5) that A be a nil group. If A and B are antiradical, sufficient conditions in terms of Hom are found for $A \oplus B$ to be antiradical (Theorem 6). We identify (Theorem 7) the divisible antiradical groups. The prime-power cyclic groups are shown to support various radical rings, and any exponent of nilpotency can be realized (Theorem 8). The antiradical direct sums of

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cyclic groups are completely determined (Theorem 9) as are the bounded antiradical groups (Theorem 10). All the countable reduced p-groups not of prime order turn out to be radical groups (Theorem 11). We show (Theorem 12) that every countable reduced p-group of Ulm type 2 supports a proper radical ring that is the epimorphic image of a proper radical ring supported by an unbounded, countable reduced p-group of Ulm type 1, and that these roles can be reversed.

Although there seems to be no literature directly on this subject, K. Eldridge [**2**] has discussed the quasi-regular groups of the rings $\mathfrak{S}$, $J(\mathfrak{S})$, and $\mathfrak{S}/J(\mathfrak{S})$. In a private communication, C. Yohe has given a different proof for Lemma 2, and Dr. Eldridge has kindly called our attention to [**10**].

Notations, such as l.q.r. for left quasi regular and r.q.i. for right quasi inverse, are standard. In general, we follow [**5**], although most references are to [**4**]. The symbol ι is the identity map. I_n is the n-by-n identity matrix; $\mathfrak{R}^n$ stands for Euclidean n-space; $\mathfrak{R}$, for the real field; $Q = \mathfrak{Q}^+$, for the additive group of rationals; $\mathfrak{Z}$, for the ring of integers with additive group $\mathfrak{Z}^+ = Z$; $\mathfrak{Q}_n$, for the n-by-n matrices over $\mathfrak{Q}$; $Z(n)$, for the cyclic group of order n; $Z[a]$, for the cyclic group with generator a. If $a \in A$, a p-group, and if $|a| = p^n$ then $n = E(a)$ is called the exponent of a. If s is in a ring $\mathfrak{S}$ then s^* denotes the quasi inverse (q.i.) of s (if it exists). If $s_1, s_2 \in \mathfrak{S}$ then $s_1 \circ s_2 = s_1 + s_2 - s_1 s_2$. A proper ring is a ring in which some product xy fails to be zero. Rings that are not proper are called zero rings.

2. **Bimultipliers.** A pair of maps $\Gamma = (\Gamma_L, \Gamma_R)$, where both Γ_L and Γ_R lie in Hom (A, Hom (A, A)), is called a *bimultiplier* on a group A if

(i) $\Gamma_L(a_1)a_2 = \Gamma_R(a_2)a_1$, and

(ii) $\Gamma_L(a_1)\Gamma_R(a_2) = \Gamma_R(a_2)\Gamma_L(a_1)$

for all $a_1, a_2 \in A$. Each bimultiplier Γ on A allows us to construct a ring on A, (A, Γ), where $(A, \Gamma)^+ = A$ and multiplication is given by $\Gamma_L(a_1)a_2 = a_1 a_2$. Indeed, the familiar associativity condition, $\Gamma_L(\Gamma_L(a_1)a_2) = \Gamma_L(a_1)\Gamma_L(a_2)$ (or this identity with Γ_R replacing Γ_L), comes from (i) and (ii). Conversely, suppose that $\mathfrak{A}$ is a ring supported by A, and that Δ_L (Δ_R) is the function that carries each $a \in A$ onto the left (right) multiplication $a_L : b \mapsto ab$ ($a_R : b \mapsto ba$) for every $b \in A$. Then $\Delta = (\Delta_L, \Delta_R)$ is a bimultiplier on A such that $\mathfrak{A} = (A, \Delta)$. Observe that if Γ is a bimultiplier on A, then, for each $a \in A$, the pair of maps $(\Gamma_L(a), \Gamma_R(a))$ is an inner bimultiplication [**6**] on the ring (A, Γ).

If Γ is a bimultiplier on A then both Γ_L and Γ_R may be viewed as ring homomorphisms from (A, Γ) to the ring $\mathrm{Hom}(A, A)$ so that $\mathrm{Im}\,\Gamma_L$ and $\mathrm{Im}\,\Gamma_R$ are subrings of $\mathrm{Hom}(A, A)$.

LEMMA 1. (i) $\Gamma_L{}^{-1}J(\mathrm{Im}\,\Gamma_L) = J(A, \Gamma) = \Gamma_R{}^{-1}J(\mathrm{Im}\,\Gamma_R)$.
(ii) (A, Γ) *is a radical ring precisely if the elements of* $\mathrm{Im}\,\Gamma_L$ $(\mathrm{Im}\,\Gamma_R)$ *are all q.r. in the ring* $\mathrm{Hom}(A, A)$.

PROOF. (i) Each of the following statements is equivalent to its neighbors. (1) $x \in \Gamma_L{}^{-1}J(\mathrm{Im}\,\Gamma_L)$. (2) $\Gamma_L(y)\Gamma_L(x)$ is l.q.r. in $\mathrm{Im}\,\Gamma_L$ for every $y \in A$. (3) If $y \in A$ there exists $c \in A$ such that $c + yx - cyx \in \ker \Gamma_L$. (4) There exists $g \in \ker \Gamma_L$ such that $c - g + yx - (c - g)yx = 0$. (5) yx is l.q.r. in (A, Γ) for each $y \in A$. (6) $x \in J(A, \Gamma)$.
(ii) If (A, Γ) is a radical ring $\mathrm{Im}\,\Gamma_L \leqq J(\mathrm{Im}\,\Gamma_L)$ so that each $\Gamma_L(a)$ is q.r. in $\mathrm{Im}\,\Gamma_L$, hence in $\mathrm{Hom}(A, A)$, with $\Gamma_L(a)^* = \Gamma_L(a^*)$. Conversely, if each $\Gamma_L(a)$ is q.r. in $\mathrm{Hom}(A, A)$, $\iota - \Gamma_L(a) \in \mathrm{Aut}\, A$. Let $a^{\#} = -(\iota - \Gamma_L(a))^{-1}a$ for $a \in A$. Then $-(\iota - \Gamma_L(a))a^{\#} = a$ so that $\Gamma_L(a)a^{\#} = a + a^{\#}$, and $a^{\#}$ is a r.q.i. for a. Since each member of (A, Γ) is r.q.r., (A, Γ) must be a radical ring. ∎

LEMMA 2. *Z is an antiradical group.*

PROOF. If (Z, Γ) is a proper radical ring Lemma 1(ii) provides that each $\Gamma_L(n)$ is q.r. whence each $\iota - \Gamma_L(n) \in \mathrm{Aut}\, Z$. Since Γ is nontrivial, there exists $m \in Z$ such that $\Gamma_L(m)$ is nontrivial so that $\iota - \Gamma_L(m) = -\iota$, the only available nonunity automorphism on Z. Thus, $\iota - \Gamma_L(2m) = -3\iota \in \mathrm{Aut}\, Z$, an impossibility. ∎

LEMMA 3. *If A is a proper subgroup of Z, then Z supports no ring with radical supported by A.*

PROOF. Since, as a group, $A \cong Z$, the only possible radical on A would, by Lemma 2, be the zero ring. But the only possible bimultipliers on Z are those Γ with $\Gamma_L(x)y = xyk$ for fixed $k \in Z$. Such multiplications never reduce to the zero multiplication on any proper subgroup unless $k = 0$. In that case, the radical would be all of Z and not just A. ∎

Let $\mathfrak{F}$ be a division ring with the property that each nontrivial bimultiplier Γ on $\mathfrak{F}^+$ is so related to the multiplication on $\mathfrak{F}$ that $\Gamma(x)y = xyk$ for some nonzero $k \in \mathfrak{F}$ (depending only on Γ). Then $\mathfrak{F}^+$ is antiradical; for, if not, $\Gamma(k^{-1})k^{-1} = k^{-1}$, contradicting the exclusion of nonzero idempotents from proper radical rings. In particular, Q and $Z(p)$ are antiradical. Further, no proper subgroup of Q can be the radical of any ring supported by Q. For, the only nontrivial bimultipliers on Q are the Γ for which $\Gamma_L(x)y = xyk$, $k \neq 0$, and k^{-1} is thus the unity of (Q, Γ) so that Q supports only division rings, devoid of proper ideals. The referee notes that Lemmas 2 and 3 and the remarks just above are known.

3. **Extensions.**

THEOREM 1. *Let* $\mathfrak{B} \rightarrowtail \mathfrak{A} \overset{\lambda}{\twoheadrightarrow} \mathfrak{C}$ *be an exact sequence of rings where* $J(\mathfrak{B}) = \mathfrak{B}$. *Then* $J(\mathfrak{B}) \rightarrowtail J(\mathfrak{A}) \overset{\lambda \mid J(\mathfrak{A})}{\twoheadrightarrow} J(\mathfrak{C})$ *is also exact.*

PROOF. We may denote the elements of $\mathfrak{A}$ by ordered pairs (b, c) ($b \in \mathfrak{B}$ and $c \in \mathfrak{C}$). It is assumed that the left and right actions of $\mathfrak{C}$ on $\mathfrak{B}^+$ are known: $c_L(b) = cb$, and $c_R(b) = bc$. We also write $b_L(b') = bb'$ for all $b, b' \in \mathfrak{B}$. In $\mathfrak{A}$, addition is given by

$$(b_1, c_1) + (b_2, c_2) = (b_1 + b_2 + \sigma_1(c_1, c_2), c_1 + c_2)$$

for some normalized cocycle σ_1; and multiplication has the form

$$(b_1, c_1)(b_2, c_2) = (b_1 b_2 + c_1 b_2 + b_1 c_2 + \sigma_2(c_1, c_2), c_1 c_2)$$

for a normalized function σ_2 from $\mathfrak{C} \times \mathfrak{C}$ to $\mathfrak{B}$. (See [3], [6], [7], and [9] for precise conditions and details.)

It is clear that $(b, c) \in J(\mathfrak{A})$ implies that $c \in J(\mathfrak{C})$. Conversely, if $c \in J(\mathfrak{C})$ and if $(b, d) \in \mathfrak{A}$ then $(0, c)(b, d) = (cb + \sigma_2(c, d), cd)$. If only we could show that this last is r.q.r. in $\mathfrak{A}$ then $(0, c) \in J(\mathfrak{A})$ from which $J(\mathfrak{A}) = \{(x, c) \mid x \in \mathfrak{B} \text{ and } c \in J(\mathfrak{C})\}$, and the proof would be complete. We shall show a bit more, namely that each (e, g), where $e \in \mathfrak{B}$ and $g \in J(\mathfrak{C})$, has a r.q.i. in $\mathfrak{A}$.

First, one proves (using, say, [9, (6)–(15)]) that the operator $\iota - g_L$ on $\mathfrak{B}^+$ has the inverse

$$(\iota - [\sigma_1(g^*, g) - \sigma_2(g^*, g)]_L^*)(\iota - g_L^*).$$

Then one shows that, as operators on $\mathfrak{B}^+$, $(\iota - g_L^*)(\iota - e_L - g_L) = \iota - y_L$ where $y = e + g^*e + \sigma_2(g^*, g) - \sigma_1(g^*, g) \in \mathfrak{B}$. Thus, $(\iota - g_L)^{-1}(\iota - e_L - g_L) = \iota - x_L$ where $x = [\sigma_1(g^*, g) - \sigma_2(g^*, g)]^* \circ y \in \mathfrak{B}$; and $(\iota - e_L - g_L)^{-1} = (\iota - x_L^*)(\iota - g_L)^{-1}$. A short computation shows that a r.q.i. for (e, g) is (h, g^*) where $h = (\iota - e_L - g_L)^{-1}[\sigma_2(g, g^*) - \sigma_1(g, g^*) - (\iota - g_R^*)e] \in \mathfrak{B}$. ∎

COROLLARY 1. *A ring extension of a radical ring by a radical ring is a radical ring.*

PROOF. Use the notation of the theorem. To each $a \in \mathfrak{A}$ there exists $j \in J(\mathfrak{A})$ such that $\lambda(a) = \lambda(j)$ since both λ and $\lambda \mid J(\mathfrak{A})$ are onto $J(\mathfrak{C}) = \mathfrak{C}$. Hence $a - j \in \ker \lambda = \mathfrak{B} = J(\mathfrak{B}) \subseteq J(\mathfrak{A})$ so that $a \in J(\mathfrak{A})$. ∎

This result is well known.

COROLLARY 2. *Let* $B \rightarrowtail A \overset{\lambda}{\twoheadrightarrow} Z$ *be an exact sequence of groups that supports an exact sequence of rings* $\mathfrak{B} \rightarrowtail \mathfrak{A} \overset{\Lambda}{\twoheadrightarrow} \mathfrak{C}$ *where* $\mathfrak{B}^+ = B$, $\mathfrak{A}^+ = A$, $\mathfrak{C}^+ = Z$, *and the morphisms* λ *and* Λ *induce the same set map. Then if* $J(\mathfrak{B}) = \mathfrak{B}$, *either* $J(\mathfrak{A}) = \mathfrak{A}$ *or* $J(\mathfrak{A}) = \mathfrak{B}$. *If B supports a radical ring* $\mathfrak{B}$ *which has an element with a nonzero square, then A will support a radical ring extension* $\mathfrak{A}$ *of* $\mathfrak{B}$ *by* $\mathfrak{C}$ *in a nontrivial way.*

PROOF. By Lemma 3, $J(\mathfrak{C}) = 0$ or $\mathfrak{C}$. Since $J(\mathfrak{A})$ consists of all $(b, c) \in \mathfrak{A}$ where $b \in B$ and $c \in J(\mathfrak{C})$, the first statement of the corollary follows. As for the second, since A is an abelian extension of B by Z, $A = B \oplus Z$ [8, 9.5.5], and σ_1 is trivial. Introduce the zero ring $\mathfrak{C}$ on Z. Denote the members of $B \oplus Z$ by ordered pairs (b, n), $b \in B$ and $n \in Z$, and introduce multiplication via $(b_1, n_1)(b_2, n_2) = ((b_1 + n_1 z_0)(b_2 + n_2 z_0), 0)$ where $z_0 \in \mathfrak{B}$ with $z_0^2 \neq 0$. It is easy to check that we have a ring extension $\mathfrak{A}$ (where $\mathfrak{A}^+ = B \oplus C$) of the radical ring $\mathfrak{B}$ by the zero ring $\mathfrak{C}$. Since each product with factors in $\mathfrak{A}$ has the form $(b', 0)$, a radical element in $\mathfrak{A}$, each $(b, n) \in \mathfrak{A}$ has all its right multiples r.q.r. so that $(b, n) \in J(\mathfrak{A})$. (Observe that $(b, n)^* = (nz_0 + (b + nz_0)^*, -n)$.) Since $(0, 1)^2 = (z_0^2, 0) \neq (0, 0)$, the extension is not trivial. ∎

4. Torsion-free groups.

THEOREM 2. *Let A be a torsion-free divisible group of finite dimension n as a* $\mathfrak{Q}$*-module. Let* $\{(ijk)\}$, $1 \leqq i, j, k \leqq n$, *be a set of* n^3 *members of* $\mathfrak{Q}$ *(repetitions allowed) subject to the conditions*

(a)
$$\sum_{t=1}^{n} \begin{vmatrix} (ltk) & (tjl) \\ (itk) & (ijt) \end{vmatrix} = 0,$$

where $1 \leqq k, l, i, j \leqq n$ *(giving* n^4 *equations); and*

(b) *in* $\mathfrak{R}^n$ *the locus given by*

$$\det\left(-\delta_{ij} + \sum_{t=1}^{n} (ijt)x_t \right) = 0$$

has no rational points.

Then the map Γ_R *from* $A = \mathfrak{Q}^n$ *to* $\mathrm{Hom}(A, A) = \mathfrak{Q}_n$ *given by*

(c)
$$\Gamma_R(x_1, \cdots, x_n) = \left(\sum_{t=1}^{n} (ijt)x_t \right) \in \mathfrak{Q}_n$$

determines a bimultiplier $\Gamma = (\Gamma_L, \Gamma_R)$ *such that* (A, Γ) *is a radical ring with* Γ_R *given by* (c).

Conversely, suppose that (A, Γ) *is a radical ring with* Γ_R *given by* (c). *Then the* n^3 *coefficients* $(ijk) \in \mathfrak{Q}$ *obey* (a) *and* (b).

PROOF. Suppose that Γ is a bimultiplier for which (A, Γ) is a radical ring. Let $u_k = (0, \cdots, 0, 1, 0, \cdots, 0) \in A$ where the 1 is in the kth position. Associativity yields the equivalent special conditions $\Gamma_R(\Gamma_R(u_k)u_l) = \Gamma_R(u_k)\Gamma_R(u_l)$ for all integers k and l subject to $1 \leqq k,\ l \leqq n$. If $q \in \mathfrak{Q}$ and if $u \in \mathfrak{Q}^n = A$ then $\Gamma_R(qu) = q\Gamma_R(u)$ since Q is torsion-free divisible. If $\Gamma_R(u_k) = ((ijk)) \in \mathfrak{Q}_n$ then the special associativity conditions provide that $\Gamma_R((l1k), \cdots, (lnk)) = ((ijk))((ijl))$, from which

$$\text{(d)} \qquad \sum_{t=1}^{n} (ltk)(ijt) = \sum_{t=1}^{n} (itk)(tjl),$$

n^4 such equations since $1 \leqq i, j, k, l \leqq n$. A rewriting of (d) produces (a).

Since (A, Γ) is radical, $\Gamma_R(x)$ has to be q.r. in Hom(A, A), by Lemma 1(ii), for each $x = (x_1, \cdots, x_n) \in A$. But $\Gamma_R(x) = \sum_{t=1}^{n} x_t((ijt))$ so that $I_n - \sum_{t=1}^{n} x_t((ijt)) \in \operatorname{Aut} A$, whence $\det[I_n - \sum_{t=1}^{n} x_t((ijt))] \neq 0$ for all rational points, the $(x_1, \cdots, x_n) \in \mathfrak{Q}^n$. At once (b) follows. The converse is immediate. ∎

If $n > 2$, the process given by the theorem is not feasible for computing the radical rings on torsion-free divisible groups of rank n. If $n = 2$, a cumbersome check shows that the only multiplications turning $Q \oplus Q$ into a radical ring are those given by

$$\text{(e)} \qquad \begin{aligned} &(x_1, x_2)(y_1, y_2) \\ &= (bb'(x_1 + b'x_2)(y_1 + b'y_2), b(x_1 + b'x_2)(y_1 + b'y_2)) \\ &\qquad\qquad (x_1, x_2, y_1, y_2 \in \mathfrak{Q}) \end{aligned}$$

for fixed $b, b' \in \mathfrak{Q}$ (and similar cases arising from the exchange of components). Except for the zero ring, each radical ring on $Q \oplus Q$ has exponent of nilpotency 3.

THEOREM 3. *A subgroup* B *of a torsion-free divisible group* A *supports the radical of some ring on* A *if and only if* B *is a* $\mathfrak{Q}$-*submodule of the* $\mathfrak{Q}$-*module* A.

PROOF. Since A is a $\mathfrak{Q}$-module, each (A, Γ) is a $\mathfrak{Q}$-algebra so that $(ra)x = a(rx)$ for $a, x \in (A, \Gamma)$ and $r \in \mathfrak{Q}$. In particular, if $a \in J(A, \Gamma)^+$, $a(rx)$ is r.q.r. in (A, Γ); hence, so is $(ra)x$ for every $x \in (A, \Gamma)$. That is, $ra \in J(A, \Gamma)^+$, and this last is a subspace of the $\mathfrak{Q}$-module A.

Conversely, let B be a $\mathfrak{Q}$-submodule of the $\mathfrak{Q}$-module A. One can find a Q-submodule C of A such that $A = B \oplus C$, a module-direct sum. On B place any radical ring structure (B, Γ). (If dim $B \geqq 2$, and only then, Γ can be chosen to be nontrivial.) On C place any semisimple ring structure (C, Λ). (Since C is a direct sum of copies of Q use the corresponding ring-direct sum of $\mathfrak{Q}$'s to obtain a semisimple ring supported by C.) Endow A with the direct-sum ring structure $(A, \Delta) = (B, \Gamma) \oplus (C, \Lambda)$. Clearly, $(B, \Gamma) \leqq J(A, \Delta)$ so that $B \leqq J(A, \Delta)^+ = B \oplus K$, a direct sum of $\mathfrak{Q}$-modules for some submodule K of C. Hence $K \leqq J(A, \Delta)^+ \cap C = J(C, \Lambda)^+$; for, (C, Λ) is an ideal in (A, Δ). Since, however, (C, Λ) is semisimple, $B = J(A, \Delta)^+$. ■

Theorem 4. *The torsion-free groups of rank* 1 *that are radical groups are precisely those of type represented by* $(k_1, k_2, \cdots)$ *where each* k_i *is either* 0 *or* ∞, *and where almost all, but not all, these* k_i *are* ∞. *Each such group supports at least one nonradical proper ring, the radical of which is supported by a subgroup also of type* $(k_1, k_2, \cdots)$. *The torsion-free, rank* 1 *antiradical groups that support proper rings are precisely those of type represented by* $(l_1, l_2, \cdots)$ *where each* l_i *is* 0 *or* ∞, *and where none or an infinite number of the* l_i*'s consists of zeros. The remaining torsion-free, rank* 1 *groups support only zero rings.*

Proof. By the Rédei-Szele theorem [4, p. 270, Theorem 70.1], unless the type numbers in some representative of the type are chosen from the set $\{0, \infty\}$ only the zero ring is supported. If a representative of the type has only ∞'s we have Q, an antiradical group. Suppose, now, that a representative of the type of A has an infinite number of 0's. By the Rédei-Szele theorem, each proper ring $\mathfrak{A}$ on A is, to within a ring isomorphism, a subring of $\mathfrak{Q}$ consisting precisely of the elements of the form mkv^{-1} where $m(\mathfrak{A}) = m > 0$ is an integer, not divisible by the primes from some set $\Pi(\mathfrak{A}) = \Pi$ (possibly void) of positive primes. Also, k and v $(\neq 0)$ are relatively prime integers if $k \neq 0$; and if $v \neq \pm 1$ all the positive prime factors of v lie in Π. If Π is void then $A \cong Z$, an antiradical group.

Now suppose that Π is nonvoid. Since the hypothesis provides an infinite number of 0's in a representative of the type of A, there must be at least one prime $p \notin \Pi$ such that $(m, p) = 1$, and $am + bp = 1$ for appropriate integers a and b. Since $ma \in \mathfrak{A}$ any q.i. in $\mathfrak{A}$ of this element would have the form $mk'v'^{-1}$ which reduces to $-ma(bp)^{-1}$ provided $b \neq 0$. But such an element does not lie in $\mathfrak{A}$ since $p \notin \Pi$. If $b = 0$, $ma = 1 \in \mathfrak{A}$. In neither case can $J(\mathfrak{A}) = \mathfrak{A}$. Thus A is antiradical if an infinite number of 0's can appear in the type.

Suppose that at most a finite number $r \geqq 1$ of zeros can occur in the type of A. If $m = \pm 1$, or if $m \neq \pm 1$ and there exists a prime $p \notin \Pi$ such that $p \nmid m$, then, as before, $\mathfrak{A}$ is not a radical ring. Suppose, however, that m is so chosen that it is divisible by each of the r primes at which the type can be 0, and that Π is nonvoid. The formal q.i. of the typical element $mkv^{-1} \in \mathfrak{A}$ is $mk(mk - v)^{-1}$. Since no prime factor in m can divide v, the prime factors (if any) of $mk - v$ must lie in Π. Since the formal q.i. of mkv^{-1} thus lies in $\mathfrak{A}$, this last is a radical ring.

Consider any radical group A of type represented by $(k_1, k_2, \cdots)$ where almost all, but not all, the k_i are ∞, and the rest are zero. As before, let Π be the set of primes at which the k_i are ∞. Decompose the set of r primes at which the k_i's are 0 into two disjoint subsets $\{p_1, \cdots, p_s\}$ and $\{q_1, \cdots, q_t\}$ where the first set may be void but the second set not. Let m be a positive integer, the prime factors (if any) of which are chosen from the p_i. (If there are no p_i's take $m = 1$.) Let $\mathfrak{U}$ be the set of all rational numbers mkv^{-1} where k and v $(\neq 0)$ are integers, relatively prime if $k \neq 0$, and where the prime factors (if any) of v lie in Π. We saw that $\mathfrak{U}$ is a nonradical ring, a subring of Let $\mathfrak{B} = \{u \mid u \in \mathfrak{U}$ and $u = mq_1 \cdots q_t kv^{-1}\}$ (k and v as above), an ideal in $\mathfrak{U}$. Since the formal q.i. of $mq_1 \cdots q_t kv^{-1}$ is $mq_1 \cdots q_t k(mq_1 \cdots q_t k - v)^{-1}$ which lies in $\mathfrak{B}$, we have $\mathfrak{B} \leqq J(\mathfrak{U})$.

If $mkv^{-1} \in \mathfrak{U} \setminus \mathfrak{B}$, at least one q_i fails to divide k so that $mka + bq_i = 1$ for appropriate integers a and b. If such an $mkv^{-1} \in J(\mathfrak{U})$ then $mkv^{-1}(av) = mka = 1 - bq_i$ would be q.r. in $\mathfrak{U}$. If $b = 0$ then 1 is q.r. in $\mathfrak{U}$, an impossibility. If $b \neq 0$, $1 - bq_i$ has the formal q.i. $-(1 - bq_i)b^{-1}q_i^{-1}$, an irreducible rational. Since $q_i \notin \Pi$, this q.i. $\notin \mathfrak{U}$. Consequently, $J(\mathfrak{U}) \leqq \mathfrak{B}$, and $J(\mathfrak{U}) = \mathfrak{B}$. Finally, type $(\mathfrak{B}^+)$, type (A), and type $(\mathfrak{U}^+)$ are all represented by $(k_1, k_2, \cdots)$. ∎

If A is a torsion-free rank 2 group, consider it in its representation [1] as a subdirect sum of two groups of rank 1, a subgroup of $Q \oplus Q$. If Γ is a bimultiplier on $Q \oplus Q$ such that $\Gamma_L(a_1)a_2 \in A$ for every $a_1, a_2 \in A$, then Γ induces a bimultiplier $\Gamma \mid A$ on A, and $(A, \Gamma \mid A)$ is a subring of $(Q \oplus Q, \Gamma)$. It is not hard to see [1, p. 106, (4)] that any ring $\mathfrak{A}$ on A must arise in this way. The only radical rings on A are the $(A, \Gamma \mid A)$ where each $a \in A$ is q.r. in $(Q \oplus Q, \Gamma)$ with $a^* \in A$. Thus, to find all the radical rings supported by the torsion-free rank 2 groups, first determine all bimultipliers Γ on $Q \oplus Q$ (these being fairly easy to classify); then find criteria for membership in the set of q.r. elements of $(Q \oplus Q, \Gamma)$ (somewhat harder to do); finally, look for those subrings $\mathfrak{S}$ of $(Q \oplus Q, \Gamma)$, each element of which is q.r. in $(Q \oplus Q, \Gamma)$ with its q.i. in $\mathfrak{S}$ (not always apparent). We shall discuss this method elsewhere; it suffices here to give some examples.

Let $\mathfrak{A} = \{(r, m(2n+1)^{-1}) \mid r \in \mathfrak{Q} \text{ and } m, n \in \mathfrak{Z}\}$. The set $\mathfrak{A}$ is an abelian group under componentwise addition, a torsion-free group of rank 2. For $(r_1, s_1), (r_2, s_2) \in \mathfrak{A}$, let $(r_1, s_1)(r_2, s_2) = (0, 2s_1s_2) \in \mathfrak{A}$, and $\mathfrak{A}$ is a proper radical ring with

$$(r, m(2n+1)^{-1})^* = (-r, -m[2(n-m)+1]^{-1}).$$

Let $\mathfrak{B}$ be the set of all $(u+2v, u)$ with $u = m(2n+1)^{-1}$, $v = m'(2n'+1)^{-1}$, and $m, n, m', n' \in \mathfrak{Z}$. Then $\mathfrak{B}$, an ideal in the ring $\mathfrak{A}$ above, is a proper radical ring where $(u+2v, u)^* = (-u-2v, -u -2(u+2v)^2)$. The multiplication on $\mathfrak{B}$ is a special case of (e) above, while $\mathfrak{A}$ arises from the consideration of another species of multiplication on $Q \oplus Q$.

5. **Direct sums.**

Lemma 4. *If* $\mathrm{Hom}(A \otimes A, B)$ *is nontrivial then* $A \oplus B$ *is a radical group.*

Proof. If $f \in \mathrm{Hom}$, $f \neq 0$, define multiplication on $A \oplus B$ by setting $(a_1, b_1)(a_2, b_2) = (0, f(a_1 \otimes a_2))$, turning $A \oplus B$ into a ring forthwith. The q.i. of (a, b) is $(-a, -b - f(a \otimes a))$, and the exponent of nilpotency of the resulting proper radical ring is 3. ∎

Corollary. *If* $\mathfrak{S}$ *and* $\mathfrak{T}$ *are rings, and if there exists* $\varphi \in \mathrm{Hom}(\mathfrak{S}, \mathfrak{T})$ *such that* $\mathrm{Im}\,\varphi$ *is a proper ring as a subring of* $\mathfrak{T}$ *then* $\mathfrak{S}^+ \oplus \mathfrak{T}^+$ *is a radical group. If* $\mathfrak{S}$ *is a proper ring* $\mathfrak{S}^+ \oplus \mathfrak{S}^+$ *is a radical group.*

Proof. Define $f \in \mathrm{Hom}(\mathfrak{S}^+ \otimes \mathfrak{S}^+, \mathfrak{T}^+)$ by setting $f(s_1 \otimes s_2) = \varphi(s_1s_2)$. ∎

From this corollary it is immediate that each of the following groups supports at least one proper radical ring (where A is any group):

$$Z(n) \oplus Z \oplus A, \quad Q \oplus Z \oplus A, \quad Z(n) \oplus Z(n) \oplus A,$$

$$Z \oplus Z \oplus A, \quad Q \oplus Q \oplus A$$

(in particular [5, p. 105], the additive groups of the real numbers and of the complex numbers, and the group of reals modulo 1).

Recall that a *nil group* [4, p. 272] is a group A such that $\mathrm{Hom}(A, \mathrm{Hom}(A, A))$ is trivial.

Theorem 5. *A nontrivial group* A *is a nil group if and only if* $A \oplus A$ *is antiradical.*

Proof. If $A \oplus A$ is antiradical, Lemma 4 shows that $\mathrm{Hom}(A \otimes A, A)$ is trivial.

Conversely, if $A \oplus A$ is a radical group it has a nontrivial bimultiplier Γ. For each $a \in A$, $\Gamma_R(a, 0)$ is some endomorphism

$$\begin{pmatrix} \alpha(a) & \beta(a) \\ \gamma(a) & \delta(a) \end{pmatrix}$$

of $A \oplus A$ where the four entries of the matrix are in $\mathrm{Hom}(A, A)$. We lose no generality in assuming that, for some $a \in A$, $\Gamma_R(a, 0)$ is nontrivial so that at least one of $\alpha, \beta, \gamma, \delta \in \mathrm{Hom}(A, \mathrm{Hom}(A, A))$ is nontrivial. By definition, A is nonnil. ∎

COROLLARY. *If A is a mixed group, then $A \oplus A$ is a radical group.*

PROOF. [4, p. 272, Theorem 71.1]. ∎

If A and B are antiradical then their direct sum need not be (e.g., $Z \oplus Z$). A partial converse to Lemma 4 does, however, exist.

THEOREM 6. *Let A and B be antiradical groups.* (i) *If* $\mathrm{Hom}(A \otimes B, A)$, $\mathrm{Hom}(B \otimes A, B)$, $\mathrm{Hom}(B \otimes B, A)$, *and* $\mathrm{Hom}(A \otimes A, B)$ *are all* 0 *then $A \oplus B$ is antiradical.* (ii) *If* $\mathrm{Hom}(A, B) = 0 = \mathrm{Hom}(B, A)$ *then $A \oplus B$ is antiradical.*

PROOF. (i) Suppose that $(A \oplus B, \Gamma)$ is a radical ring. For $b \in B$, the endomorphism $\Gamma_R(b)$ of $A \oplus B$ has the representation

$$\begin{pmatrix} \alpha_1(b) & \alpha_2(b) \\ \alpha_3(b) & \alpha_4(b) \end{pmatrix}$$

where $\alpha_1 \in \mathrm{Hom}(B, \mathrm{Hom}(A, A))$, $\alpha_2 \in \mathrm{Hom}(B, \mathrm{Hom}(A, B))$, $\alpha_3 \in \mathrm{Hom}(B, \mathrm{Hom}(B, A))$, and $\alpha_4 \in \mathrm{Hom}(B, \mathrm{Hom}(B, B))$. By hypothesis, the first three double "Homs" vanish so that $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$. Similarly, if $a \in A$, $\Gamma_R(a)$ can be represented as

$$\begin{pmatrix} \beta_1(a) & \beta_2(a) \\ \beta_3(a) & \beta_4(a) \end{pmatrix}$$

where $\beta_1 \in \mathrm{Hom}(A, \mathrm{Hom}(A, A))$, $\beta_2 \in \mathrm{Hom}(A, \mathrm{Hom}(A, B))$, $\beta_3 \in \mathrm{Hom}(A, \mathrm{Hom}(B, A))$, and $\beta_4 \in \mathrm{Hom}(A, \mathrm{Hom}(B, B))$, so that all but β_1 vanish. Thus, $(a_1 \oplus b_1)(a_2 \oplus b_2) = a_1a_2 + b_1b_2$ for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

For $a \in A$, $a^* = a' \oplus b$ for some $a' \in A$ and $b \in B$. Then $(a + a') \oplus b = a(a' \oplus b) = (a' \oplus b)a = aa' = a'a$. Let Π_A be the projection of $A \oplus B$ onto A, so that, here, $a + a' = \Pi_A(aa') = \Pi_A(a'a)$. Since, however, $a_1a_2 = \Gamma_L(a_1)a_2 = \beta_1(a_1)a_2 \in A$ for all $a_1, a_2 \in A$, in particular $a + a' = aa' = a'a \in A$, and $(A, \Gamma \mid A)$ is a radical ring. But A is antiradical so that this ring is the zero ring on A. Hence, each $a_1a_2 = 0$. Likewise, each $b_1b_2 = 0$, and $(A \oplus B, \Gamma)$ is the zero ring on $A \oplus B$. We now have (i), from which (ii) follows. ∎

THEOREM 7. *The only divisible antiradical groups are Q and the torsion divisible groups.*

PROOF. By Szele's theorem [4, p. 272, Theorem 71.1], the torsion divisible groups are precisely the torsion groups that support only zero rings, so that all these groups are antiradical. If D is divisible, $D = C \oplus T$ where C is trivial or is some $\Sigma \oplus Q$, and where T is trivial or is some $\Sigma_i \oplus (\Sigma \oplus Z(p_i^{\infty}))$. If more than one summand Q appears, an earlier remark shows that D is a radical group. Hence consider the case $D = Q \oplus T$. It is well known [4, pp. 25–26] that the nontrivial homomorphic images H of Q are all possible direct sums of quasi-cyclic groups with no repetitions of primes allowed. If, therefore, T is nontrivial then T has such an H as a direct summand, and Hom(Q, T) is nontrivial. Since, as $\mathfrak{Z}$-modules, $Q \cong Q \otimes Q$, we have Hom($Q \otimes Q$, T) nontrivial, so that, by Lemma 4, $D = Q \oplus T$ is a radical group. ∎

THEOREM 8. *For each positive integer n and for each prime p, $Z(p^n)$ supports exactly p^{n-1} radical rings (including the zero ring). The proper radical rings on $Z(p^n)$ fall into $n - 1$ isomorphism classes, and the members of each such class are commutative nilpotent with fixed exponent of nilpotency $1 - [-n/j]$, for $j = 1, 2, \cdots, n-1$.*

PROOF. Each bimultiplier Γ on $Z(p^n)$ corresponds to a unique integer k, $0 \leqq k < p^n : \Gamma_L(m_1{}')m_2{}' = (m_1 m_2 k)'$ where $m' \in Z(p^n)$ is the residue class, modulo p_n, in which the integer m lies. Each such k determines a ring $\mathfrak{Z}(p^n; k)$ supported by $Z(p^n)$. For this ring to be radical the equation $\Gamma_L(a')x' = (a + x)'$ must have a solution $x \in Z$, once $a \in Z$ is given. That is, the congruence $(ak - 1)x \equiv a(p^n)$ must be solvable. If $p \mid k$ then $p \nmid (ak - 1)$, and the congruence has a solution. If $p \nmid k$ then there exist integers c_1 and c_2 such that $c_1 k + c_2 p = 1$ so that $p \nmid c_1$. Now if we choose $a = c_1$, the congruence reduces to $-c_2 px \equiv c_1(p^n)$ so that $p \mid c_1$ if a solution exists, contradicting $p \nmid c_1$. Thus, for a radical ring, k must be one of the p^{n-1} multiples of p on the interval $0 \leqq k < p^n$. Of these, only $k = 0$ provides us with the zero ring. If such a $k \neq 0$ then $k = p^j t$ where $1 \leqq j < n$ and $(p, t) = 1$. It is easy to show that $\mathfrak{Z}(p^n; p^{j_1}t_1)$ and $\mathfrak{Z}(p^n; p^{j_2}t_2)$ are ring isomorphic if and only if $j_1 = j_2$.

Suppose that $k = p^j t$ where $1 \leqq j < n$ and where $(p, t) = 1$. Denote the multiplication on $\mathfrak{Z}(p^n; p^j t)$ by $\Gamma_L(m_1{}')m_2{}' = m_1{}' \# m_2{}'$. Then $m_1{}' \# \cdots \# m_r{}' = (m_1 \cdots m_r p^{j(r-1)} t^{r-1})'$. The least positive integer $r = r_j$ for which $(r - 1)j \geqq n$ must be the exponent of nilpotency. Thus, $r_j = 1 - [-n/j]$ with minimum value 3 realized for all j such that $-[-n/2] \leqq j < n$. It has maximum value $n + 1$

at $j = 1$, so that to construct a proper radical ring of exponent of nilpotency $m \geqq 3$ take $n = m - 1$ and $k = p$. ∎

THEOREM 9. *If $\{p_i\}$ is a (finite or infinite) set of distinct primes then $\Sigma_i \oplus Z(p_i)$ is antiradical. Such sums and Z are the only antiradical direct sums of cyclic groups.*

PROOF. We could handle the finite case by Theorem 6. In general, however, if $(\Sigma_i \oplus Z(p_i), \Gamma)$ is a proper ring, there exists at least one $q \in \Sigma_i$ such that $\Gamma_L(q) \neq 0$. Since the orders of the $Z(p_i)$ are all coprime, $\Gamma_L(q) \mid Z(p_i) = (t_i)_L$, a left multiplication on $Z(p_i)$ by some integer t_i, $0 \leqq t_i < p_i$. At least one $t_i \neq 0$ since $\Gamma_L(q) \neq 0$. If the ring is radical, for each $x \in \Sigma_i$, $\iota - \Gamma_L(x) \in \text{Aut}\, \Sigma_i$, by Lemma 1(ii). Thus, for each $k \in \mathfrak{Z}$, $(\iota - \Gamma_L(kq)) \mid Z(p_i) = (1 - kt_i)_L$, an induced automorphism on $Z(p_i)$. That is, $(1 - kt_i, p_i) = 1$ for every k, a contradiction since the congruence $t_i x \equiv 1(p_i)$ always has a solution when $t_i \neq 0$. The second statement of the theorem follows from Lemma 4, Corollary, and Theorem 8. ∎

Thus, if $\{p_i\}$ and $\{q_i\}$ are two disjoint, nonempty sets of positive primes where the first set may not have repetitions, but where the second set may, then $(\Sigma_i \oplus Z(p_i)) \oplus (\Sigma_j \oplus Z(q_j^{\infty}))$ is antiradical, by Theorems 6, 7, and 9. Likewise, $(\Sigma_i \oplus Z(p_i)) \oplus Q$ is antiradical, although in this case all positive primes p_i without repetitions may be used.

THEOREM 10. *The bounded antiradical groups are precisely the $Z(m)$ where m is a product of distinct primes.*

PROOF. That these $Z(m)$ are antiradical follows from Theorem 9. Bounded groups are the direct sums $A = \Sigma_p \oplus [\Sigma_i \oplus (\Sigma \oplus Z(p^i))]$ where each $\Sigma \oplus Z(p^i)$ is a direct sum of copies of $Z(p^i)$, where i is bounded for each prime p, and where only a finite number of primes p can appear. If A is to be antiradical then Theorem 8 shows that no $i > 1$, and A reduces to $\Sigma_p \oplus (\Sigma \oplus Z(p))$. By an earlier remark, $\Sigma \oplus Z(p)$ reduces to $Z(p)$ if it is nontrivial. Consequently, $A = \Sigma_j \oplus Z(p_j)$ for a finite number t of distinct primes. ∎

6. p-groups.

LEMMA 5. *For $\alpha, \beta \in \text{End}\, A$, suppose that* (i) $\bigcup_{i=1}^{\infty} \ker \alpha^i = A = \bigcup_{j=1}^{\infty} \ker \beta^j$, *and that* (ii) *there exists a nontrivial*

$$\gamma \in \text{Hom}\left(A, \text{Hom}\left(A, \bigcap_{i=1}^{\infty} [(\text{Im}\, \alpha^i) \cap (\text{Im}\, \beta^i)] \right)\right)$$

such that, for every $a \in A$, $\gamma(\beta a) = \gamma(a)\alpha$. Then A is a radical group that supports a proper radical ring of exponent of nilpotency 3.

PROOF. For $a_1, a_2 \in A$, let $a_1a_2 = \gamma(a_1)a_2$ from which two distributive laws hold. From (i), $a_1 \in \ker \beta^n$ for some n. Since $\gamma(a_2)a_3 \in \bigcap_{i=1}^{\infty} \operatorname{Im} \alpha^i$, $\gamma(a_2)a_3 = \alpha^n(b)$ for some $b \in A$. Then $a_1(a_2a_3) = a_1\alpha^n(b) = \gamma(a_1)\alpha^n b = (\gamma\beta^n a_1)b = 0$, from (ii). Similarly, $(a_1a_2)a_3 = 0$. The q.i. for $a \in A$ is $a^* = -a - a^2$. ∎

Recall that A^1 is the subgroup of elements of infinite height in the p-group A.

COROLLARY. *If, for a p-group A,* $\operatorname{Hom}(A, \operatorname{Hom}(A, A^1)) \neq 0$ *then A is a radical group.*

PROOF. Let $\alpha = p_L = \beta$ where $p_L : a \mapsto pa$. ∎

THEOREM 11. (i) *Each p-group that has a nonzero basic subgroup and a nonzero subgroup of elements of infinite height is a radical group.* (ii) *The only countable reduced p-group that is antiradical is* $Z(p)$.

PROOF. (i) Let G be a p-group with nontrivial basic subgroup B. Then $G \otimes G \cong B \otimes B$, a direct sum of cyclic p-groups, since B is nontrivial. Now $\operatorname{Hom}(B \otimes B, G^1)$ is nontrivial; for, we can map all but one cyclic summand of $B \otimes B$ onto 0 and the remaining one onto some cyclic subgroup of order p of G^1. But $\operatorname{Hom}(G, \operatorname{Hom}(G, G^1)) \cong \operatorname{Hom}(B \otimes B, G^1)$ so that we can now apply Lemma 5, Corollary.

(ii) Let G be a nontrivial, countable, reduced p-group, and suppose, first, that $G^1 \neq 0$. If the basic subgroups of G were to be trivial, then G would be divisible, a contradiction, so that G is a radical group by (i). If $G^1 = 0$ then, by Prüfer's theorem [4, p. 44], G is a direct sum of cyclic p-groups. From earlier results, all such examples but $Z(p)$ support proper radical rings. ∎

THEOREM 12. (i) *Let G be a countable reduced p-group of Ulm type* 2. *Then there exists an unbounded, countable, reduced p-group H of Ulm type* 1 *such that G and H support respective, proper, radical commutative rings $\mathfrak{G}$ and $\mathfrak{H}$ for which there is a ring epimorphism* $\mathfrak{H} \twoheadrightarrow \mathfrak{G}$.

(ii) *Let G be an unbounded, countable, reduced p-group of Ulm type* 1. *Then there exists a countable, reduced p-group H of Ulm type* 2 *such that G and H support respective, proper, radical commutative rings $\mathfrak{G}$ and $\mathfrak{H}$ for which there is a ring epimorphism* $\mathfrak{H} \twoheadrightarrow \mathfrak{G}$.

PROOF. (i) Since G is of type 2 its Ulm sequence consists of G_0, G_1 where $G_0 = \Sigma_i \oplus Z[b_i]$ for suitable $b_i \in G_0$, each of order p^{n_i} where $E(b_i) = n_i \geqq 1$, and the set N of the n_i's is unbounded. See [4, pp. 117–123]. Since G_1 is a direct sum of cyclic groups, assume first that $G_1 = Z[a]$ where $E(a) = n \geqq 1$. As in the proof of Zippin's theorem

[4, loc. cit.], let $H = \Sigma_i \oplus Z[x_i]$ where $E(x_i) = n + n_i$. The only significant member of the Ulm sequence for H is H, itself. Let K be that subgroup of H which is generated by all the $p^{n_i}x_i - p^{n_j}x_j$ $(i, j = 1, 2, \cdots)$. Each group $Z[x_i]$ supports the proper, radical commutative ring $\mathfrak{Z}(p^{n+n_i}; p^n)$, as in the proof of Theorem 8. Let $\mathfrak{H} = \Sigma_i \oplus \mathfrak{Z}(p^{n+n_i}; p^n)$ so that $\mathfrak{H}^+ = H$, and $\mathfrak{H}$ is a proper, radical commutative ring. The subgroup K supports an ideal $\mathfrak{K}$ of $\mathfrak{H}$ since the constructed multiplication (denoted by #) on $\mathfrak{H}$ introduces a numerical factor p^n that nullifies each $p^{n_i}x_i - p^{n_j}x_j$. Further, at least one product in $\mathfrak{H}$ fails to lie in the ideal $\mathfrak{K}$. For, choose $n_t > n$, which is always possible since N is unbounded. Then $x_t \# x_t = p^n x_t$, and if the latter were in $\mathfrak{K}$ there would exist a finite set of integers $\{a_{ij}\}$ such that $\sum a_{ij}(p^{n_i}x_i - p^{n_j}x_j) = p^n x_t$. Matching coefficients, we obtain $(\sum_j a_{tj} - \sum_i a_{it})p^{n_t} \equiv p^n \bmod(p^{n_t+n})$, which is impossible since $n_t > n$.

By the proof of Zippin's theorem, H/K has Ulm sequence G_0, G_1 so that, by Ulm's theorem, $H/K \cong G$. Since $\mathfrak{H}$ is a proper, radical commutative ring with at least one product not in $\mathfrak{K}$, $\mathfrak{H}/\mathfrak{K}$ is a proper radical ring supported, to within an isomorphism, by G.

If $G_1 = \Sigma_r \oplus Z(p^{s_r})$ for positive integers s_r, it is possible to construct a countable reduced p-group G' of type 2 with Ulm sequence G_0, G_1 where $G' = \Sigma_r \oplus G^{(r)}$, each $G^{(r)}$ of type 2 where $G^{(r)} = H^{(r)}/K^{(r)}$, each $H^{(r)}$ of type 1, and each $G^{(r)}$ with Ulm sequence G_0, $Z(p^{s_r})$. Further, $G' = H/K$ where $H = \Sigma_r \oplus H^{(r)}$ is of type 1, and $K = \Sigma_r \oplus K^{(r)}$. Each $H^{(r)}$ supports a proper, commutative radical ring $\mathfrak{H} = \Sigma_i \oplus \mathfrak{H}^{(r)}$ with ideal $\mathfrak{K} = \Sigma_r \oplus \mathfrak{K}^{(r)}$ supported by K. Also, $\mathfrak{G} = \mathfrak{H}/\mathfrak{K}$ is a proper, commutative radical ring; and $\mathfrak{G}^+ \cong G'$. But G and G' have the same Ulm sequence so that $\mathfrak{G}^+ \cong G$, and (i) holds.

(ii) For any countable, reduced p-group G of Ulm type 3, let $H(G)$ be the group of type 2, and let $K(G)$ be the subgroup of $H(G)^1$, provided by the proof of Zippin's theorem, such that $H(G)$ has the Ulm sequence G_0, $H(G)^1$, and such that $G \cong H(G)/K(G)$. As a group of type 2, $H(G)$ can be given a proper, commutative, radical ring structure $\mathfrak{H}(G)$ by the proof of (i), but it remains an open question whether a suitable radical ring structure can be imposed on $H(G)$ in such a way that $K(G)$ will support an ideal. If, however, a p-group U supports a ring $\mathfrak{U}$ then U^1 supports an ideal $\mathfrak{U}^1$ of $\mathfrak{U}$, so that $H(G)^1$ supports an ideal $\mathfrak{H}(G)^1$ of $\mathfrak{H}(G)$. As in the proof of (i), $H(G)$ is a direct sum of the form $\Sigma_t \oplus H_{[t]}/K_{[t]}$ (one summand for each summand $Z(p^{n_t})$ of G_2), and each summand supports a proper, commutative radical ring. In fact, $H_{[t]} = \Sigma_i \oplus Z[x_{ti}]$ where $E(x_{ti}) = n_t + n_{ti}$ $(n_t, n_{ti} \geqq 1$, and $N_t = \{n_{ti}\}$ unbounded). Each $Z[x_{ti}]$ supports the radical ring $\mathfrak{Z}(p^{n_{ti}+n_t}; p^{n_t})$, and the ring-direct sum $\mathfrak{H}_{[t]}$ of these last is reduced by the ideal $\mathfrak{K}_{[t]}$ on $K_{[t]}$, the group on all $p^{n_{ti}}x_{ti} - p^{n_{tj}}x_{tj}$. Since N_t is unbounded, choose any $n_{tr} \in N_t$ such that $n_{tr} > n_t$. In $\mathfrak{H}_{[t]}/\mathfrak{K}_{[t]}$, $(x_{tr} + \mathfrak{K}_{[t]})^2 = p^{n_t}x_{tr} + \mathfrak{K}_{[t]}$. If this square were to lie in $(H_{[t]}/K_{[t]})^1$ then, in particular, one could solve for the coset $y + K_{[t]}$ in the group equation

$$p^{n_t+1}(y + K_{[t]}) = p^{n_t}x_{tr} + K_{[t]}.$$

We can assume that $y + K_{[t]} = \sum_i c_{ti}x_{ti} + K_{[t]}$ for integers c_{ti} so that $(\sum_i c_{ti}p^{n_t+1}x_{ti}) - p^{n_t}x_{tr} \in K_{[t]}$. Since, however, the elements of $K_{[t]}$ are nullified by p^{n_t}, $\sum_i c_{ti}p^{2n_t+1}x_{ti} - p^{2n_t}x_{tr} = 0$. The coefficient of x_{tr} reduces to $p^{2n_t}(pc_{tr} - 1)$. But direct sum considerations show that this coefficient must nullify x_{tr}. Since $p \nmid (pc_{tr} - 1)$, $2n_t \geqq n_t + n_{tr}$, contradicting the assumption that $n_{tr} > n_t$. Thus, $\mathfrak{H}_{[t]}/\mathfrak{K}_{[t]}$ has at least one product not in its subgroup of elements of infinite height. Hence $\mathfrak{H}(G)/\mathfrak{H}(G)^1$ is a proper, radical commutative ring.

Since $H(G)/H(G)^1 \cong G_0$ we have proved that, for groups G of type 3, G_0 supports a proper, commutative radical ring $\mathfrak{G}_0$, a ring epimorph of the proper, commutative radical ring $\mathfrak{H}(G)$ supported by the type 2 group $H(G)$. But any unbounded, countable, reduced, type 1 p-group G_0 is the initial member of the Ulm sequence for a countable, reduced, type 3 p-group. ∎

Precisely, because it is not clear how one would turn $K(G)$ into an ideal, a suitable generalization of (i) for groups of type $\geqq 3$ remains to be found. It is true that an unbounded, type 1 p-group G_0 can be represented as $H(G)/H(G)^1$ where $H(G)$ has finite type $\geqq 3$ chosen at will, that $H(G)$ supports some proper, radical ring, and that $H(G)^1$ supports an ideal of the latter; but it is not apparent how we would show that the resulting radical ring on G_0 is proper, so that (ii), also, awaits an extension.

Added in proof. Professor K. E. Eldridge has kindly indicated that the results in Theorem 8 of this paper overlap those of [**11**].

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