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ON THE POWER MAP IN COMPACT GROUPS. II

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Let G be a topological group with identity component G_0. For an integer $k \geqq 2$, the *power map* $p_k : G \to G$ is defined by $p_k(g) = g^k$. Given $g \in G$, define the automorphism $C_g : G_0 \to G_0$ by $C_g(x) = g^{-1}xg$. In this note we examine some consequences of the following observation:

PROPOSITION 1. *Let G be a compact Lie group and let g be an element of G. If $p_k(gG_0) = G_0$ for some $k \geqq 2$, then C_g is an inner automorphism.*

PROOF. By hypothesis, we may choose g so that g^k generates a maximal torus T of G_0. Consequently, $C_g(x) = x$ for all $x \in T$. If G_0 is semisimple, then C_g is an inner automorphism by [2, Proposition 2.5, p. 334]. The generalization to all G_0 follows by standard arguments. ∎

A topological group G is a *central extension* (*of G_0 by G/G_0*) if every component of G contains an element of the centralizer of G_0 in G.

PROPOSITION 2. *Let G be a compact Lie group; then $p_k(gG_0) = g^kG_0$ for all $g \in G$ and $k \geqq 2$ if and only if G is a central extension.*

PROOF. The identity component of a compact group is divisible so $p_k(gG_0) = g^kG_0$ for all g and k when G is a central extension. To prove the converse, suppose $p_k(gG_0) = G_0$ whenever $g^k \in G_0$. Then C_g is an inner automorphism by Proposition 1. Therefore, $C_{g'}$ is the identity map for some $g' \in gG_0$. ∎

QUESTION 1. The statement "$p_k(gG_0) = g^kG_0$ for all $g \in G$ and $k \geqq 2$" is a necessary condition for a compact topological group to be a central extension. Is it a sufficient condition?

By a theorem of Kostant, the real homology $H_*(G)$ of a compact Lie group is the smash product $H_*(G/G_0) \# H_*(G_0)$ (see [3, 7.2, 8.1]).

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PROPOSITION 3. *The real homology of a compact Lie group G is isomorphic, as a Hopf algebra, to the tensor product $H_*(G/G_0) \otimes H_*(G_0)$ if and only if G is a central extension.*

PROOF. If G is a central extension, then $H_*(G)$ is isomorphic to $H_*(G/G_0) \otimes H_*(G_0)$ by Kostant's theorem and the definition of smash product. Conversely, if $H_*(G)$ is isomorphic to $H_*(G/G_0) \otimes H_*(G_0)$ and $g \in G$ is any element, then the degree of $p_k : gG_0 \to g^kG_0$ is k^r, where r is the rank of G_0. Therefore, G is a central extension by Proposition 2. ∎

PROPOSITION 4. *Let K be a compact connected Lie group. If h is an automorphism of K such that h_* is the identity function on $H_*(K)$, then h is an inner automorphism.*

PROOF. First assume that K is semisimple; then we may suppose that h is of finite order m. Let G be the semidirect product of the cyclic group of order m and K induced by h, then $K = G_0$ and $h = C_g$ for some g in G such that $g^m \in G_0$. Since C_{g*} is the identity function, Theorem 2.3 of [1] implies that $p_m : gG_0 \to G_0$ has nonzero degree. Proposition 1 completes the argument in this case. A standard technique then establishes the result for all K. ∎

QUESTION 2. If h is an automorphism of a compact connected topological group G such that h_* is the identity function on $H_*(G)$, is h an inner automorphism?

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