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RADICALS OF ENDOMORPHISM NEAR-RINGS

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Several radical properties have been defined for a distributively generated (d.g.) near-ring R with identity — the radical $J(R)$, the quasi-radical $N(R)$, the ideal-radical $I(R)$, the radical-subgroup, the primitive-radical $P(R)$, and the nil-radical $L(R)$. The order of containment of the various radicals is $L(R) \subseteq I(R) \subseteq N(R) \subseteq J(R) \subseteq P(R)$ (cf. [1], [2]). The radical-subgroup is also contained in $J(R)$, but it is not known how it compares with $N(R)$ in general. If R is a ring, the radical, quasi-radical, ideal-radical, and radical-subgroup are all equal to the Jacobson radical. If R is a near-ring which is not a ring, then the above radicals are not equivalent in general, even if R is finite (cf. [2], [7]).

The purpose of this paper is to examine these radicals for the particular (left) d.g. near-ring $E(G)$, the near-ring generated by the endomorphisms of G, where G is a finite group. We show that $L(E(G)) = I(E(G)) = N(E(G)) = J(E(G)) = P(E(G))$. If G is the sum of its minimal fully invariant subgroups, then $J(E(G))$ and hence all of the radicals of $E(G)$ are $\{0\}$. If G is not the sum of its minimal fully invariant subgroups, the radical $J(E(G))$ is a proper nonzero ideal of $E(G)$. In §5 we give examples to show that in the latter situation, the radical-subgroup of $E(G)$ may or may not be equal to $J(E(G))$.

1. **Definitions.** It is assumed that the reader is familiar with the definitions of a (left) d.g. near-ring and of $E(G)$, the near-ring generated by the endomorphisms of a group G (cf. [8]). Note that all functions of G are written on the right and hence $E(G)$ is a left d.g. near-ring.

Let R be a (left) d.g. near-ring. The concepts of R-group, right module of R, ideal and right ideal are all defined in [7]. The radical properties which need to be defined for this paper are given below.

The radical $J(R)$ of R is the intersection of all annihilating ideals of the minimal R-groups (cf. [6]).

The nil-radical $L(R)$ of R is the sum of all nilpotent ideals of R (cf. [2]).

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The radical-subgroup of R is the intersection of all maximal R-groups of R (cf. [3]).

For definitions of the remaining radical properties consult the references.

2. Relationship between $J(E(G))$ and the minimal fully invariant subgroups of G. Since we are interested in radicals of near-rings which are not rings, G will always denote a finite group which is not commutative. Let $\{H_i \mid i = 1, \cdots, n\}$ be the collection of minimal fully invariant subgroups of G.

LEMMA 1. *Let $\beta \in E(G)$; $h_i \in H_i$, $i = 1, \cdots, n$. Then $(\sum_{i=1}^n h_i)\beta = \sum_{i=1}^n (h_i)\beta$.*

PROOF. Since $E(G)$ is d.g., there exists a positive integer m and maps s_j, $j = 1, \cdots, m$, such that $\beta = \sum_{j=1}^m s_j$, where s_j is either an endomorphism or an anti-endomorphism for all j. Then

$$\left(\sum_{i=1}^n h_i\right)\beta = \left(\sum_{i=1}^n h_i\right)\left(\sum_{j=1}^m s_j\right) = \sum_{j=1}^m \left(\sum_{i=1}^n h_i\right)s_j$$

$$(*) \qquad = \sum_{i=1}^n \left(\sum_{j=1}^m (h_i)s_j\right) = \sum_{i=1}^n (h_i)\left(\sum_{j=1}^m s_j\right) = \sum_{i=1}^n (h_i)\beta$$

Equality $(*)$ holds since elements of distinct minimal fully invariant subgroups commute with each other.

If K is a subset of G, define $A(K) = \{\alpha \in E(G) \mid (k)\alpha = 0, \ \forall k \in K\}$. Using Lemma 1, we obtain the following.

PROPOSITION 2. $A(\sum_{i=1}^n H_i) = \bigcap_{i=1}^n A(H_i)$.

Since $J(E(G))$ is defined to be the intersection of all annihilating ideals of minimal $E(G)$-groups and since minimal fully invariant subgroups of G are minimal $E(G)$-groups, we have

PROPOSITION 3. $J(E(G)) \subseteq \bigcap_{i=1}^n A(H_i) = A(\sum_{i=1}^n H_i)$.

3. G equals sum of minimal fully invariant subgroups. Suppose G is a noncommutative finite group which is the sum of its minimal fully invariant subgroups. As an immediate consequence of Proposition 3, $J(E(G)) = \{0\}$. By a theorem of Beidleman (cf. [1, Theorem 4]) if G is a finite group, then $J(E(G)) = P(E(G))$. Hence, $L(E(G)) = I(E(G)) = N(E(G)) = J(E(G)) = P(E(G)) = \{0\}$. Also, the radical-subgroup is $\{0\}$ since it is contained in the radical.

4. **G does not equal sum of minimal fully invariant subgroups.** Now consider the case when G is a finite noncommutative group which is not the sum of its minimal fully invariant subgroups. Again we show the equality of the nil-radical, ideal-radical, quasi-radical, radical, and primitive-radical. In addition, we show that the radical is nonzero.

Since $J(E(G)) \subseteq \bigcap_{i=1}^{n} A(H_i)$ by Proposition 3, the first step is to show that $\bigcap_{i=1}^{n} A(H_i)$ is nonzero.

LEMMA 4. *Let M be a nilpotent right module of $E(G)$ and let $\alpha \in M$. Then $\alpha \in \bigcap_{i=1}^{n} A(H_i)$.*

PROOF. Suppose $(h)\alpha \neq 0$, where $h \in H_p$ for some p, $1 \leqq p \leqq n$. Note that $(h)E(G) = \{(h)\beta \mid \beta \in E(G)\}$ is a fully invariant subgroup of G. Since H_p is minimal, there exists $\gamma \in E(G)$ such that $(h)\alpha\gamma = h$. Then $\alpha\gamma \in M$, but $\alpha\gamma$ is not nilpotent. This contradiction establishes the lemma.

If K is a subset of $E(G)$, define $\mathrm{Im}(K) = \{(g)\beta \mid g \in G,\ \beta \in K\}$.

PROPOSITION 5. *$I = \{\alpha \in E(G) \mid \mathrm{Im}(\alpha) \subseteq \sum_{i=1}^{n} H_i\}$ is a proper nonzero ideal of $E(G)$.*

PROOF. Since $\sum_{i=1}^{n} H_i$ is a fully invariant subgroup of G, it is easy to check that I is an ideal of $E(G)$. Since $G \neq \sum_{i=1}^{n} H_i$, the identity map ι is not in I. We show $I \neq \{0\}$.

Let $x_1, x_2, \cdots, x_m$ be the nonzero elements of G. Since $(x_p)E(G)$ is a fully invariant subgroup of G and hence must contain a minimal fully invariant subgroup, $(x_p)E(G) \cap \sum_{i=1}^{n} H_i \neq \{0\}$, for all $p = 1, \cdots, m$.

Define maps β_p inductively as follows. If $x_1 \in \sum_{i=1}^{n} H_i$, let $\beta_1 = \iota$. If $x_1 \notin \sum_{i=1}^{n} H_i$, let z_1 be a nonzero element of $(x_1)E(G) \cap \sum_{i=1}^{n} H_i$ and let β_1 be a map in $E(G)$ such that $(x_1)\beta_1 = z_1$.

Now suppose β_k has been defined for all $k \leqq t$. If $(x_{t+1})\prod_{p=1}^{t}\beta_p \in \sum_{i=1}^{n} H_i$, let $\beta_{t+1} = \iota$. If $(x_{t+1})\prod_{p=1}^{t}\beta_p \notin \sum_{i=1}^{n} H_i$, let z_{t+1} be a nonzero element of $((x_{t+1})\prod_{p=1}^{t}\beta_p)E(G) \cap \sum_{i=1}^{n} H_i$ and let β_{t+1} be a map in $E(G)$ such that $(x_{t+1})\prod_{p=1}^{t+1}\beta_p = z_{t+1}$.

Then $\prod_{p=1}^{m}\beta_p$ is a nonzero map in $E(G)$ whose image is contained in $\sum_{i=1}^{n} H_i$ and hence is a nonzero element of the ideal I.

PROPOSITION 6. *Let K be a minimal right module of $E(G)$. Then $\mathrm{Im}(K) \subseteq \sum_{i=1}^{n} H_i$.*

PROOF. Let β be a nonzero element of K. By the procedure used in Proposition 5 we can define a map γ such that $\beta\gamma \neq 0$ and $\mathrm{Im}(\beta\gamma) \subseteq \sum_{i=1}^{n} H_i$.

Let I be defined as in Proposition 5. Then $\beta\gamma \in K \cap I$, so $K \cap I$ is a nonzero right module. But K is minimal, so $K \subseteq I$ and thus $\mathrm{Im}(K) \subseteq \sum_{i=1}^{n} H_i$.

THEOREM 7. $\bigcap_{i=1}^{n} A(H_i) \neq \{0\}$.

PROOF. Suppose $\bigcap_{i=1}^{n} A(H_i) = \{0\}$. Then, by Lemma 4, $E(G)$ contains no nonzero nilpotent right modules. Hence by a result of Blackett (cf. [4, Theorem 3]), $E(G)$ is a direct sum of minimal nonzero modules. This means by Proposition 6 that $\mathrm{Im}(E(G)) \subseteq \sum_{i=1}^{n} H_i$. In particular if ι is the identity map, then $G = (G)\iota \subseteq \sum_{i=1}^{n} H_i$, so $G = \sum_{i=1}^{n} H_i$. This is a contradiction, since by hypothesis G is not the sum of its minimal fully invariant subgroups. Hence $\bigcap_{i=1}^{n} A(H_i) \neq \{0\}$.

THEOREM 8. *The nil-radical* $L(E(G)) \neq \{0\}$.

PROOF. By definition $L(E(G))$ is the sum of all nilpotent ideals of $E(G)$. Hence we must find a nonzero nilpotent ideal of $E(G)$.

Define $B = A(\sum_{i=1}^{n} H_i) \cap \{\gamma \mid \mathrm{Im}(\gamma) \subseteq \sum_{i=1}^{n} H_i\}$. Clearly B is a nilpotent ideal. Since $A(\sum_{i=1}^{n} H_i)$ is nonzero, by the procedure used in Proposition 5, we can show the existence of a nonzero element in B. Hence $L(E(G)) \neq \{0\}$.

In order to show the equality of the various radicals, we use the following definition and theorem of Beidleman (cf. [2]).

DEFINITION. A proper ideal D of a d.g. near-ring R with identity is called a *strong radical-ideal* of R if and only if every nonzero right ideal of R/D contains a minimal right ideal which contains an idempotent element.

THEOREM 9 (BEIDLEMAN). *Let R be a d.g. near-ring with identity. If the nil-radical $L(R)$ is a strong radical-ideal, then $L(R)$ is the radical of R.*

Hence, our goal is to show that $L(E(G))$ is a strong radical-ideal.

LEMMA 10. *Let B be a nilpotent right module of $E(G)$. Let $\beta \in B$; let x be a nonzero element of G. Then the fully invariant subgroup generated by $(x)\beta$ is properly contained in the fully invariant subgroup generated by x.*

PROOF. Let X, Y be the fully invariant subgroups generated by x and $(x)\beta$, respectively. Since $(x)\beta \in X$, then $Y \subseteq X$.

Suppose $Y = X$. Then there exists $\gamma \in E(G)$ such that $(x)\beta\gamma = x$. But then $\beta\gamma \in B$ and $\beta\gamma$ is not nilpotent. This contradiction establishes the lemma.

Lemma 11. *Let B be a nilpotent right module of $E(G)$. Let $\beta \in B$; $\alpha \in E(G)$. Then the right ideal C of $E(G)$ generated by $\alpha\beta$ is a nilpotent right ideal.*

Proof. Since $E(G)$ is d.g., C is the set of all finite sums of elements of the form $\lambda + \alpha\beta\nu - \lambda$, where ν, $\lambda \in E(G)$. Since C is finite, it suffices to show that every element in C is nilpotent.

Let $\omega \in C$. Then there exists a positive integer m and maps $\delta_i, \gamma_i \in E(G)$, $i = 1, \cdots, m$, such that $\omega = \sum_{i=1}^{m}(\delta_i + \alpha\beta\gamma_i - \delta_i)$.

Let $x \in G$; let K be the fully invariant subgroup generated by $(x)\alpha\beta$. Then $(x)\omega = (x)(\sum_{i=1}^{m}(\delta_i + \alpha\beta\gamma_i - \delta_i)) \in K$. By Lemma 10, K is properly contained in the fully invariant subgroup generated by x. Since G is finite, there exists a positive integer p such that $(x)\omega^p \in \sum_{j=1}^{n} H_j$. Since $\beta \in A(\sum_{j=1}^{n} H_j)$ by Lemma 4, then $(x)\omega^{p+1} = 0$.

Since G is finite, there exists a positive integer q such that $(y)\omega^q = 0$, for all $y \in G$. Hence ω is nilpotent.

Laxton (cf. [6]) has shown that the sum of a finite number of nilpotent right ideals is a nilpotent right ideal. Using Laxton's theorem and Lemma 11 we have

Proposition 12. *The sum of all nilpotent right ideals of $E(G)$ is a nilpotent ideal of $E(G)$.*

Combining Lemma 11 and Proposition 12, we obtain

Corollary 13. *If M is a nilpotent right module of $E(G)$, then $M \subseteq L(E(G))$.*

The next two results are routine.

Lemma 14. *Let B be a nilpotent ideal of $E(G)$ and let D be a right module of $E(G)$ such that D/B is nilpotent. Then D is nilpotent.*

Lemma 15. *$E(G)/L(E(G))$ contains no nonzero nilpotent right modules.*

Recall that Beidleman's definition of strong radical-ideal involves minimal right ideals. For reference we state the definition of minimal right ideal below.

Definition. A minimal right ideal of a near-ring R is a right ideal P which contains no proper nonzero R-groups; i.e. P is minimal as an R-group (cf. [1]).

Theorem 16. $L(E(G)) = J(E(G))$.

PROOF. Since $E(G)/L(E(G))$ contains no nonzero nilpotent right modules, we can apply two theorems of Blackett. By the first result, Theorem 2 of [4], every right ideal of $E(G)/L(E(G))$ contains a minimal right ideal. By Theorem 1 of [4] every minimal right module and hence every minimal right ideal of $E(G)/L(E(G))$ contains an idempotent element. Therefore $L(E(G))$ is a strong radical-ideal, so, by Theorem 9, $L(E(G)) = J(E(G))$.

From §3 since G is a finite group, $J(E(G)) = P(E(G))$. Hence $L(E(G)) = I(E(G)) = N(E(G)) = J(E(G)) = P(E(G))$.

Using Lemmas 10 and 11 and the fact that $J(E(G))$ is a nilpotent right module of $E(G)$ which contains all nilpotent right modules of $E(G)$, we see that a map α is contained in $J(E(G))$ if and only if the fully invariant subgroup generated by $(x)\alpha$ is properly contained in the fully invariant subgroup generated by x, for all $x \in G$. In particular if G is a finite, noncommutative group which contains a unique proper fully invariant subgroup H, then $J(E(G)) = A(H) \cap \{\alpha \in E(G) \mid \mathrm{Im}(\alpha) \subseteq H\}$.

5. **Radical-subgroup of $E(G)$.** In §3 we showed that if a group G is equal to the sum of its minimal fully invariant subgroups, both the radical $J(E(G))$ and the radical-subgroup of $E(G)$ are $\{0\}$.

If G is not equal to the sum of its minimal fully invariant subgroups, the radical-subgroup of $E(G)$ need not equal $J(E(G))$. For example, consider $E(S_3)$, where S_3 is the symmetric group on three elements. In Table III of [8] Malone and Lyons have listed the elements of $E(S_3)$. By examining this table, we see that the radical-subgroup of $E(S_3)$ consists of three elements, (00000), (ddd00) and (eee00), which is a proper subset of $J(E(S_3))$.

On the other hand, if G is a finite, nonabelian p-group, where p is a prime number, then Beidleman has shown that $J(E(G))$ is equal to the radical-subgroup of $E(G)$ (cf. [3]).

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